A Note on Semistable Barsotti-Tate Groups

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Abstract. We show that the Dieudonné crystal associated to a Barsotti-Tate group with potentially semistable reduction over a smooth curve is overconvergent. As a corollary, we obtain the rationality of the $L$-function associated to this group.

1. Introduction

Let $U/\mathbb{F}_p$ be a smooth curve and $G/U$ a Barsotti-Tate group. Assume $G/U$ has potentially semistable reduction (see 4.2 for a precise definition). We show that the Dieudonné crystal as defined in [1] is overconvergent in the sense of Berthelot. As a corollary we get the rationality of the $L$-function associated to $G/U$. In the third section we study the local situation, that is, semistable Barsotti-Tate groups over a complete discrete valuation field of equal characteristic $p$. Using Extension groups in the category of Dieudonné crystals and their interpretation in terms of syntomic cohomology (as defined in [13]) we prove that the Dieudonné crystal associated to such group extends to a log Dieudonné crystal over the ring of integers. Using the gluing properties of overconvergent $F$-isocrystals over smooth curves proved in [14], we deduce from section three the overconvergence of the Dieudonné crystal associated to $G/U$ and the rationality of its $L$-function in the last section. We end both sections three and four by some open questions.

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3. Semistable Barsotti-Tate Groups and Extensions

In this section, we extend the Dieudonné crystal of a semistable Barsotti-
Tate group over a complete discrete valuation field of equal characteristic
$p > 0$ to a log Dieudonné crystal.

3.1. Let $k$ be a perfect field of characteristic $p$ endowed with its Frobe-
nius $\sigma$, $W := W(k)$ the ring of Witt vectors and $K = Frac(W)$. We denote
$\eta := \text{Spec}(k((t)))$. Let $G_{\eta}/\eta$ a Barsotti-Tate group. Following [5]:

3.2 Definition. The Barsotti-Tate group $G_{\eta}/\eta$ is called semistable if
there exists a filtration:

$$0 \subset G^\mu \subset G^f \subset G_{\eta}$$

by Barsotti-Tate groups such that the following conditions hold:

1. $G^f$ and $G_{\eta}/G^\mu$ extend to Barsotti-Tate groups $G_1$ and $G_2$ over $k[[t]]$.
   In this case, the composed map
   $$G^f \hookrightarrow G_{\eta} \rightarrow G_{\eta}/G^\mu$$

   extends to a map $G_1 \rightarrow G_2$.

2. $G^\mu_1 := \ker(G_1 \rightarrow G_2)$ and $G^\text{\acute{e}t}_2 := \text{coker}(G_1 \rightarrow G_2)$ are Barsotti-Tate
   groups over $k[[t]]$.

3. $G^\mu_1$ is of multiplicative type and $G^\text{\acute{e}t}_2$ is étale.

3.3 Remark. It has been shown in [5], 2.5, that an abelian variety $A$
over $\eta$ has semistable reduction if and only if its associated Barsotti-Tate
group $G_{\eta} := \lim_{\rightarrow n} A[p^n]$ is semistable.

3.4. Let $S$ be a fine log-scheme over $\text{Spec}(k)$ endowed with the trivial
log-structure. We denote the absolute Frobenius of $S$ by $\sigma_S$, lying above $\sigma$.
We work on the log crystalline site with the étale topology, denoted
A crystal $E$ on $\text{Crys}(S/W)$ is called a Dieudonné crystal if it is a finite locally free crystal endowed with linear operators $F : \sigma_S^*E \to E$ and $V : E \to \sigma_S^*E$ called respectively Frobenius and Verschiebung such that $FV = p$ and $VF = p$. If $(D, F_D, V_D)$ is a Dieudonné crystal on $\text{Crys}(S/W)$, its $\mathcal{O}_{S/W}$-dual $D^\vee$ is endowed with a structure of Dieudonné crystal such that $F_{D^\vee} = (V_D)^\vee$ and $V_{D^\vee} = (F_D)^\vee$.

