An $SO(3)$-Version of 2-Torsion Instanton Invariants

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Abstract. We construct an invariant for non-spin 4-manifolds by using 2-torsion cohomology classes of moduli spaces of instantons on $SO(3)$-bundles. The invariant is an $SO(3)$-version of Fintushel-Stern’s 2-torsion instanton invariant. We show that this $SO(3)$-torsion invariant is non-trivial for $2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, while it is known that any known invariant of $2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ coming from the Seiberg-Witten theory is trivial since $2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has a positive scalar curvature metric.

1. Introduction

The purpose of this paper is to construct an $SO(3)$-version of Fintushel-Stern’s torsion invariants [FS]. R. Fintushel and R. Stern constructed a variant of Donaldson invariants for spin 4-manifolds by using 2-torsion cohomology classes of the moduli spaces of instantons on $SU(2)$-bundles. They used cohomology classes of degree one and two. S. K. Donaldson gave another construction by using a class of degree 3 [D4]. As is well known, the usual Donaldson invariant is trivial for the connected sum of 4-manifolds with $b^+$ positive ([D3]). On the other hand, Fintushel and Stern showed that their torsion invariant is not necessarily trivial for the connected sum of the form $Y \# S^2 \times S^2$ in general.

In this paper, we define an invariant of 4-manifolds using 2-torsion cohomology classes of $SO(3)$-moduli spaces and show that our invariant is not necessarily trivial for $Y \# S^2 \times S^2$ as in the case of Fintushel-Stern’s invariant. We basically follow the argument in [FS] and modify it to extend the definition to non-spin 4-manifolds.

The outline of the construction is as follows. Let $X$ be a closed, oriented, simply connected, non-spin Riemannian 4-manifold and $P$ be an $SO(3)$-bundle over $X$ satisfying

$$w_2(P) = w_2(X) \in H^2(X; \mathbb{Z}_2), \quad p_1(P) \equiv \sigma(X) \mod 8.$$
Here \( \sigma(X) \) is the signature of \( X \). Let \( \mathcal{B}_P^\ast \) be the space of gauge equivalence classes of irreducible connections on \( P \). In [AMR], S. Akbulut, T. Mrowka and Y. Ruan showed that \( H^1(\mathcal{B}_P^\ast; \mathbb{Z}_2) \) is isomorphic to \( \mathbb{Z}_2 \). We denote the generator by \( u_1 \). On the other hand, for homology class \( [\Sigma] \in H_2(X; \mathbb{Z}) \) with self-intersection number even, we have an integral cohomology class \( \mu([\Sigma]) \in H^2(\mathcal{B}_P^\ast; \mathbb{Z}) \). Suppose that the dimension of the moduli space \( M_P \) of instantons on \( P \) is \( 2d + 1 \) for some non-negative integer \( d \). In general \( M_P \) is not compact. However for homology classes \( [\Sigma_1], \ldots, [\Sigma_d] \in H_2(X; \mathbb{Z}) \) with self-intersection numbers even, we can define the pairing

\[
q^\mu_X([\Sigma_1], \ldots, [\Sigma_d]) = \langle u_1 \cup \mu([\Sigma_1]) \cup \cdots \cup \mu([\Sigma_d]), [M_P] \rangle \in \mathbb{Z}_2
\]

in an appropriate sense. We show that this number depends only on the homology classes \( [\Sigma_i] \) and gives a differential-topological invariant of \( X \).

We will show a gluing formula of torsion invariants for \( Y \# S^2 \times S^2 \), which is an \( SO(3) \)-version of Theorem 1.1 in [FS]. By using this gluing formula and D. Kotschick’s calculation in [K1, K2], we prove that \( q^\mu_1 CP^2 \# CP^2 \) is non-trivial. This example exhibits two interesting aspects explained below.

The first aspect is related to vanishing theorem. We have a description of \( X = 2CP^2 \# \overline{CP^2} \) as the connected sum of \( Y_1 = CP^2 \) and \( Y_2 = CP^2 \# \overline{CP^2} \). Since the second Stiefel-Whitney class \( w_2(P) \) is equal to \( w_2(X) \), both of \( w_2(P)|_{Y_1} \) and \( w_2(P)|_{Y_2} \) are non-trivial. In such a situation, the usual Donaldson invariants are trivial by the dimension-count argument ([MM]). Hence the non-triviality of \( q^\mu_1 CP^2 \# \overline{CP^2} \) implies that the dimension-count argument can not be applied directly to proving such a vanishing theorem in our case. If each homology class \( [\Sigma_i] \) is in \( H_2(Y_1; \mathbb{Z}) \) or \( H_2(Y_2; \mathbb{Z}) \), then we can show that our invariant vanishes. However we can not reduce the argument to this case because of the condition that \( [\Sigma_i] \cdot [\Sigma_i] \) must be even to define our invariant.

The next aspect is related to the Seiberg-Witten theory. In [Wi], E. Witten introduced invariants, called the Seiberg-Witten invariants, of 4-manifolds using monopole equations. He conjectured that the invariants are equivalent to the Donaldson invariants and explicitly wrote a formula which should give a relation between the Donaldson invariants and the Seiberg-Witten invariants. In [PT], V. Pidstrigach and A. Tyurin proposed a program to give a rigorous mathematical proof of the formula by using non-abelian monopoles. The theory of non-abelian monopoles has been de-
veloped by P. Feehan and T. Leness ([FL1, FL2, FL3]). Feehan and Leness recently announced that they completed the proof of Witten’s formula for 4-manifolds of simple type with $b_1 = 0$ and $b^+ > 1$ in [FL4].

The non-triviality of $d_{2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}}^{u_1}$ is quite a contrast to the equivalence of the Donaldson invariants and Seiberg-Witten invariants. If a 4-manifold has a positive scalar curvature metric and satisfies $b^+(X) \geq 1$, then the moduli space of solutions of the monopole equations with respect to the metric is empty for some perturbation. Hence any known invariant of $2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ coming from the monopole equations (the Seiberg-Witten invariant and a refinement due to S. Bauer and M. Furuta [BF]) is trivial since $2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ has a positive scalar curvature metric.

