**Gap Conjecture for 3-Dimensional Canonical Thresholds**

By Yuri Prokhorov

**Abstract.** We prove that the interval $(5/6, 1)$ contains no 3-dimensional canonical thresholds.

1. Introduction

We work over the complex number field $\mathbb{C}$.

Let $(X \ni P)$ be a three-dimensional canonical singularity and let $S \subset X$ be a $\mathbb{Q}$-Cartier divisor. The *canonical threshold* of the pair $(X, S)$ is

$$\text{ct}(X, S) := \sup \{ c \mid \text{the pair } (X, cS) \text{ is canonical} \}.$$ 

It is easy to see that $\text{ct}(X, S)$ is rational and non-negative. Moreover, if $S$ is effective and integral, then $\text{ct}(X, S) \in [0, 1]$. Define the subset $T^\text{can}_n \subset [0, 1]$ as follows

$$T^\text{can}_n := \{ \text{ct}(X, S) \mid \dim X = n, S \text{ is integral and effective} \}.$$ 

The following conjecture is an analog of corresponding conjectures for log canonical thresholds and minimal discrepancies, see [Sho88], [Kol92], [Kol97], [MP04], [Kol08].

**Conjecture 1.1.** The set $T^\text{can}_n$ satisfies the ascending chain condition.

The conjecture is interesting for applications to birational geometry, see, e.g., [Cor95]. It was shown in [BS06] that much more general form of 1.1 follows from ACC for minimal log discrepancies and weak Borisov-Alexeev conjecture. The important particular case of 1.1 is the following

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Conjecture 1.2 (cf. [Kol08]). \( \epsilon_n^{\text{can}} := 1 - \sup(T_n^{\text{can}} \setminus \{1\}) > 0. \)

The aim of this note is to prove Conjecture 1.2 for \( n = 3 \) in a precise form:

Theorem 1.3. \( \epsilon_3^{\text{can}} = 1/6. \)

An analog of this theorem for log canonical thresholds was proved by J. Kollár [Kol94]: \( \epsilon_3^{\text{lc}} = 1/42. \)

Note that replacing \((X \ni P)\) with its terminal \(\mathbb{Q}\)-factorial modification we may assume that \((X \ni P)\) is terminal. Thus the following is a stronger form of Theorem 1.3:

Theorem 1.4. Let \((X \ni P)\) be a three-dimensional terminal singularity and let \(S \subset X\) be an (integral) effective Weil \(\mathbb{Q}\)-Cartier divisor such that the pair \((X, S)\) is not canonical. Then \(\text{ct}(X, S) \leq 5/6\) and this bound is sharp. Moreover, if \((X \ni P)\) is singular, then \(\text{ct}(X, S) \leq 4/5.\)

In Section 3 we give examples where the values 5/6 and 4/5 in the above theorem are achieved (see Examples 3.10 and 3.11).

The proof is rather standard. We use the classification of terminal singularities and weighted blowups techniques, cf. [Kaw92], [Kol94], [Mar96].

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2. Preliminaries

2.1. Notation. For a polynomial \( \phi \), \( \text{ord}_0 \phi \) denotes the order of vanishing of \( \phi \) at 0 and \( \phi_d \) is the homogeneous component of degree \( d \).

Throughout this paper we let \((X \ni P)\) be the germ of a three-dimensional terminal singularity and let \(S \subset X\) be an effective Weil \(\mathbb{Q}\)-Cartier divisor such that the pair \((X, S)\) is not canonical. Put \( c := \text{ct}(X, S) > 0. \) Since \((X, S)\) is not canonical, \( c < 1. \) We assume that \( c > 1/2. \)

Lemma 2.2. In the above notation the singularity \((S \ni P)\) is not Du Val.
Proof. Assume that $(S \ni P)$ is Du Val. Since $X \ni P$ is an isolated singularity, by the inversion of adjunction [Sho93, §3] we see that the pair $(X, S)$ is PLT. Further, since $K_S$ is Cartier lifting its nowhere vanishing section to $X$ we can show that $K_X + S$ is also Cartier. Hence, the pair $(X, S)$ is canonical. □

Lemma 2.3. In the above notation $S$ is reduced, irreducible and normal.