3.6. Let $G$ be a Barsotti-Tate group over $S$. By the crystalline Dieudonné theory (see for example [1], [2], [4]), the Dieudonné crystal $\mathbb{D}(G)$ on $\text{Crys}(S/W)$ is defined by forgetting the log structures of objects of $\text{Crys}(S/W)$ ($\mathbb{D}$ is a contravariant functor). More precisely, let $\pi$ denote the canonical morphism from $S$ to $S_{\text{triv}}$, the scheme $S$ endowed with the trivial log-structure. Then $\pi^*\mathbb{D}(G)$ is a Dieudonné crystal on $\text{Crys}(S/W)$ that we still denote $\mathbb{D}(G)$. The $\mathcal{O}_{S/W}$-dual of $\mathbb{D}(G)$ will be denoted by $\mathbb{D}(G)$, so that $D(\cdot)$ becomes a covariant functor. We will furthermore, denote by $\mathbf{1} := D(\mathbb{Q}_p/\mathbb{Z}_p)$ the Dieudonné crystal $(\mathcal{O}_{S/W}, F = p, V = \text{id})$ and by $\mathbf{1}(1) := D(\mu_{p^\infty})$ the Dieudonné crystal $(\mathcal{O}_{S/W}, F = \text{id}, V = p)$. The Dieudonné crystals $\mathbf{1}$ and $\mathbf{1}(1)$ are dual to each other.

3.7. We recall the construction of the syntomic cohomology as defined in [13] in the case $S = \text{Spec}(k[[t]])$ with the log structure associated to
the closed point. Let $D$ be a Dieudonné crystal over $S/W$. The syntomic complex $S_D$ is the total complex associated to the bicomplex

$$
\begin{array}{cc}
D^0 & D \otimes \Omega^1_{\mathcal{Y}} \\
1 - F_1 & 1 - F_2 \\
D & D \otimes \Omega^1_{\mathcal{Y}}
\end{array}
$$

We explain the notations: $\mathcal{Y} = Spf(W[[t]])$ is endowed with the log-structure associated to $\mathbb{N} \to W[[t]]$ sending $n$ to $t^n$. It is a log smooth formal lifting of $S$ and we denote $\sigma_\mathcal{Y}$ a lifting of the Frobenius of $S$ sending the variable $t$ to $t^p$. By abuse of notation, we still denote $(D, \nabla, F_D, V_D)$ the realization of the Dieudonné crystal $D$ at the p.d. thickening $(S \subset \mathcal{Y})$ endowed with its connection, Frobenius and Verschiebung. Consider the composed map

$$
D \xrightarrow{\iota} \sigma_\mathcal{Y}^* D \to \sigma_\mathcal{Y}^* D / V_D(D)
$$

where $\iota$ is the map sending $x \to 1 \otimes x$. Set $\text{Lie}(D)$ to be the image of the above map. Then $\text{Lie}(D)$ is a locally free $O_S$-module (see [13], 5.3) and we denote $D^0$, the kernel of the surjective map $D \to \text{Lie}(D)$. Finally, we explain the Frobenius operators. The map $F_1 : D^0 \to D$ is constructed as follows: the composed map

$$
\tilde{F}_1 : D^0 \xrightarrow{1} D \xrightarrow{\iota} \sigma_\mathcal{Y}^* D \xrightarrow{F_D} D,
$$

is in $p.D$ (see [13], 5.8.1) and we set $F_1 := p^{-1} \tilde{F}_1$. On the other side, remark that $\sigma_\mathcal{Y}(\Omega^1_{\mathcal{Y}}) \subset p.\Omega^1_{\mathcal{Y}}$ so that we can define a map

$$
F_2 := F_D \circ \iota \otimes p^{-1} \sigma_\mathcal{Y}.
$$

3.8 Proposition. Assume $k$ is algebraically closed and let $S = \text{Spec}(k[[t]])$ endowed with the log structure associated to the closed point. Then, we have:

$$
H^i(S, \mathcal{S}_{1(1)}) = H^i(\eta, \mathcal{S}_{1(1)}) = \begin{cases} 
\hat{k}((t))^\times, & i = 1 \\
0, & \text{otherwise},
\end{cases}
$$