The paper is organized as follows. In Section 2, we construct cohomology classes $\mu([\Sigma])$ and $u_1$, and define a torsion invariant. In Section 3, we prove a gluing formula for the connected sum of the form $Y \# S^2 \times S^2$. In Section 4, we prove that $d_{2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}}^{u_1}$ is non-trivial by using the gluing formula. We also discuss the reason why the usual vanishing theorem does not hold for our torsion invariant.

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2. Torsion Invariants

2.1. Notations

Let $X$ be a closed, oriented, simply connected 4-manifold, $g$ a Riemannian metric on $X$ and $P$ an $SO(3)$-bundle over $X$. Put

$$k = -\frac{1}{4}p_1(P) \in \mathbb{Q}, \quad w = w_2(P) \in H^2(X; \mathbb{Z}_2).$$

Let $\mathcal{A}_P^*$ be the space of irreducible connections on $P$ and $\mathcal{G}_P$ be the gauge group of $P$. We write $\mathcal{B}_P^*$ or $\mathcal{B}_{k,w,X}^*$ for the quotient space $\mathcal{A}_P^*/\mathcal{G}_P$. We denote by $M_P$ or $M_{k,w,X}$ the moduli space of instantons on $P$.

Let $A$ be an instanton on $P$. We have a sequence

$$\Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{1+}_X(\mathfrak{g}_P).$$
The condition that $A$ is an instanton implies that $d_A^+ \circ d_A = 0$. Hence the above sequence define a complex. We denote the cohomology groups by $H^0_A$, $H^1_A$, $H^2_A$.

Let $\tilde{P}$ be a $U(2)$-lift of $P$ and $\tilde{E}$ be the rank 2 complex vector bundle associated with $\tilde{P}$. Fix a connection $a_{\det}$ on $\det \tilde{E}$. We write $A_{\tilde{E}}^*$ for the space of connections on $\tilde{E}$ which induce the connection $a_{\det}$ on $\det \tilde{E}$, and write $A_{\tilde{E}}^*$ for the space of irreducible connections in $A_{\tilde{E}}^*$. Let $G_{\tilde{E}}$ be the group of bundle automorphisms on $\tilde{E}$ with determinant 1. We also introduce a subgroup $G_{\tilde{E}}^0$ of $G_{\tilde{E}}$. Fix a point $x_0$ in $X$. The subgroup $G_{\tilde{E}}^0$ is defined by

$$G_{\tilde{E}}^0 = \{ g \in G_{\tilde{E}} | g(x_0) = 1 \}.$$ 

We denote the quotient spaces by

$$B^*_E = A_{\tilde{E}}^*/G_{\tilde{E}}, \quad \tilde{B}_E = A_{\tilde{E}}/G_{\tilde{E}}^0, \quad \tilde{B}_E^* = A_{\tilde{E}}^*/G_{\tilde{E}}^0.$$ 

Since we are assuming that $X$ is simply connected, the natural map $B^*_E \to B^*_P$ is bijective.

To construct cohomology classes $u_1$ and $\mu([\Sigma])$, we need the universal bundle $\tilde{E}$ over $X \times \tilde{B}_E$. The universal bundle is defined by

$$\tilde{E} := E \times_{G_{\tilde{E}}} A_{\tilde{E}} \longrightarrow X \times \tilde{B}_E.$$ 

For a closed, oriented surface $\Sigma$ embedded in $X$, let $\nu(\Sigma)$ be a small tubular neighborhood of $\Sigma$. We define spaces of gauge equivalence classes of connections on $\nu(\Sigma)$. Let $A_{\nu(\Sigma)}$ be the space of connections on $\tilde{E}|_{\nu(\Sigma)}$ which induce the connection $a_{\det|_{\nu(\Sigma)}}$ on $\det \tilde{E}|_{\nu(\Sigma)}$. Let $G_{\nu(\Sigma)}$ be the group of automorphisms of $\tilde{E}|_{\nu(\Sigma)}$ with determinant 1. We assume that the base point $x_0$ is in $\nu(\Sigma)$. Define $G_{\nu(\Sigma)}^0$ by

$$G_{\nu(\Sigma)}^0 = \{ g \in G_{\nu(\Sigma)} | g(x_0) = 1 \}.$$ 

We denote the quotient spaces by

$$B^*_{\nu(\Sigma)} = A_{\nu(\Sigma)}^*/G_{\nu(\Sigma)}, \quad \tilde{B}_{\nu(\Sigma)} = A_{\nu(\Sigma)}/G_{\nu(\Sigma)}^0, \quad \tilde{B}_{\nu(\Sigma)}^* = A_{\nu(\Sigma)}^*/G_{\nu(\Sigma)}^0.$$ 

Restricting connections, we have a map

$$\tilde{r}_{\nu(\Sigma)} : \tilde{B}_{\nu(\Sigma)}^* \longrightarrow \tilde{B}_{\nu(\Sigma)}.$$ 

We have the universal bundle $\tilde{E}_{\nu(\Sigma)}$ over $\nu(\Sigma) \times \tilde{B}_{\nu(\Sigma)}$ defined by

$$\tilde{E}_{\nu(\Sigma)} := (\tilde{E}|_{\nu(\Sigma)}) \times_{G_{\nu(\Sigma)}} A_{\nu(\Sigma)} \longrightarrow \nu(\Sigma) \times \tilde{B}_{\nu(\Sigma)}.$$
2.2. Cohomology classes of $\mathcal{B}_E^*$

Suppose $\Sigma$ is a closed, oriented surface embedded in $X$ such that $\langle w_2(P), [\Sigma] \rangle \equiv 0 \mod 2$. In this subsection, we define a 2-dimensional integral cohomology class $\mu([\Sigma]) \in H^2(\mathcal{B}_E^*; \mathbb{Z})$. Basically we follow a standard construction in [DK, K1].

We first define the cohomology class $\tilde{\mu}_E([\Sigma]) \in H^2(\tilde{\mathcal{B}}_E^*; \mathbb{Z})$ to be the slant product $c_2(\tilde{E})/[\Sigma]$.

**Lemma 2.1.** Let $\beta : \tilde{\mathcal{B}}_E^* \to \mathcal{B}_E^*$ be the projection. Then the induced homomorphism

$$\beta^* : H^2(\mathcal{B}_E^*; \mathbb{Z}) \longrightarrow H^2(\tilde{\mathcal{B}}_E^*; \mathbb{Z})$$

is injective. Moreover for a homology class $[\Sigma] \in H_2(X; \mathbb{Z})$ with $\langle w_2(P), [\Sigma] \rangle \equiv 0 \mod 2$, the cohomology class $\tilde{\mu}_E([\Sigma])$ lies in the image of $\beta^*$.