Proof. Indeed, otherwise by blowing up a curve in the singular locus of $S$ we get $c \leq 1/2$. □

2.4. We use the techniques of weighted blowups. For definitions and basic properties we refer, for example, to [Mar96], [Rei87]. By fixing coordinates $x_1, \ldots, x_n$ we regard the affine space $\mathbb{C}^n$ as a toric variety. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a weight (a primitive lattice vector in the positive octant) and let $\sigma_\alpha: \mathbb{C}_\alpha^n \to \mathbb{C}^n$ be the weighted blowup with weight $\alpha$ ($\alpha$-blowup). The exceptional divisor $E_\alpha$ is irreducible and determines a discrete valuation $v_\alpha$ of the function field $\mathbb{C}(\mathbb{C}^n)$ such that $v_\alpha(x_i) = \alpha_i$.

2.5. Now let $X \subset \mathbb{C}^n$ be a hypersurface given by the equation $\phi = 0$ and let $X_\alpha \subset \mathbb{C}_\alpha^n$ be its proper transform. Fix an irreducible component $G$ of $E_\alpha \cap X_\alpha$ such that $X_\alpha$ is smooth at the generic point of $G$. Let $v_G$ be the corresponding discrete valuation of $\mathbb{C}(X)$. Write

$$E_\alpha |_{X_\alpha} = m_G G + \text{(other components)}.$$

Assume that $m_G = 1$ and $G$ is not a toric subvariety in $\mathbb{C}_\alpha^n$. Then the discrepancy of $G$ with respect to $K_X$ is computed by the formula

$$a(G, K_X) = |\alpha| - 1 - v_\alpha(\phi), \quad |\alpha| = \sum \alpha_i,$$

see [Mar96]. Let $S \subset X$ be a Cartier divisor and let $\psi$ be a local defining equation of $S$ in $\mathcal{O}_{0,X}$. Then $v_G(\psi) = v_\alpha(\psi)$ and the discrepancy of $G$ with respect to $K_X + cS$ is computed by the formula

$$a(G, K_X + cS) = a(G, K_X) - cv_G(\psi) = |\alpha| - 1 - v_\alpha(\phi) - cv_\alpha(\psi).$$

Therefore,

$$c \leq a(G, K_X)/v_\alpha(\psi) = (|\alpha| - 1 - v_\alpha(\phi))/v_\alpha(\psi).$$
DEFINITION 2.6 (cf. [Mar96]). A weight $\mathbf{\alpha}$ is said to be admissible if $E_\mathbf{\alpha} \cap X_\mathbf{\alpha}$ contains at least one reduced non-toric component.

3. Gorenstein Case

In this section we consider the case where $(X \ni P)$ is either smooth or an index one singularity.

**Lemma 3.1.** If $(X \ni P)$ is smooth, then $c \leq 5/6$.

**Proof.** Let $c > 5/6$. We may assume that $X = \mathbb{C}^3$. Let $\psi(x, y, z) = 0$ be an equation of $S$. Consider a weighted blowup $\sigma_\mathbf{\alpha}: \mathbb{C}^3_\mathbf{\alpha} \to \mathbb{C}^3$ with a suitable weight $\mathbf{\alpha}$. Let $E_\mathbf{\alpha}$ be the exceptional divisor. Recall that $(S \ni P)$ is not Du Val. Since $S$ is normal, up to analytic coordinate change there are the following cases (cf. [KM98, 4.25]):

3.2. **Case** $\text{ord}_0 \psi \geq 3$. Take $\mathbf{\alpha} = (1, 1, 1)$ (usual blowup of 0). Then $a(E_\mathbf{\alpha}, K_X) = 2, v_\mathbf{\alpha}(\psi) = \text{ord}_0 \psi \geq 3$. Hence $c \leq a(E_\mathbf{\alpha}, K_X)/v_\mathbf{\alpha}(\psi) \leq 2/3$, a contradiction.