where $\hat{M} = \lim \limits_{\leftarrow n} M/M^{p^n}$ for any multiplicative group $M$. 
Proof. First, we prove the claim for $H^i(\eta, S_{1(1)})$. By [13], 5.10, we have

$$H^i(\eta, S_{1(1)}) = H^i_{fl}(\eta, T_p G_m).$$

Since $k((t))$ is a $C_1$-field, we have

$$H^i_{fl}(\eta, G_m) = \begin{cases} k((t))^\times, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By using the short exact sequence

$$0 \to \mu_{p^n} \to G_m \overset{p^n}{\to} G_m \to 0$$

on the flat site, we see that

$$H^i_{fl}(\eta, T_p G_m) = \begin{cases} k((t))^\times, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

So the claim for $H^i(\eta, S_{1(1)})$ is proved.

Next we prove the claim for $H^i(S, S_{1(1)})$. In the case of the crystal $D = 1(1)$ the short exact sequence

$$0 \to D^0 \to D \to \text{Lie}(D) \to 0$$

is induced by the canonical short exact sequence in the crystalline site:

$$0 \to I_{S/W} \to O_{S/W} \to G_a \to 0$$

which induces on the pd-thichening $S \subset \mathcal{Y}$ the short exact sequence:

$$0 \to p.W[[t]] \to W[[t]] \to k[[t]] \to 0.$$
where $d: W[[t]] ightarrow W[[t]]\frac{dt}{t}$ is the map sending an element $\sum_i a_i t^i$ to $(\sum_i a_i t^i)\frac{dt}{t}$, $F_1$ is the map sending an element $p.\sum_i a_i t^i$ to $\sum_i \sigma(a_i)t^{pi}$ and $F_2$ the map sending an element $(\sum_i a_i t^i)\frac{dt}{t}$ to $(\sum_i \sigma(a_i)t^{pi})\frac{dt}{t}$. Hence, $S_{1(1)}$ is the complex concentrated in degree $0, 1, 2$:

$$[pW[[t]] \xrightarrow{d, 1-F_1} W[[t]]\frac{dt}{t} \oplus W[[t]] \xrightarrow{1-F_2, -d} W[[t]]\frac{dt}{t}] .$$

Remark that this complex is isomorphic to the complex

$$[W[[t]] \xrightarrow{pd, p-\sigma} W[[t]]\frac{dt}{t} \oplus W[[t]] \xrightarrow{1-F_2, -d} W[[t]]\frac{dt}{t}] .$$

We compute the $H^0$: By definition $H^0 = \text{Ker} (d) \cap \text{Ker} (1 - F_1)$. Since $\text{Ker}(d) = pW$, $H^0$ is equal to the set of element $p.a \in pW$ such that $pa - \sigma(a) = 0$. Since the $p$-adic valuation $v(\sigma(a))$ is equal to $v(a)$, the previous equality gives $a = 0$.

We compute the $H^2$: to show that this is zero, we just need to show that the map $\pi := (1 - F_2, -d)$ is surjective. But for any $\sum_i c_i t_i \frac{dt}{t} \in W[[t]]\frac{dt}{t}$, the element $(\sum_i b_i t_i \frac{dt}{t}, 0)$, with $b_i = c_i + \sigma(b_{i/p})$ if $p$ divide $i$ and $b_i = c_i$ else is an antecedent of $\sum_i c_i t_i \frac{dt}{t}$ by $\pi$.

We now turn to the computation of $H^1 := \text{Ker} (\pi)/\text{Im}(d, 1 - F_1)$. The group $\text{Ker}(\pi)$ is the set of elements $(\sum_i a_i t_i \frac{dt}{t}, \sum_i b_i t_i)$ such that $a_0 \in \mathbb{Z}_p$ and for $n$, any positive integer with $p$-adic valuation $r$, $a_n = nb_n + (n/p)\sigma(b_{n/p}) + ... + (n/p^r)\sigma^r(b_{n/p^r})$. We get this way an isomorphism

$$\text{Ker} (\pi) \simeq \mathbb{Z}_p \frac{dt}{t} \oplus W[[t]]$$

by sending $(\sum_i a_i t_i \frac{dt}{t}, \sum_i b_i t_i)$ to $(a_0, \sum_i b_i t_i)$, which induces an isomorphism

$$\text{Im}(d, 1 - F_1) \simeq 0 \oplus \text{Im}(1 - F_1),$$

since the elements in $\text{Im}(d)$ have no constant terms.