**Proof.** Since $H^1(SO(3); \mathbb{Z}) = 0$, the spectral sequence associated with the fibration $SO(3) \to \tilde{\mathcal{B}}_E^* \to \mathcal{B}_E^*$ induces an exact sequence

$$0 \longrightarrow H^2(\mathcal{B}_E^*; \mathbb{Z}) \xrightarrow{\beta^*} H^2(\tilde{\mathcal{B}}_E^*; \mathbb{Z}) \longrightarrow H^2(SO(3); \mathbb{Z}),$$

which implies the injectivity of $\beta^*$.

Let $\eta$ be a complex line bundle over $SO(3)$ defined by

$$\eta := SU(2) \times_{\{\pm 1\}} \mathbb{C} \longrightarrow SO(3).$$

Here the action of $\{\pm 1\}$ on $\mathbb{C}$ is the multiplication. Then it is easy to obtain the identification

$$\tilde{E}|_{\Sigma \times SO(3)} = (E|_{\Sigma}) \times_{\{\pm 1\}} SU(2) = (E|_{\Sigma}) \boxtimes \eta \longrightarrow \Sigma \times SO(3),$$

and we have

$$c_2(\tilde{E}|_{\Sigma \times SO(3)}/[\Sigma]) = (\pi_1^*c_2(E|_{\Sigma}) + \pi_2^*c_1(E|_{\Sigma}) \cup \pi_2^*c_1(\eta))/[\Sigma]$$

$$= \langle c_1(E), [\Sigma] \rangle c_1(\eta)$$

$$\in H^2(SO(3); \mathbb{Z}) \cong \mathbb{Z}_2,$$

where

$$\pi_1 : \Sigma \times SO(3) \longrightarrow \Sigma, \quad \pi_2 : \Sigma \times SO(3) \longrightarrow SO(3)$$
are the projections. If \( \langle w_2(P), [\Sigma] \rangle \) is zero, the pairing \( \langle c_1(\widetilde{E}), [\Sigma] \rangle \) is even, and hence the restriction of \( c_2(\widetilde{E})/\Sigma \) to \( SO(3) \) is trivial. From the exact sequence (1), \( \bar{\mu}_{\widetilde{E}}([\Sigma]) \) is in the image of \( \beta^* \). □

By Lemma 2.1, there is a unique element of \( H^2(B^*_p; \mathbb{Z}) \) such that the image by \( \beta^* \) is \( \bar{\mu}_{\widetilde{E}}([\Sigma]) \). Through the natural identification between \( B^*_p \) and \( B^*_E \), we have a 2-dimensional cohomology class of \( B^*_p \). We denote it by \( \mu_{\widetilde{E}}([\Sigma]) \).

**Lemma 2.2.** Let \( X \) be a closed, oriented, simply connected 4-manifold and \( P \) be an \( SO(3) \)-bundle over \( X \). Suppose that \( [\Sigma] \) is a 2-dimensional homology class in \( X \) with \( \langle w_2(P), [\Sigma] \rangle \equiv 0 \mod 2 \). Then the cohomology class \( \mu_{\widetilde{E}}([\Sigma]) \in H^2(B^*_p; \mathbb{Z}) \) is independent of the choice of \( \widetilde{E} \).

This lemma will be shown in §2.4 as a corollary of Lemma 2.15. Under the assumption in Lemma 2.2, we define \( \mu([\Sigma]) \in H^2(B^*_p; \mathbb{Z}) \) as follows.

**Definition 2.3.** For a homology class \( [\Sigma] \in H_2(X; \mathbb{Z}) \) with \( \langle w_2(P), [\Sigma] \rangle \equiv 0 \mod 2 \), the cohomology class \( \mu([\Sigma]) \in H^2(B^*_p; \mathbb{Z}) \) is defined to be \( \mu_{\widetilde{E}}([\Sigma]) \).

**Remark 2.4.** Let \( \mathbb{P} := P \times_{\mathcal{G}_P} \mathcal{A}^*_P \rightarrow X \times B^*_p \) be the universal bundle of \( P \). Then the usual definition of \( \mu \)-map is given by

\[
\mu_Q: \quad H_2(X; \mathbb{Z}) \rightarrow H^2(B^*_p; \mathbb{Q}) \\
[\Sigma] \quad \mapsto \quad -\frac{1}{4} p_1(\mathbb{P})/\Sigma.
\]

In general, \( \mu_Q([\Sigma]) \) does not have an integral lift. Under our assumptions, it is easy to see that \( \mu([\Sigma]) \) is an integral lift of \( \mu_Q([\Sigma]) \).

Next we define a torsion cohomology class \( u_1 \in H^1(B^*_p; \mathbb{Z}_2) \). We write \( \sigma(X) \) for the signature of \( X \). Akbulut, Mrowka and Ruan showed the following in [AMR].
Proposition 2.5 ([AMR]). Let $X$ be a closed, oriented, simply connected 4-manifold and $P$ be an $SO(3)$-bundle over $X$. Then we have

$$\pi_1(B_P^*) = \begin{cases} 
Z_2 & \text{if } w_2(P) = w_2(X), \ p_1(P) \equiv \sigma(X) \mod 8 \\
1 & \text{otherwise.} 
\end{cases}$$

Remark 2.6. Suppose $P$ is an $SO(3)$-bundle over $X$ with $w_2(P)$ equal to $w_2(X)$ and let $\bar{P}$ be a $U(2)$-lift of $P$. Then $p_1(P)$ is equal to $\sigma(X)$ modulo 8 if and only if $c_2(\bar{P})$ is equal to 0 modulo 2. This equivalence is a consequence of the formulas

$$p_1(P) = -4c_2(\bar{P}) + c_1(\bar{P})^2, \ w_2(X)^2 \equiv \sigma(X) \mod 8.$$ 

When $w_2(P) = w_2(X)$ and $p_1(P) \equiv \sigma(X) \mod 8$, we have $H^1(B_P^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$ from Proposition 2.5.