3.3. **Case** $\psi = x^2 + \eta(y, z)$, where $\text{ord}_0 \eta \geq 4$. Take $\mathbf{\alpha} = (2, 1, 1)$. Then $a(E_\mathbf{\alpha}, K_X) = 3, v_\mathbf{\alpha}(\psi) = 4$. Hence $c \leq a(E_\mathbf{\alpha}, K_X)/v_\mathbf{\alpha}(\psi) \leq 3/4$, a contradiction.

3.4. **Case** $\psi = x^2 + y^3 + \eta(y, z)$, where $\text{ord}_0 \eta \geq 4$. We may assume that $\eta(y, z) = u_ayz^a + u_bz^b$ (see, e.g., [KM98, 4.25]). Since he singularity $(S \ni P)$ is not Du Val, we have $a \geq 4, b \geq 6$ and $u_a, u_b$ are either units or zero. Take $\mathbf{\alpha} = (3, 2, 1)$. Then $a(E_\mathbf{\alpha}, K_X) = 5, v_\mathbf{\alpha}(\psi) = 6$. Hence $c \leq a(E_\mathbf{\alpha}, K_X)/v_\mathbf{\alpha}(\psi) = 5/6$, a contradiction. $\square$

**Lemma 3.5.** Assume that $(X \ni P)$ is a Gorenstein terminal singularity and $(X \ni P)$ is not smooth. Then $c \leq 4/5$.

**Proof.** Let $c > 4/5$. We may assume that $X$ is a hypersurface in $\mathbb{C}^4$ (it is an isolated cDV-singularity [Rei80]). Let $\phi(x, y, z, t) = 0$ be the equation of $X$. Since $(X \ni P)$ is a cDV-singularity, $\text{ord}_0 \phi = 2$. According to [Mar96], in a suitable coordinate system $(x, y, z, t)$, there is an admissible weighted blowup $\sigma_\mathbf{\alpha}: \mathbb{C}^4_\mathbf{\alpha} \to \mathbb{C}^4$ such that at least for one component $G$ of
$E_\alpha \cap X_\alpha$ we have $a(G, K_X) = 1$. Then $c \leq 1/v_\alpha(\psi)$, so $v_\alpha(\psi) = 1$. This means, in particular, that $\text{ord}_0 \psi = 1$. Up to coordinate change we may assume that $\psi = t$. Write

$$\phi = \eta(x, y, z) + t\zeta(x, y, z, t).$$

Then $S$ is a hypersurface in $\mathbb{C}^3_{x,y,z}$ given by $\eta(x, y, z) = 0$. As in the proof of Lemma 3.1, using Morse Lemma we get the following cases:

3.6. Case $\text{ord}_0 \eta \geq 3$. Since $\text{ord}_0 \phi = 2$, $\zeta$ contains a linear term. Take $\alpha = (1, 1, 1, 2)$. By the终端性 condition [Rei87, Th. 4.6], we have $4 = v_\alpha(xyzt) - 1 > v_\alpha(\phi)$.

First we assume that $\zeta$ contains at least one of the terms $x, y, z$. By symmetry we may assume that $\zeta$ contains $x$. After the analytic coordinate change $x \mapsto \zeta(x, y, z, t)$ we obtain

$$\phi = \eta(x, y, z) + tx.$$

In the affine chart $U_x := \{ x \neq 0 \}$ the map $\sigma^{-1}_\alpha$ is given by

(3.7) \quad x \mapsto x', \quad y \mapsto y'x', \quad z \mapsto z'x', \quad t \mapsto t'x'^2.$

$E_\alpha \cap X_\alpha$ is given in $\sigma^{-1}_\alpha(U_x) \simeq \mathbb{C}^4$ by

$$x' = \eta_3(1, y', z') + t' = 0.$$

Hence $\alpha$ is admissible, i.e., $E_\alpha \cap X_\alpha$ has a reduced non-toric component $G$. Then $a(G, K_X) = 1$, $v_G(\psi) = 2$ and $c \leq a(G, K_X)/v_G(\psi) = 1/2$, a contradiction.