We get

$$H^1 = \mathbb{Z}_p \frac{dt}{t} \oplus W[[t]]/\text{Im}(1 - F_1).$$

On the other hand, $k((t))^x \simeq t^\mathbb{Z} \times k^x \times (1 + tk[[t]])$ and $(k((t))^x)^{p^n} = k((t^{p^n}))^x \simeq t^{p^n\mathbb{Z}} \times k^x \times (1 + tp^n k[[t^{p^n}]]).$ So, we are reduced to identify
W[[t]]/Im(1 − F_1) and \( \lim_{n \to \infty} (1 + tk[[t]])/(1 + t^{p^n} k[[t^{p^n}]]) \). We first prove that the lefthand side is \( p \)-adically complete. By, [15], chapter 8, it is enough to prove that \( I = \text{Im}(1 - F_1) \) is closed and in particular complete. Let \( (f_m(t) = \sum_i b_i^{(m)} t_i)_{m \in \mathbb{N}} \) a sequence of elements in \( I \) converging to \( f(t) = \sum_i b_i t_i \in W[[t]] \). We want to show that \( f(t) \) is in fact in \( I \). Since \( f_m(t) \in I \), for any \( m \), there exists some sequence \( (a_i^{(m)})_{i \in W} \) such that \( b_i^{(m)} = p a_i^{(m)} - \sigma(a_i^{(m)})^p \), if \( p \) divides \( i \) and \( b_i^{(m)} = p a_i^{(m)} \) else. We construct by induction on the \( p \)-adic valuation of \( i \), a sequence \( (a_i)_{i \in W} \) such that \( (1 - F_1)(\sum_i p a_i t_i) = f(t) \). For \( v_p(i) = 0 \), that is when \( p \) does not divide \( i \), \( p a_i \) converges when \( m \) goes to infinity to \( b_i = p a_i \). Assume now that for any \( i \) such that \( v_p(i) \leq r \), \( (a_i)_{i \in W} \) converges to an element \( a_i \in W \). Then, if \( v_p(i) = r + 1 \), we have \( b_i^{(m)} = p a_i^{(m)} - \sigma(a_i^{(m)})^p \), with \( (b_i^{(m)})_m \) converging to an element \( b_i \) and by induction hypothesis, \( (\sigma(a_i^{(m)})^p)_m \) converging to an element \( \sigma(a_i^p) \) and so we deduce that \( (a_i^{(m)})_p \) converges to an element \( a_i \in W \).

Let \( D = 1(1) \). We compute now \( H^1(S, S_D)/p^n \): we have a short exact sequence

\[
0 \to S_D \overset{\times p^n}{\to} S_D \to S_{D,n} \to 0,
\]

which induces an exact sequence

\[
H^1(S, S_D) \overset{\times p^n}{\to} H^1(S, S_D) \to H^1(S, S_{D,n}) \to H^2(S, S_D).
\]

Since we already have proved that \( H^2(S, S_D) = 0 \), we deduce for any \( n \) the isomorphisms

\[
H^1(S, S_D)/p^n \simeq H^1(S, S_{D,n}).
\]

By [13], 5.14.6, we also have

\[
H^1(\eta, S_D)/p^n \simeq H^1(\eta, S_{D,n}).
\]

Again, by using the short exact sequence:

\[
0 \to S_{D,1} \to S_{D,n+1} \overset{\times p}{\to} S_{D,n} \to 0
\]
and the 5-lemma, we are reduced by induction to prove that

\[ H^1(S,S_{D,1}) \simeq H^1(\eta,S_{D,1}). \]