Definition 2.7. Let $X$ be a closed, oriented, simply connected 4-manifold and $P$ be an $SO(3)$-bundle over $X$ satisfying $w_2(P) = w_2(X)$, $p_1(P) \equiv \sigma(X) \mod 8$. We write $u_1$ for the generator of $H^1(B_P^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

2.3. Construction of $q_{X}^{u_1}$

Let $X$ be a closed, oriented, simply connected 4-manifold. Suppose $b^+(X) = 2a$ for a positive integer $a$. Let $P$ be an $SO(3)$-bundle over $X$. Assume that $P$ satisfies the condition

(2) \quad \begin{align*}
w_2(P) &= w_2(X) \in H^2(X; \mathbb{Z}_2), \quad p_1(P) \equiv \sigma(X) \mod 8.
\end{align*}

The virtual dimension of $M_P$ is given by

$$\dim M_P = -2p_1(P) - 3(1 + b^+(X)) = 8k - 3(1 + 2a).$$

If we put $d = -p_1(P) - 3a - 2 = 4k - 3a - 2$, then we have

$$\dim M_P = 2d + 1.$$ 

From the condition (2), we have

$$d \equiv -\sigma(X) - 3a - 2 \mod 8.$$
Suppose that \( d \geq 0 \) and take 2-dimensional homology classes \([\Sigma_1], \ldots, [\Sigma_d]\) of \( X \) satisfying
\[
\langle w_2(P), [\Sigma_i] \rangle \equiv 0 \pmod{2} \quad (i = 1, \ldots, d).
\]
The assumption \( \langle w_2(P), [\Sigma_i] \rangle \equiv 0 \pmod{2} \) is equivalent to \([\Sigma_i] \cdot [\Sigma_i] \equiv 0 \pmod{2}\) since \( w_2(P) \) is equal to \( w_2(X) \). We want to define the pairing
\[
\langle u_1 \cup \mu([\Sigma_1]) \cup \cdots \cup \mu([\Sigma_d]), M_P \rangle \in \mathbb{Z}.
\]

The moduli space \( M_P \) is not compact in general and the pairing is not well-defined in the usual sense. To define the pairing, we need submanifolds \( V_{\Sigma_i} \) dual to \( \mu([\Sigma_i]) \) which behave nicely near the ends of \( M_P \). We briefly explain how the submanifolds are constructed. See [D3, DK] for the details.

We use the following three things. The first is that when \( b^+(X) \) and \( k = -\frac{4}{3}p_1(P) \) are positive \( M_P \) lies in \( \mathcal{B}_P \) and has a natural smooth structure for generic metrics on \( X \). The second is that the restrictions of irreducible instantons to open subsets are also irreducible. The third is that the cohomology class \( \mu([\Sigma]) \) comes from \( \mathcal{B}_{\nu(\Sigma)}^* \). More precise statement of the third is as follows.

Let \([\Sigma] \in H_2(X; \mathbb{Z})\) be a homology class with \([\Sigma] \cdot [\Sigma] \equiv 0 \pmod{2} \). Since the following diagram is commutative
\[
\begin{align*}
\tilde{E}|_{\nu(\Sigma) \times \mathcal{B}_E^*} = (\tilde{E}|_{\nu(\Sigma)}) \times g_0^\nu \mathcal{A}_E^* & \xrightarrow{\text{id}_E \times \tilde{r}_\nu(\Sigma)} \tilde{E}_\nu(\Sigma) = (\tilde{E}|_{\nu(\Sigma)}) \times g_0^\nu \mathcal{A}_\nu(\Sigma) \\
\nu(\Sigma) \times \tilde{B}_E^* & \xrightarrow{\text{id}_{\nu(\Sigma)} \times \tilde{r}_\nu(\Sigma)} \nu(\Sigma) \times \tilde{B}_\nu(\Sigma)
\end{align*}
\]
we obtain
\[
\tilde{\mu}_E([\Sigma]) = c_2(\tilde{E})/[\Sigma] = \tilde{r}_\nu(\Sigma)^* (c_2(\tilde{E}_\nu(\Sigma))/[\Sigma]) \in H^2(\tilde{B}_E^*; \mathbb{Z}).
\]
We apply Lemma 2.1 to the restriction of \( P \) on \( \nu(\Sigma) \), instead of \( P \) itself. Then we see that there exists a unique 2-dimensional cohomology class \( \mu_{\nu(\Sigma),\tilde{E}}([\Sigma]) \) of \( \mathcal{B}_{\nu(\Sigma)}^* \) such that the pull-back by the natural projection
\[
\tilde{B}_{\nu(\Sigma)}^* \to \mathcal{B}_{\nu(\Sigma)}^* \text{ is equal to } c_2(\tilde{E}_\nu(\Sigma))/[\Sigma].
\]
We define \( V_{\Sigma} \) as follows.

**Definition 2.8.** Take a homology class \([\Sigma] \in H_2(X; \mathbb{Z})\) with \([\Sigma] \cdot [\Sigma] \) even. We write \( L_{\Sigma} \) for a complex line bundle over \( \mathcal{B}_{\nu(\Sigma),\tilde{E}} \) with first Chern
class $\mu_{\nu(\Sigma), \bar{E}}([\Sigma]) \in H^2(\mathcal{B}^*_{\nu(\Sigma)}, \bar{E}; \mathbb{Z})$. Fix a section $s_\Sigma$ of $\mathcal{L}_\Sigma$. We denote the zero locus of $s_\Sigma$ by $V_\Sigma \subset \mathcal{B}^*_{\nu(\Sigma)}$. Suppose that $b^+(X)$ and $k = -\frac{1}{4}p_1(P)$ are positive. For a generic metric $g$, we define

$$M_P \cap V_\Sigma := \{ [A] \in M_P \mid [A|_{\nu(\Sigma)}] \in V_\Sigma \}.$$

We will show that the pairing $\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle$ is well-defined under some condition.

**Remark 2.9.** We give some remarks on the line bundle $\mathcal{L}_\Sigma$. We refer to [D3, DK] for details.

- As is well-known, we are also able to construct the line bundle $\mathcal{L}_\Sigma$ by using a family of twisted Dirac operators on $\Sigma$.