Now we assume that $\zeta$ does not contain any of the terms $x, y, z$. Then $\zeta$ contains $t$. So,

$$\phi = \eta(x, y, z) + t^2 + t\xi(x, y, z, t), \quad \text{ord}_0 \xi \geq 2.$$

Further, $v_\alpha(\eta) \leq 3$ and $\eta_3 \neq 0$. We claim that $\alpha$ is admissible. Using (3.7) we see that $E_\alpha \cap X_\alpha$ is given in $\sigma^{-1}_\alpha \simeq \mathbb{C}^4$ by the equations $x' = \eta_3(1, y', z') = 0$. If $\eta_3$ is not a cube of a linear form, then $E_\alpha \cap X_\alpha$ has a reduced non-toric component $G$. Then, as above, $c \leq 1/2$, a contradiction.
Finally assume that $\zeta$ does not contain any of the terms $x, y, z$ and $\eta_3$ is a cube of a linear form. Then, as above, $\eta_3 \neq 0$ and up to linear coordinate change we have $\eta_3(x, y, z) = y^3$. So,

$$\phi = y^3 + \eta^*(x, y, z) + t^2 + t\xi(x, y, z, t), \quad \text{ord}_0 \xi \geq 2, \quad \text{ord}_0 \eta^* \geq 4.$$ 

Put $\alpha' = (2, 2, 2, 3)$. Again, in the affine chart $U_x := \{x \neq 0\}$ the map $\sigma_{\alpha'}^{-1}$ is given by $x \mapsto x'^2$, $y \mapsto y'x'^2$, $z \mapsto z'x'^2$, $t \mapsto t'x'^3$, where $\sigma_{\alpha'}^{-1}(U_x) \simeq \mathbb{C}/\mu_2(1, 0, 0, 1)$ and

$$E_{\alpha'} \cap X_{\alpha'} \cap \sigma_{\alpha'}^{-1}(U_x) = \{x' = 0, \ y'^3 + t'^2 = 0\}.$$ 

Thus $\alpha'$ is admissible and for some component $G'$ of $X_{\alpha'} \cap E_{\alpha'}$ we have $a(G', K_X) = 2$, $v_{G'}(\psi) = 3$, $c \leq 2/3$, a contradiction.

**3.8. Case $\eta = x^2 + \zeta(y, z)$, where $\text{ord}_0 \xi \geq 4$.** By Morse Lemma we may assume that $\zeta$ does not depend on $x$. Write the linear part of $\zeta$ in the form $\zeta_1 = \delta_1y + \delta_2z + \delta_3t$, $\delta_i \in \mathbb{C}$. Take $\alpha = (2, 1, 1, 3)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\alpha}^{-1}$ is given by $x \mapsto x'y^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y^3$ and

$$E_{\alpha} \cap X_{\alpha} \cap \sigma_{\alpha}^{-1}(U_y) = \{y' = 0, \ x'^2 + \zeta_4(1, z') + \delta_1t' + \delta_2t'z' = 0\}.$$ 

If either $\delta_1 \neq 0$ or $\delta_2 \neq 0$ or $\zeta_4 \neq 0$, then $E_{\alpha} \cap X_{\alpha}$ is reduced (at least over $U_y$). Hence, $\alpha$ is admissible and for some component $G$ of $E_{\alpha} \cap X_{\alpha}$ we have $c \leq a(G, K_X)/v_G(\psi) = 2/3$, a contradiction. Thus $\delta_1 = \delta_2 = 0$ and $\zeta_4 = 0$. Then we can write

$$\phi = x^2 + \zeta(y, z) + \delta_3t^2 + t\zeta^*(y, z, t), \quad \text{ord}_0 \xi \geq 5, \quad \text{ord}_0 \zeta^* \geq 2.$$ 