Using the second description of the syntomic complex, we have the quasi-isomorphisms:

\[ S_{1(1),S} \otimes \mathbb{Z}/p \simeq [k[[t]] \frac{dt}{t} \oplus k[[t]] \frac{\pi t}{t}], \]

\[ S_{1(1),\eta} \otimes \mathbb{Z}/p \simeq [k((t)) \frac{dt}{t} \oplus k((t)) \frac{\pi t}{t}] \]

and the map \( H^1(S,S_{1(1)})/p \to H^1(\eta,S_{1(1)})/p \) is induced by the natural inclusion

\[ k[[t]] \frac{dt}{t} \oplus k[[t]] \to k((t)) \frac{dt}{t} \oplus k((t)). \]

Now, we compute \( H^1(\eta,S_{1(1)})/p \). For any element \( (\sum_i a_it^i, \sum_i b_it^i) \in \text{Ker}(\pi_\eta) \) we find the same conditions that \( a_0 \in \mathbb{F}_p \) and for \( n \), any positive integer with \( p \)-adic valuation \( r \), \( a_n = nb_n+(n/p)\sigma(b_{n/p})+...+(n/p^r)\sigma^r(b_{n/p^r}) \). For negative integers and working modulo \( \text{Im}(\sigma) = k((t^p)) \), we claim that only the \( b_j \)'s with \( b_{-j} = 0 \) for any \( j \) prime to \( p \), gives a solution. Namely, for such \( j \) we have \( a_{-j} = -jb_{-j} \) but then \( a_{-jp^k} = \sigma(-jb_{-j}) \) for any positive integer \( k \). But since \( \sum_i a_i t^i \in k((t)) \), we must have \( a_{-jp^k} = 0 \) for \( k \) big enough. Therefore, the canonical inclusion

\[ k[[t]] \frac{dt}{t} \oplus k[[t]] \to k((t)) \frac{dt}{t} \oplus k((t)) \]

induces the identity map

\[ H^1(S,S_D)/p = \mathbb{F}_p \frac{dt}{t} \oplus k[[t]]/k[[t^p]] \to \mathbb{F}_p \frac{dt}{t} \oplus k[[t]]/k[[t^p]] = H^1(\eta,S_D)/p. \]

Hence, we proved the canonical isomorphism \( H^1(S,S_D) \simeq H^1(\eta,S_D) \) and so the proof of the proposition is finished. \( \square \)

3.9. Let \( D_1, D_2 \) some Dieudonné crystals over \( S/W \). We will denote \( \text{Ext}_{S/W}(D_1, D_2) \) (or \( \text{Ext}(D_1, D_2) \) if there is no ambiguity) the isomorphism classes of extensions

\[ 0 \to D_2 \to ? \to D_1 \to 0. \]
in the category of Dieudonné crystals over \( S/W \). Any commutative diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
\text{Spec}(W) & \xrightarrow{g} & \text{Spec}(W)
\end{array}
\]

induces in the crystalline topos a functor \( f^*: (S'/W)_{\text{crys}} \to (S/W)_{\text{crys}} \) allowing to define for any Dieudonné crystals \( D_1 \) and \( D_2 \) over \( S/W \) a canonical map

\[
f^*: \text{Ext}^1(D_1, D_2) \to \text{Ext}^1(f^*D_1, f^*D_2),
\]

sending the isomorphism class of an extension:

\[
0 \to D_2 \to ? \to D_1 \to 0
\]

to the isomorphism class of the extension

\[
0 \to f^*D_2 \to f^*? \to f^*D_1 \to 0.
\]

(The exactness of this sequence follows from the local freeness of \( D_1 \).)

**3.10.** Let \( G_{\eta}/\eta \) a semistable Barsotti-Tate group and denote as in 3.2 \( G^\ell_\eta, G^\mu_\eta, G_1, G^\mu_1, G_2 \) and \( G^\ell_2 \) its associated Barsotti-Tate groups. We denote \( S := \text{Spec}(k[[t]]) \) endowed with the log-structure induced by its closed point. We also denote \( j: \eta \to \text{Spec}(k[[t]]) \) the open immersion. Then there is a commutative diagram of exact sequences:

\[
\begin{array}{ccc}
\text{Ext}(D(G^\ell_2), D(G^\mu_1)) & \xrightarrow{f^\log} & \text{Ext}(D(G^\ell_2), D(G_1)) \\
\downarrow h_1 & & \downarrow h_2 \\
\text{Ext}(D(G_\eta/G^\mu_1), D(G^\mu_1)) & \xrightarrow{g^\log} & \text{Ext}(D(G^\ell_1), D(G_1/G^\mu_1)) \\
\downarrow h_3 & & \downarrow h_4 \\
\text{Ext}(D(G_\eta/G^\mu_1), D(G_\eta)) & \xrightarrow{j^*} & \text{Ext}(D(G_\eta/G^\mu_1), D(G^\mu_1)) \\
\end{array}
\]

where the horizontal maps are defined by applying the functor \( \mathbb{R}\text{Hom}(D(G^\ell_2), .) \) and \( \mathbb{R}\text{Hom}(D(G_\eta/G^\mu_1), .) \) to the short exact sequences:

\[
0 \to D(G^\mu_1) \to D(G_1) \to D(G_1/G^\mu_1) \to 0,
\]

and

\[
0 \to D(G^\mu_1) \to D(G^\ell_1) \to D(G^\ell_1/G^\mu_1) \to 0
\]

of Dieudonné crystals over \( (S/W)_{\text{crys}} \) and \( (\eta/W)_{\text{crys}} \) respectively. The vertical maps are induced by the functor \( j^*: (S/W)_{\text{crys}} \to (\eta/W)_{\text{crys}} \).
3.11 Lemma. Assume $k$ is algebraically closed. Then the map $g_{\log}$ is surjective.

Proof. Since $k$ is algebraically closed, $G_2^{\text{et}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^a$ and we can reduce to the the case $a = 1$, that is to the case $D(G_2^{\text{et}}) = 1$. By [13], 5.9, $\text{Ext}(1, D(G_1)) \simeq H^1(k[[t]], S_{D(G_1)})$. Similarly, we have

$$\text{Ext}(1, D(G_1/G_1^\mu)) \simeq H^1(k[[t]], S_{D(G_1/G_1^\mu)})$$

so that the cokernel of $g_{\log}$ is $H^2(k[[t]], S_{D(G_1^\mu)})$. Again, since $k$ is algebraically closed, we can reduce to the case $D(G_1^\mu) = 1(1)$ and the assertion results from 3.8. □

3.12 Lemma. Assume that $k$ is algebraically closed, then $h_1$ is an isomorphism.

Proof. We are reduced to prove that

$$\text{Ext}_{S/W}(1, 1(1)) \simeq \text{Ext}_{\eta/W}(1, 1(1)).$$

Using [13], 5.9 and 5.10, it is enough to prove that the map

$$H^1(S, S_{1(1)}) \to H^1(\eta, S_{1(1)})$$

is an isomorphism but this has already been proved in 3.8. □

3.13 Theorem. Assume $k$ is algebraically closed.

Let $\alpha \in \text{Ext}(D(G_\eta/G_\eta^\ell), D(G_\eta^f))$ be the isomorphism class of the extension:

$$0 \to D(G_\eta^f) \to D(G_\eta) \to D(G_\eta/G_\eta^f) \to 0.$$

There exists a short exact sequence of Dieudonné crystals over $S/W$:

$$0 \to D(G_1) \to D_{\log} \to D(G_2^{\text{et}}) \to 0,$$

such that its isomorphism class $\beta$ is sent by $h_2$ to $\alpha$.

As a corollary, we get:

3.14 Corollary. Let $G_\eta/\eta := k((t))$ be a semistable Barsotti-Tate group. Then its Dieudonné crystal $D(G_\eta)$ extends to a Dieudonné crystal
Let $\gamma \in Ext(D(G_{\xi}^{\text{et}}), D(G_1/G_1^\mu))$ be the isomorphism class of the extension:

$$0 \to D(G_1/G_1^\mu) \to D(G_2) \to D(G_{\xi}^2) \to 0$$

such that we have $g(\alpha) = h_3(\gamma)$. Since $g_{log}$ is surjective, there exists $\tilde{\gamma} \in Ext(D(G_2^\xi), D(G_1))$ such that $g_{log}(\tilde{\gamma}) = \gamma$. Since $g(\alpha - h_2(\tilde{\alpha})) = 0$, there exists some $\delta \in Ext(D(G_\eta/G_\eta^f), D(G_\eta^\mu))$, corresponding by 3.12 to a unique $\tilde{\delta} \in Ext((D(G_\eta^f), D(G_\eta^\mu)))$, such that $f(\delta) = \alpha - h_2(\tilde{\alpha})$. Then $\beta := f_{log}(\tilde{\delta}) + \tilde{\gamma}$ is sent by $h_2$ to $\alpha$. □

3.15 Definition. Let $G_\eta/\eta$ be a Barsotti-Tate group. We say that it is overconvergent if its associated Dieudonné isocrystal, corresponding to a $(\varphi, \nabla)$ over

$$E = \{a = \sum_{i=-\infty}^{+\infty} a_i x^i | a_i \in K, \sup_i |a_i| < \infty, |a_i| \to 0 \ (i \to -\infty)\}$$

(see [14]) admits a lattice as $(\varphi, \nabla)$-module over

$$E^+ = \{a \in E ||a_i|r^i \to 0 \ (i \to -\infty) \text{ for a certain } r, 0 < r < 1\}.$$ 

As a corollary of 3.14, we have:

3.16 Corollary. Any semistable Barsotti-Tate group $G_\eta/\eta$ is overconvergent.

3.17 Remark.

1. Any Barsotti-Tate group coming from an abelian variety is overconvergent: the abelian variety has potentially semistable reduction and in consequence it has been shown in [13] that the Dieudonné crystal of the abelian variety (which coincides with the Dieudonné crystal of the associated Barsotti-Tate group) comes from a log Dieudonné crystal after taking some finite étale base change.
2. Barsotti-Tate groups associated to $p$-adic representations of the absolute Galois group of $\eta$ with infinite monodromy are not overconvergent ([18]).

3. We have shown that if $G_{\eta}/\eta$ is semistable then it is overconvergent. Reciprocally, if $G_{\eta}/\eta$ is overconvergent, can we conclude that it is potentially semistable? Since $G_{\eta}/\eta$ is overconvergent its associated isocrystal will be quasi-unipotent by the local $p$-adic monodromy theorem of André-Kedlaya-Mebkhout. So we know that it will come from some log-Dieudonné crystal after considering some finite étale base change. The previous question can thus be rephrased as: Is there a log Dieudonné functor from the category (still to be defined) of log $p$-divisible groups to the category of log Dieudonné modules over $k[[t]]$ and if yes, is this functor an equivalence of categories (as this is the case without log-structure by [4])?

4. Recall (see [19]) that $G_{\eta}/\eta$ is endowed with a unique Frobenius slope filtration, whose quotients are isoclinic Barsotti-Tate groups. Assume each quotients to be overconvergent. Does it imply that $G_{\eta}/\eta$ is overconvergent?

4. **Semistable Barsotti-Tate Groups over Smooth Curves**

   4.1. In this section, we consider a dense open subset $U$ of a proper smooth connected curve $C/\mathbb{F}_p$. For any closed point $x \in C$, we denote $\eta_x := \text{Spec}(k(x)((t)))$ and $K_x := \text{Frac}(W(k(x)))$. For any $\mathbb{F}_p$-scheme $T$, we will denote $\bar{T} := T \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$.

   4.2 Definition. A Barsotti-Tate group $G/U$ is called semistable if at any closed point $x \in Z := C \setminus U$, $G_{\eta_x} := G \times_U \eta_x/\eta_x$ is semistable. We say that a Barsotti-Tate group $G/U$ is potentially semistable if and only if there exists some finite Galois covering $U' \to U$ such that $G' := G \times_U U'/U'$ is semistable.