- Assume that $\langle w_2(P), [\Sigma] \rangle$ is equal to 0 modulo 2. Then $P|_{\nu(\Sigma)}$ is topologically trivial. Let $\mathcal{B}^*_{\nu(\Sigma)} + := \mathcal{B}^*_{\nu(\Sigma)} \cup \{[\Theta_{\nu(\Sigma)}]\}$. Here $\Theta_{\nu(\Sigma)}$ is the trivial connection on $\nu(\Sigma)$. It is known that $\mathcal{L}_\Sigma$ extends to $\mathcal{B}^*_{\nu(\Sigma)} +$. Hence we can assume that the section $s_\Sigma$ is non-zero near $[\Theta_{\nu(\Sigma)}]$. In the case when $w_2(P)$ is zero, we need this property to define invariants. On the other hand, when we treat an $SO(3)$-bundle $P$ with $w_2(P)$ non-trivial, we do not need this property for the definition of invariants. However we will need this property in Lemma 3.7 to prove some property of our invariant.

We prepare some lemmas. The following is well-known.

**Lemma 2.10 ([D3, DK]).** Let $X$ be a closed, oriented, simply connected 4-manifold with $b^+(X)$ positive and $P$ be an $SO(3)$-bundle with $k = -\frac{1}{4}p_1(P) > 0$. Take homology classes $[\Sigma_1], \ldots, [\Sigma_{d'}] \in H_2(X; \mathbb{Z})$ with self-intersection numbers even. For generic sections $s_{\Sigma_i}$, the intersections

$$M_{k-j, w, X} \cap \left( \bigcap_{i \in I} V_{\Sigma_i} \right) \quad (I \subset \{1, \ldots, d'\}, \ 0 \leq j < k)$$

are transverse.
From now on, we require that $\Sigma_i$ are generic in the following sense.

\begin{equation}
\begin{cases}
\Sigma_i \cap \Sigma_j \quad (i, j \text{ distinct}) \\
\Sigma_i \cap \Sigma_j \cap \Sigma_k = \emptyset \quad (i, j, k \text{ distinct}).
\end{cases}
\end{equation}

**Lemma 2.11.** Let $X$ be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(X)$ positive. Let $P$ be an $SO(3)$-bundle over $X$ with $w_2(P)$ equal to $w_2(X)$. Suppose that the dimension of $M_P$ is $2d' + r$ for a non-negative integer $d'$ and $1 \leq r \leq 3$. Take $d'$ homology classes $[\Sigma_1], \ldots, [\Sigma_{d'}] \in H^2(X; \mathbb{Z})$ with

$$[\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2 \quad (i = 1, \ldots, d').$$

Moreover we assume that the surfaces $\Sigma_i$ satisfy the condition (4). Then for generic sections $s_{\Sigma_i}$, the intersection

$$M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$$

is a compact $r$-dimensional manifold.

**Proof.** Put $k = -\frac{1}{4}p_1(P)$, $w = w_2(P)$. For $[A] \in M_P$, we have

\begin{align*}
k &= -\frac{1}{4}p_1(P) \\
&= \frac{1}{8\pi^2} \int_X \text{Tr}(F_A^2) \\
&= \frac{1}{8\pi^2} \int_X |F_A^-|^2 d\mu_g - \frac{1}{8\pi^2} \int_X |F_A^+|^2 d\mu_g \\
&= \frac{1}{8\pi^2} \int_X |F_A^-| d\mu_g \geq 0.
\end{align*}

by the Chern-Weil theory. Here $d\mu_g$ is the volume form with respect to $g$. First we show $k > 0$. If not, $k = 0$ and $A$ is flat. Since $X$ is simply connected, $A$ is trivial. This contradicts the assumption that $w_2(P)$ is non-trivial. Hence we have $k > 0$. From Lemma 2.10, $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is a smooth $r$-dimensional manifold for generic sections $s_{\Sigma_i}$.

Next we prove that $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is compact. Let $\{[A^{(n)}]\}_{n \in \mathbb{N}}$ be a sequence in $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$. Uhlenbeck’s weak compactness theorem
implies that there is a subsequence \([A^{(n')}])_{n'} which is weakly convergent to
\[
([A_\infty]; x_1, \ldots, x_l) \in M_{k-l,w,X} \times X'.
\]
We also have \(k-l > 0\) in the same way as above. Let \(m\) be the number of the tubular neighborhoods \(\nu(\Sigma_i)\) which contain \(x_\alpha\) for some \(\alpha\) with \(1 \leq \alpha \leq l\). Then without loss of generality, we may suppose that
\[
[A_\infty] \in M_{k-l,w,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}-m}
\]
if we change the order of the surfaces. If we take the tubular neighborhoods \(\nu(\Sigma_i)\) to be sufficiently small, we have
\[
\nu(\Sigma_i) \cap \nu(\Sigma_j) \cap \nu(\Sigma_k) = \emptyset \quad (i,j,k \text{ distinct})
\]
from (4). Hence we have \(m \leq 2l\). Since \(k-l > 0\), the intersection \(M_{k-l,w,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}-m}\) is transverse by Lemma 2.10. From this transversality, we obtain
\[
0 \leq \dim M_{k-l,w,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}-m} = \dim M_{k,w,X} - 8l - 2(d' - m) = r - 8l + 2m \leq r - 4l.
\]
Since we suppose \(1 \leq r \leq 3\), we have \(l = 0\) and
\[
[A_\infty] \in MP \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}.
\]
Hence \(MP \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}\) is compact. □

Let \(X\) be as in Lemma 2.11 and \(P\) be an \(SO(3)\)-bundle over \(X\) satisfying (2). Suppose that \(\dim MP\) is \(2d + 1\) for a non-negative integer \(d\) and take homology classes \([\Sigma_1], \ldots, [\Sigma_d] \in H_2(X; \mathbb{Z})\) with self-intersection numbers even. From Lemma 2.11, we have the pairing
\[
\langle u_1, MP \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle \in \mathbb{Z}_2.
\]
Proposition 2.12. Let $X$ be a closed, oriented, simply connected, non-spin $4$-manifold with $b^+(X) = 2a$ for a positive integer $a$ and $P$ be an $SO(3)$-bundle over $X$ satisfying (2). Assume that the dimension of $M_P$ is $2d + 1$ for a non-negative integer $d$. Then the pairing

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle \in \mathbb{Z}_2$$

is independent of the choices of Riemannian metric $g$, $U(2)$-lift $\tilde{P}$ of $P$, sections $s_{\Sigma_i}$ of $L_{\Sigma_i}$ and surfaces $\Sigma_i$ representing the homology classes $[\Sigma_i]$. Moreover the pairing is multi-linear with respect to $[\Sigma_1], \ldots, [\Sigma_d]$.