Take $\alpha' = (2, 1, 1, 2)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\alpha'}^{-1}$ is given by $x \mapsto x'y^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y^2$ and

$$E_{\alpha'} \cap X_{\alpha'} \cap \sigma_{\alpha'}^{-1}(U_y) = \{y' = 0, \ x'^2 + \delta_3t'^2 + t'\zeta^*(2)(1, z', 0) = 0\},$$ 

where $\zeta^*(2)(y, z, t) = \zeta^*(2)(y, z, 0)$ is the degree 2 weighted homogeneous part of $\zeta^*$. If $\delta_3 \neq 0$ or $\zeta^*(2) \neq 0$, as above, $\alpha'$ is admissible and $c \leq 1/2$, a contradiction. Thus $\delta_3 = 0$, $\zeta^*(2) = 0$, and

$$\phi = x^2 + \zeta(y, z) + \delta t^3 + t\zeta^0(y, z, t), \quad \delta \in \mathbb{C}, \quad \text{ord}_0 \xi \geq 5, \quad \text{ord}_0 \zeta^0 \geq 3.$$
Applying the terminality condition [Rei87, Th. 4.6] with weight \((2,1,1,1)\) we get \(\delta \neq 0\).

Take \(\alpha''' = (3,1,1,2)\). Again by the terminality condition \(\xi_5 \neq 0\) or \(\zeta_3 \neq 0\), where \(\zeta_3\) is the degree 3 weighted homogeneous part of \(\zeta\). As above we get

\[
E_{\alpha'''} \cap X_{\alpha'''} \cap \sigma_{\alpha'''}^{-1}(U_x) = \{x' = 0, \xi_5(y',z') + t'\zeta_3(y',z',t') = 0\}.
\]

If either \(\zeta_3 \neq 0\) or \(\xi_5\) has a factor of multiplicity 1, then \(\alpha''\) is admissible and \(c \leq \frac{1}{2}\), a contradiction.

Therefore, we may assume that \(\zeta_3 = 0\) and \(\xi_5\) has only multiple factors. Up to linear coordinate change of \(y\) and \(z\) we can write \(\xi_5 = y^5\) or \(\xi_5 = y^2z^3\).

Take \(\alpha''' = (3,2,1,2)\). Then \(\alpha'''\) is admissible and \(c \leq \frac{1}{2}\), a contradiction.

### 3.9. Case \(\eta = x^2 + y^3 + \xi(y,z)\), where \(\text{ord}_0 \xi \geq 4\)

As in [KM98, 4.25] we may assume that \(\xi(y,z) = u_ayz^a + u_bz^b\). Since the singularity \((S \ni P)\) is not Du Val, we have \(a \geq 4, b \geq 6\) and \(u_a, u_b\) are either units or zero. Write the linear part of \(\xi\) in the form \(\zeta_1 = cz + \ell(x,y,t)\). Let \(\zeta_6\) is the degree 6 weighted homogeneous part of \(\xi\) with respect to \(\text{wt}(y,z) = (2,1)\). Clearly, \(\zeta_6\) is a linear combination of \(z^6\) and \(yz^4\). Take \(\alpha = (3,2,1,5)\). In the affine chart \(U_z := \{z \neq 0\}\) the map \(\sigma_{\alpha}^{-1}\) is given by \(x \mapsto x'z^3, y \mapsto y'z^2, z \mapsto z', t \mapsto t'z^5\) and

\[
E_{\alpha} \cap X_{\alpha} \cap \sigma_{\alpha}^{-1}(U_z) = \{z' = 0, x'^2 + y'^3 + \zeta_6(y',1) + \delta t' = 0\},
\]

where \(\delta\) is a constant and \(\zeta_6(y',1)\) contains no \(y'^3\). Hence \(\alpha\) is admissible, i.e., \(E_{\alpha} \cap X_{\alpha}\) has a reduced non-toric component \(G\). Then \(a(G,K_X) = 4, v_G(\psi) = 5\), and \(c \leq a(G, K_X)/v_G(\psi) \leq 4/5\), a contradiction.