   4.3. Let $G/U$ a potentially semistable Barsotti-Tate group. We associate to $G/U$ the following $L$-function:

   $$L(U, G, t) := \prod_{x \in U} \det(1 - t^{\deg(x)} F_x, D(G_x))^{-1},$$
where \((D(G_x), F_x)\) is the \(F\)-isocrystal over \(k(x)\) deduced from \(D(G)\) by restriction and \(\text{deg}(x) := [k(x) : \mathbb{F}_p]\).

We are going to show that the Dieudonné crystal associated to a potentially semistable Barsotti-Tate group is overconvergent and that its associated \(L\) function is a rational function. We will need the following lemma:

4.4 \textbf{Lemma.} ([6])

Let \(E\) be a convergent \(F\)-isocrystal over \(U\). Let \(\pi : \bar{U} \to U\) the canonical étale covering. Assume that \(\pi^*E\) is overconvergent, then \(E\) is overconvergent.

4.5 \textbf{Theorem.} Let \(G/U\) a potentially semistable Barsotti-Tate group. Then its associated Dieudonné crystal \(D(G)\) over \(U\) has a structure of overconvergent \(F\)-isocrystal \(D(G)^\dagger\) over \(U\).

\textbf{Proof.} By the previous lemma we can assume \(U = \bar{U}\) and by finite étale descent we can assume that \(G/U\) is semistable. For any closed point \(x \in Z\), we denote \(\eta_x := \text{Spec}(\text{Frac}(\hat{O}_{\bar{C},x}))\) and \(S_x := \text{Spec}(\hat{O}_{\bar{C},x})\) endowed with the log-structure induced by its closed point. By 3.14, the Dieudonné crystal \(D(G \times_U \eta_x)\) extends to a Dieudonné crystal \(D_{\log}\) over \(S_x\). Hence, the assertion follows from [14], proposition 4. □

4.6 \textbf{Corollary.} Let \(G/U\) a potentially semistable Barsotti-Tate group. Then its \(L\)-function \(L(G, U, t)\) is a rational function in \(t\). More precisely, we have:

\[
L(G, U, t) = \prod_{i=0}^{2} \det(1 - tF_iH_{rig,c}^i(U, D(G)^\dagger))(-1)^{i+1}.
\]

\textbf{Proof.} By 4.5 \(D(G)\) has a structure of overconvergent \(F\)-isocrystal and the formula results from [17], theorem 1.2. Finally, the rationaly results from the finiteness of the cohomological groups \(H_{rig,c}^i(U, D(G)^\dagger)\) which follows from [14], corollary 8 and the Poincaré duality of rigid cohomology. □

4.7 \textbf{Remark.} Let \(G_F/F\) be a Barsotti-Tate group, where \(F\) is the function field of \(C\).
1. It is a priori not always possible to extend $G_F/F$ to some Barsotti-Tate group $G$ over some dense open subset $U$ of $C$ (but this is the case when $G_F/F$ is the Barsotti-Tate group associated to an abelian variety). For example, consider the étale case. Then, we can replace Barsotti-Tate groups by $p$-adic representations. We can find an example of a $p$-adic representation of $Gal(\bar{F}/F)$ that ramifies at infinitely many places and thus don’t factorize through any fundamental group of some dense open subset $U$ of $C$. To construct such representation, it is enough to construct a $\mathbb{Z}_p$-extension $K$ of $F$ that ramifies at infinitely many places (it exists: see for example [11]). Take any extension $L/F$ with Galois group $(\mathbb{Z}/p)^\times$. Then the extension $K.L/F$ has a Galois group isomorphic to $\mathbb{Z}_p^\times$ and the natural projection $Gal(\bar{F}/F) \to Gal(K.L/F)$ gives an example of one-dimensional $p$-adic representation of $Gal(\bar{F}/F)$ that ramifies at infinitely many places.

2. If $G_1, G_2/U$ are two Barsotti-Tate groups with $G_F/F$ as generic fiber, are $G_1$ and $G_2$ isomorphic (or at least isogenous)? See [5] for some evidences on this question. If the answer to this question is yes and $G_F/F$ extends to some Barsotti-Tate group $G/U$, then we can define the Hasse-Weil $L$-function of $G_F/F$ as $L(G_F,t) := L(U,G,t)$.

References


Semistable Barsotti-Tate Group


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