We prove the above proposition in §2.4. By using this proposition, we can easily show that the following invariant $q^{u_1}_X$ is well defined.

**Definition 2.13.** Let $X$ be as in Proposition 2.12. Let $A'_d(X)$ be the subspace of $\otimes^d H^2(X; \mathbb{Z})$ generated by

$$\{ [\Sigma_1] \otimes \cdots \otimes [\Sigma_d] \mid [\Sigma_i] \in H_2(X; \mathbb{Z}), [\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2 \},$$

and we put

$$A'(X) := \bigoplus_d A'_d(X),$$

where $d$ runs over non-negative integers with $d \equiv -\sigma(X) - 3a - 2 \mod 8$. We define $q^{u_1}_X$ by

$$q^{u_1}_X : A'(X) \rightarrow \mathbb{Z}_2 \quad ([\Sigma_1], \ldots, [\Sigma_d]) \mapsto q^{u_1}_{k,w,X}([\Sigma_1], \ldots, [\Sigma_d]) := \langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle.$$

Here $P$ is an $SO(3)$-bundle over $X$ with $w_2(P) = w_2(X)$ and $p_1(P) = -d - 3a - 2$.

2.4. **Well-definedness of $q^{u_1}_X$**

In this subsection, we prove Proposition 2.12. First we show the independence of $q^{u_1}_X$ from Riemannian metric $g$ and sections $s_{\Sigma_i}$ in a standard way. Take two metrics $g$, $g'$ on $X$ and sections $s_{\Sigma_i}$, $s'_{\Sigma_i}$ of $L_{\Sigma_i}$. Choose a
path \{g_t\}_{t \in [0,1]} between \(g\) and \(g'\), and a path \{s_{\Sigma_i,t}\}_{t \in [0,1]} between \(s_{\Sigma_i}\) and \(s'_{\Sigma_i}\). Then put

\[M := \coprod_{t \in [0,1]} MP(g_t) \times \{t\},\]

\[M \cap V_{\Sigma_i} := \{([A], t) \in M \mid s_{\Sigma_i,t}([A|_{\nu(\Sigma_i)}]) = 0\}.

Using a similar argument in the proof of Lemma 2.11, we can show the following lemma:

**Lemma 2.14.** Let \(X\) and \(P\) be as in Proposition 2.12. Then for generic paths \{g_t\}_{t \in [0,1]} and \{s_{\Sigma_i,t}\}_{t \in [0,1]}, the intersection

\[M \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}\]

is a compact 2-dimensional manifold whose boundary is

\[(MP(g) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}) \coprod (MP(g') \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d}).\]

This lemma implies

\[\langle u_1, MP(g) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1, MP(g') \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d} \rangle \in \mathbb{Z}_2,

and the pairing \(\langle u_1, MP \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle\) is independent of the choices of \(g\) and \(s_{\Sigma_i}\).

Next we see the independence of \(q_X' u_1\) from the choice of \(U(2)\)-lift \(\tilde{P}\) of \(P\). Take two \(U(2)\)-lifts \(\tilde{P}\) and \(\tilde{P}'\) of \(P\). The associated vector bundle \(\tilde{E}'\) with \(\tilde{P}'\) is topologically isomorphic to \(\tilde{E} \otimes L\) for some complex line bundle \(L\) over \(X\). Fix connections \(a_{\text{det}}, a_L\) on \(\text{det} \tilde{E}, L\) and an isomorphism

\[\varphi : \tilde{E}' \xrightarrow{\cong} \tilde{E} \otimes L.\]

We have a connection \(a'_{\text{det}}\) on \(\text{det} \tilde{E}'\) induced by \(a_{\text{det}}, a_L\) and \(\varphi\). We consider connections on \(\tilde{E} \otimes L\) and \(\tilde{E}'\) which are compatible with \(a_{\text{det}} + 2a_L\) and \(a'_{\text{det}}\) respectively. By tensoring \(a_L|_{\nu(\Sigma)}\), we have maps

\[t_A : A_{\nu(\Sigma), \tilde{E}} \xrightarrow{\cong} A_{\nu(\Sigma), \tilde{E} \otimes L},\]

\[t_B : B_{\nu(\Sigma), \tilde{E}}^* \xrightarrow{\cong} B_{\nu(\Sigma), \tilde{E} \otimes L}^*;\]

\[t_{\tilde{B}} : \tilde{B}_{\nu(\Sigma), \tilde{E}} \xrightarrow{\cong} \tilde{B}_{\nu(\Sigma), \tilde{E} \otimes L}.\]
Moreover the pull-back by $\varphi$ induces identifications

$$\psi_{\mathcal{B}^*} : \mathcal{B}^*_\nu(\Sigma) \otimes \mathcal{L} \xrightarrow{\cong} \mathcal{B}^*_\nu(\Sigma), \quad \psi_{\mathcal{B}} : \tilde{\mathcal{B}}\nu(\Sigma) \otimes \mathcal{L} \xrightarrow{\cong} \tilde{\mathcal{B}}\nu(\Sigma).$$

**Lemma 2.15.** Suppose $\mathcal{L}_\Sigma, \mathcal{L}_\Sigma'$ are complex line bundles over $\mathcal{B}^*_\nu(\Sigma), \mathcal{E}', \mathcal{B}^*_\nu(\Sigma), \mathcal{E}'$ corresponding to the cohomology classes $\mu_{\nu(\Sigma)}, \mathcal{E}([\Sigma]) \in H^2(\mathcal{B}^*_\nu(\Sigma), \mathcal{E}; \mathbb{Z})$, $\mu_{\nu(\Sigma)}, \mathcal{E}'([\Sigma]) \in H^2(\mathcal{B}^*_\nu(\Sigma), \mathcal{E}'; \mathbb{Z})$. Then we have

$$(\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}})^* \mathcal{L}_\Sigma \cong \mathcal{L}_\Sigma.$$

**Proof.** It is sufficient to show that $(\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}})^*(c_2(\tilde{\mathcal{B}}^*_\nu(\Sigma))/[\Sigma])$ is equal to $c_2(\tilde{\mathcal{B}}^*_\nu(\Sigma))/[\Sigma]$ since $H^2(\mathcal{B}^*_\nu(\Sigma), \mathcal{E}; \mathbb{Z}) \to H^2(\tilde{\mathcal{B}}^*_\nu(\Sigma), \mathcal{E}'; \mathbb{Z})$ is injective.