The following examples show that bounds \(\text{ct}(X,S) \leq 5/6\) and \(\leq 4/5\) in Theorem 1.4 are sharp.

**Example 3.10.** Let \(X = \mathbb{C}^3\) and let \(S = S^d\) be given by \(x^2 + y^3 + z^d\), \(d \geq 6\). Then \(\text{ct}(\mathbb{C}^3, S^d) = 5/6\). We prove this by descending induction on \([d/6]\). Take \(\alpha = (3,2,1)\) and consider the \(\alpha\)-blowup \(\sigma_{\alpha}: \mathbb{C}^3_{\alpha} \to \mathbb{C}^3\). Let \(S_{\alpha} \subset X_{\alpha}\) be the proper transform of \(S\). We have \(a(E_{\alpha}, K_X) = 5\) and \(v_{\alpha}(\psi) = 6\). Hence, \(\text{ct}(\mathbb{C}^3, S^d) \leq 5/6\). Further,

\[
K_{\mathbb{C}^3_{\alpha}} + \frac{5}{6} S_{\alpha} = \sigma_{\alpha}(K_{\mathbb{C}^3} + \frac{5}{6} S).
\]
Thus it is sufficient to show that $\text{ct}(X_\alpha, S_\alpha)$ is canonical. We have three affine charts:

- $U_x := \{ x \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto x^3, y \mapsto y'x^2, z \mapsto z'x', S_\alpha$ is given in $\sigma^{-1}_\alpha(U_x) \simeq \mathbb{C}^3/\mu_3(-1,2,1)$ by the equation $1 + y^3 + z^d x^{d-6} = 0$. Hence, in this chart, $S_\alpha$ is smooth and does not pass through a (unique) singular point of $\sigma^{-1}_\alpha(U_x)$.

- $U_y := \{ y \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto x'y^3, y \mapsto y'^2, z \mapsto z'y', S_\alpha$ is given in $\sigma^{-1}_\alpha(U_y) \simeq \mathbb{C}^3/\mu_2(3,-1,1)$ by the equation $x'^2 + 1 + z'y'^d = 0$. Again, in this chart, $S_\alpha$ is smooth and does not pass through a (unique) singular point of $\sigma^{-1}_\alpha(U_y)$.

- $U_z := \{ z \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto x'z^3, y \mapsto y'z^2, z \mapsto z', S_\alpha$ is given in $\sigma^{-1}_\alpha(U_z) \simeq \mathbb{C}^3$ by the equation $x'^2 + y'^3 + z'^d = 0$. In this chart, $(X_\alpha, S_\alpha)$ is canonical (moreover, $(X_\alpha, S_\alpha)$ is canonical if $d \leq 11$). Therefore, $\text{ct}(X, S) = 5/6$.

Example 3.11. Let $X \subset \mathbb{C}^4$ is given by $x^2 + y^3 + z^d + tz = 0$, $d \geq 7$ and let $S$ cut out by $t = 0$. Take $\alpha = (3,2,1,5)$ and consider the $\alpha$-blowup $\sigma_\alpha: X_\alpha \to X$. Let $S_\alpha \subset X_\alpha$ be the proper transform of $S$. We see below that $\alpha$ is admissible. Moreover, the exceptional divisor $G := E_\alpha \cap X_\alpha$ is reduced and irreducible. We have four charts:

- $U_x := \{ x \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto x^3, y \mapsto yx^2, z \mapsto zx, t \mapsto tx^5$, $X_\alpha$ is given in $\sigma^{-1}_\alpha(U_x) \simeq \mathbb{C}^4/\mu_3(-1,2,1,5)$ by the equation $1 + y^3 + z^d x^{d-6} + tz = 0$ and $S_\alpha$ by two equations $x = 1 + y^3 + tz = 0$. Hence, in this chart, both $X_\alpha$ and $S_\alpha$ are smooth.

- $U_y := \{ y \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto xy^3, y \mapsto y^2, z \mapsto zy, t \mapsto ty^5$, $\sigma^{-1}_\alpha(U_y) \simeq \mathbb{C}^4/\mu_2(3,-1,1,5)$, $X_\alpha = \{ x^2 + 1 + z^d y^{d-6} + tz = 0 \}$, and $S_\alpha = \{ y = x^2 + 1 + tz = 0 \}$. As above, both $X_\alpha$ and $S_\alpha$ are smooth in this chart.

- $U_z := \{ z \neq 0 \}$. Here $\sigma^{-1}_\alpha: x \mapsto xz^3, y \mapsto yz^2, z \mapsto z, t \mapsto tz^5$, $\sigma^{-1}_\alpha(U_z) \simeq \mathbb{C}^4$, $X_\alpha = \{ x^2 + y^3 + z^{d-6} + t = 0 \}$, and $S_\alpha = \{ z = x^2 + y^3 + t = 0 \}$. As above, both $X_\alpha$ and $S_\alpha$ are smooth in this chart.
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There is an analytic \( \sigma^{-1}_a : x \mapsto xt^3, y \mapsto yt^2, z \mapsto zt, t \mapsto t^5 \), \( \sigma^{-1}_a(U_t) \simeq \mathbb{C}^4/\mu_5(3, 2, 1, -1) \), \( X_\alpha = \{ x^2 + y^3 + z^d t^d - 6 + z = 0 \} \), and \( S_\alpha = \{ t = x^2 + y^3 + z = 0 \} \). The variety \( X_\alpha \) has a unique singular point \( Q \) at the origin and this point is terminal of type \( \frac{1}{5}(3, 2, -1) \). In this case, \( S_\alpha \in -K_{U_t} \) and the pair \( (U_t, S_\alpha) \) is canonical.

Thus we have \( a(G, K_X) = 4 \), \( v_\alpha(\psi) = 5 \), and \( a(G, K_X + \frac{4}{5} S) = 0 \). Therefore,

\[
K_{X_\alpha} + \frac{4}{5} S_\alpha = \sigma^{-1}_a(K_X + \frac{4}{5} S).
\]

Since the pair \( K_{X_\alpha} + \frac{4}{5} S_\alpha \) is canonical, \( \text{ct}(X, S) = 4/5 \).

4. Non-Gorenstein Case

Now we assume that \( (X \ni P) \) is a (terminal) point of index \( r > 1 \). Let \( \pi : (X^2 \ni P^2) \rightarrow (X \ni P) \) be the index-one cover and let \( S^2 := \pi^{-1}(S) \).

**Lemma 4.1.** If \( (X \ni P) \) is a cyclic quotient singularity, then \( \text{ct}(X, S) \leq 1/2 \).

**Proof.** By our assumption we have \( X \simeq \mathbb{C}^3/\mu_r(a, -a, 1) \) for some \( r \geq 2 \), \( 1 \leq a < r \), \( \gcd(a, r) = 1 \). Assume that \( c = \text{ct}(X, S) > 1/2 \). Let \( \psi = 0 \) be a defining equation of \( S^2 \). Consider the weighted blowup \( \sigma_\alpha : X_\alpha \rightarrow X \) with weights \( \alpha = \frac{1}{r}(a, r-a, 1) \). Then \( a(E_\alpha, K_X) = 1/r \). Since \( a(E_\alpha, K_X) - cv_\alpha(\psi) \geq 0 \), we have \( v_\alpha(\psi) \leq a(E_\alpha, K_X)/c < 2a(E_\alpha, K_X) = 2/r \) and so \( v_\alpha(\psi) = 1/r \). Thus we may assume that \( \psi \) contains \( x_3 \) (if \( a \equiv \pm 1 \) we possibly have to permute coordinates). Then \( S^2 \simeq \mathbb{C}^2 \) is smooth and \( S \simeq \mathbb{C}^2/\mu_r(a, -a) \), i.e., \( S \) is Du Val of type \( A_{r-1} \). \( \square \)

**Lemma 4.2.** If \( (X \ni P) \) is a terminal singularity of index \( r > 1 \) and \( \text{ct}(X, S) > 1/2 \), then \( K_X + S \sim 0 \).