Let $\pi_1 : \nu(\Sigma) \times \tilde{\mathcal{B}}^*_\nu(\Sigma) \to \nu(\Sigma)$ be the projection. We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{B}}^*_\nu(\Sigma) \otimes \pi_1^*(L|\nu(\Sigma)) & \cong & \tilde{\mathcal{B}}^*_\nu(\Sigma) \\
\downarrow & & \downarrow \\
(\tilde{\mathcal{B}}^*_\nu(\Sigma) \otimes \pi_1^*(L|\nu(\Sigma))) \times \mathcal{A}_{\nu(\Sigma), \mathcal{E}} & \xrightarrow{\varphi^{-1} \times (\varphi^* \circ t_{\mathcal{A}})} & (\tilde{\mathcal{B}}^*_\nu(\Sigma) \otimes \pi_1^*(L|\nu(\Sigma))) \times \mathcal{A}_{\nu(\Sigma), \mathcal{E}'} \\
\downarrow & & \downarrow \\
\nu(\Sigma) \times \tilde{\mathcal{B}}^*_\nu(\Sigma) & \xrightarrow{id_{\nu(\Sigma)} \times (\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}})} & \nu(\Sigma) \times \tilde{\mathcal{B}}^*_\nu(\Sigma) \\
\end{array}
\]

Hence we have

$$(\text{id}_{\nu(\Sigma)} \times (\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}}))^* \tilde{\mathcal{B}}^*_\nu(\Sigma) \cong \tilde{\mathcal{B}}^*_\nu(\Sigma) \otimes \pi_1^*(L|\nu(\Sigma))$$

and we obtain

\begin{align*}
(\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}})^*(c_2(\tilde{\mathcal{B}}^*_\nu(\Sigma))/[\Sigma]) \\
= c_2(\tilde{\mathcal{B}}^*_\nu(\Sigma) \otimes \pi_1^*(L|\nu(\Sigma)))/[\Sigma] \\
= \{c_2(\tilde{\mathcal{B}}^*_\nu(\Sigma)) + \pi_1^*c_1(L|\nu(\Sigma)) \cup c_1(\tilde{\mathcal{B}}^*_\nu(\Sigma)) + \pi_1^*c_1(L|\nu(\Sigma))^2\}/[\Sigma] \\
&= [\tilde{\mathcal{B}}^*_\nu(\Sigma)]/\Sigma + \{\pi_1^*c_1(L|\nu(\Sigma)) \cup c_1(\tilde{\mathcal{B}}^*_\nu(\Sigma))\}/[\Sigma] \\
&\in H^2(\tilde{\mathcal{B}}^*_\mathcal{E}; \mathbb{Z}).
\end{align*}
By the Künneth formula, we can write
\[ c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma)) = c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma))_B + c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma))_{\tilde{\mathcal{B}}_{\nu}(\Sigma), E; \mathbb{Z}) \]
since \( \tilde{\mathcal{B}}_{\nu}(\Sigma), \tilde{\mathcal{E}} \) is simply connected ([AB]). The action of \( G^0_{\nu(\Sigma)}, E \) on \( \Lambda^2 \tilde{\mathcal{E}}_{\nu}(\Sigma) \) is trivial, since the determinants of elements of \( G^0_{\nu(\Sigma)}, \tilde{\mathcal{E}} \) are equal to 1 by definition. Hence \( \Lambda^2 \tilde{\mathcal{E}}_{\nu}(\Sigma) \) is the pull-back \( \pi^* \Lambda^2 \tilde{\mathcal{E}}_{\nu}(\Sigma) \) of \( c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma)) = c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma))_{\tilde{\mathcal{B}}_{\nu}(\Sigma), E; \mathbb{Z}} \) which implies that the \( \tilde{\mathcal{B}}_{\nu}(\Sigma) \)-part \( c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma))_B \) of \( c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma)) = c_1(\Lambda^2 \tilde{\mathcal{E}}_{\nu}(\Sigma)) \) is 0 and we have
\[ \{ \pi^*_1 c_1(L_{\nu(\Sigma)}) \cup c_1(\tilde{\mathcal{E}}_{\nu}(\Sigma)) \} / [\Sigma] = 0 \in H^2(\tilde{\mathcal{B}}_{\nu(\Sigma)}; \mathbb{Z}). \]

From the equation (5), we obtain
\[ (\psi_B \circ t_B)^*(c_2(\tilde{\mathcal{E}}'_{\nu(\Sigma)}) / [\Sigma]) = c_2(\tilde{\mathcal{E}}_{\nu(\Sigma)}) / [\Sigma] \in H^2(\tilde{\mathcal{B}}_{\nu(\Sigma)}; \mathbb{Z}). \]

**Proof of Lemma 2.2.** Lemma 2.2 follows from (6) and the following commutative diagram:

![Diagram](image)

**Proof of independence of \( q_X^{ij} \) from \( \tilde{P} \).** Take homology classes \( [\Sigma_i] \in H_2(X; \mathbb{Z}) \) with \( [\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2 \) for \( i = 1, \ldots, d \) and choose \( U(2) \)-lifts \( \tilde{P} \) and \( \tilde{P}' \) of \( P \). Then we obtain line bundles \( \mathcal{L}_{\Sigma_i} \) and \( \mathcal{L}'_{\Sigma_i} \) over \( \mathcal{B}_{\nu(\Sigma_i), E} \).
and $B^*_{\nu(\Sigma_i),E'}$. We denote the zero locus of sections $s_{\Sigma_i}, s'_{\Sigma_i}$ of $L_{\Sigma_i}, L'_{\Sigma_i}$ by $V_{\Sigma_i}, V'_{\Sigma_i}$. By Lemma 2.15, $(\psi_{B'} \circ t_{B'})^* L'_{\Sigma_i}$ is isomorphic to $L_{\Sigma_i}$. We fix an isomorphism and regard the section $s'_{\Sigma_i}$ of $L'_{\Sigma_i}$ as a section of $L_{\Sigma_i}$ through the identifications

$$\psi_{B'} \circ t_{B'} : B^*_{\nu(\Sigma_i),E} \xrightarrow{\sim} B^*_{\nu(\Sigma_i),E'}, \quad (\psi_{B'} \circ t_{B'})^* L'_{\Sigma_i} \simeq L_{\Sigma_i}.$$