**Proof.** By Lemma 4.1 \( (X \ni P) \) is not a cyclic quotient singularity. There is an analytic \( \mu_r \)-equivariant embedding \( (X^2, P^2) \subset (\mathbb{C}^4, 0) \). Let \( (x_1, x_2, x_3, x_4) \) be coordinates in \( \mathbb{C}^4 \), let \( \phi = 0 \) be an equation of \( X^2 \), and let \( \psi = 0 \) be an equation of \( S^2 \). We can take \( (x_1, x_2, x_3, x_4) \) and \( \phi \) to be semi-invariants such that one of the following holds \( [\text{Mor85}] \) (see also \( [\text{Rei87}] \)):

- **Main series.** \( \text{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (a, -a, 1, 0; 0) \mod r \), where \( \gcd(a, r) = 1 \).
- **Case** $cAx/4$. $r = 4$, $\text{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (1, 3, 1, 2; 2) \mod 4$.

In both cases $\text{wt}(x_1x_2x_3x_4) - \text{wt} \phi \equiv \text{wt} x_3 \mod r$. According to [Kaw92] there is a weight $\alpha$ such that for the corresponding $\alpha$-blowup $\sigma_\alpha : X_\alpha \subset W \rightarrow X \subset \mathbb{C}^4/\mu_r$ the exceptional divisor $E_\alpha \cap X_\alpha$ has a reduced component $G$ of discrepancy $a(G, K_X) = 1/r$. Moreover, $r\alpha_i \equiv \text{wt} x_i \mod r$, $i = 1, 2, 3, 4$. Since $c > 1/2$, we have $1/r - cv_\alpha(\psi) \geq 0$, i.e., $rv_\alpha(\psi) < 2$, so $rv_\alpha(\psi) = 1$. In particular, $\text{wt} \psi \equiv 1 \mod r$.

Let $\omega$ be a section of $\mathcal{O}_X(-K_X)$. Then $\omega$ can be written as

$$\omega = \lambda(\partial \phi/\partial x_4)(dx_1 \wedge dx_2 \wedge dx_3)^{-1},$$

where $\lambda$ is a semi-invariant function with

$$\text{wt} \lambda - \text{wt}(x_1x_2x_3x_4) + \text{wt} \phi \equiv \text{wt} \omega \equiv 0 \mod r.$$

Thus, $\text{wt} \psi \equiv \text{wt} \lambda \mod r$. Hence, $S \sim -K_X$. □

**Lemma 4.3.** If $(X \ni P)$ is a terminal singularity of index $r > 1$, then $c \leq 4/5$.

**Proof.** Since $\pi$ is étale in codimension one, we have $K_{X^\sharp} + cS^\sharp = \pi^*(K_X + cS)$. Hence the pair $(X^\sharp, cS^\sharp)$ is canonical (see, e.g., [Kol97, 3.16.1]). Assume that $c > 4/5$. By Lemma 4.1 the point $(X^\sharp \ni P^\sharp)$ is singular. Then by Lemma 3.5 the pair $(X^\sharp, S^\sharp)$ is canonical. Therefore, $(S^\sharp \ni P^\sharp)$ is a Du Val singularity. Then the singularity $(S \ni P) = (S^\sharp \ni P^\sharp)/\mu_r$ is log terminal. On the other hand, by Lemma 4.2 the divisor $K_S$ is Cartier. Hence, $(S \ni P)$ is Du Val, a contradiction. □

**References**

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