We take paths $\{s_{\Sigma_i,t}\}_{t \in [0,1]}$ between $s_{\Sigma_i}$ and $s'_{\Sigma_i}$. In the same way as Lemma 2.14, we have a bordism between $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ and $M_P \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d}$. Hence we obtain

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1, M_P \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d} \rangle \in \mathbb{Z}_2. \quad \Box$$

Lastly we show that $q^u_X$ is independent of the choice of surfaces $\Sigma_i$ representing the homology classes $[\Sigma_i]$ and that $q^u_X$ is multi-linear with respect to $[\Sigma_1], \ldots, [\Sigma_d]$. It follows from the following lemma directly.

**Lemma 2.16.** Let $X$ and $P$ be as in Proposition 2.12. Take homology classes $[\Sigma_1], \ldots, [\Sigma_d] \in H_2(X; \mathbb{Z})$ with self-intersection numbers even. Moreover assume that $[\Sigma_1] = [\Sigma'_1] + [\Sigma''_1] \in H_2(X; \mathbb{Z}), \quad [\Sigma'_1] \cdot [\Sigma'_1] \equiv [\Sigma''_1] \cdot [\Sigma''_1] \equiv 0 \mod 2$.

Then we have

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1, M_P \cap V'_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle + \langle u_1, M_P \cap V'_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle \in \mathbb{Z}_2.$$

**Proof.** By definition, we have

$$\tilde{\mu}_E([\Sigma_1]) = c_2(\mathbb{E})/|\Sigma_1| = c_2(\mathbb{E})/|\Sigma'_1| + c_2(\mathbb{E})/|\Sigma''_1|$$

$$= \tilde{\mu}_E([\Sigma'_1]) + \tilde{\mu}_E([\Sigma''_1]) \in H^2(\mathbb{E}; \mathbb{Z}).$$
The homomorphism $\beta^* : H^2(B^*_E; \mathbb{Z}) \to H^2(\tilde{B}^*_E; \mathbb{Z})$ is injective and $\tilde{\mu}_E([\Sigma_1]), \tilde{\mu}_E([\Sigma'_{i}]), \tilde{\mu}_E([\Sigma''_{i}])$ lie in the image $\beta^*$ from Lemma 2.1. Hence we have

$$\mu([\Sigma_1]) = \mu([\Sigma'_{i}]) + \mu([\Sigma''_{i}]) \in H^2(B^*_P; \mathbb{Z}).$$

Since $MP \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}$ is compact from Lemma 2.11, we have

$$\langle u_1, MP \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1 \cup \mu([\Sigma_1]), MP \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1 \cup (\mu([\Sigma'_{i}]) + \mu([\Sigma''_{i}])), MP \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle$$

$$+ \langle u_1 \cup \mu([\Sigma''_{i}]), MP \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle = \langle u_1, MP \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle$$

$$+ \langle u_1, MP \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle . \Box$$

3. A Connected Sum Formula for $Y \# S^2 \times S^2$

3.1. Statement of the result

As is well known Donaldson invariants vanish for the connected sum $X_1 \# X_2$ provided $b^+(X_i) > 0$ for $i = 1, 2$ ([D3]). In [FS], however, Fintushel and Stern defined some torsion invariants by using instantons on $SU(2)$-bundles and they showed that their $SU(2)$-torsion invariants are non-trivial for the connected sum of the form $Y \# S^2 \times S^2$. In this section, we show a similar non-vanishing theorem for our $SO(3)$-torsion invariants.

Let $Y$ be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(Y) = 2a - 1$ for $a > 1$. Let $Q$ be an $SO(3)$-bundle with $w_2(Q)$ equal to $w_2(Y)$ and $p_1(Q)$ equal to $\sigma(Y) + 4$ modulo 8. Suppose that the dimension of $M_Q$ is $2d$ for a non-negative integer $d$. When we fix an orientation on the space $\mathcal{H}^+(Y)$ of self-dual harmonic 2-forms on $Y$ and an lift $c \in H^2(Y; \mathbb{Z})$ of $w_2(Q) \in H^2(Y; \mathbb{Z}_2)$, we have the Donaldson invariant

$$q_{k-1,w,Y} : \otimes^d H_2(Y; \mathbb{Z}) \longrightarrow \mathbb{Q}$$

where

$$k - 1 = -\frac{1}{4} p_1(Q) \in \mathbb{Q}, \quad w = w_2(Q) \in H^2(Y; \mathbb{Z}_2).$$

When $[\Sigma_i] \cdot [\Sigma_i]$ are even for $i = 1, \ldots, d$, then $q_{k-1,w,Y}([\Sigma_1], \ldots, [\Sigma_d])$ is in $\mathbb{Z}$. 
We consider an $SO(3)$-bundle $P$ over $X = Y \# S^2 \times S^2$ satisfying

$$w_2(P) = w_2(X), \quad p_1(P) = p_1(Q) - 4,$$

so that $P$ satisfies (2). The dimension of $M_P$ is given by $2d + 5$.

We define surfaces $\Sigma, \Sigma'$ embedded in $S^2 \times S^2$ by

$$\Sigma = S^2 \times \{pt\}, \quad \Sigma' = \{pt\} \times S^2 \subset S^2 \times S^2.$$

Then we have

$$[\Sigma] \cdot [\Sigma] \equiv [\Sigma'] \cdot [\Sigma'] \equiv 0 \mod 2.$$

Now $q_{k,w,Y \# S^2 \times S^2}^u([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma'])$ is defined for homology classes $[\Sigma_i]$ of $Y$ with self-intersection numbers even. The following is an $SO(3)$-version of Theorem 1.1 in [FS].

**Theorem 3.1.** In the above situation, we have

$$q_{k,w,Y \# S^2 \times S^2}^u([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma']) \equiv q_{k-1,w,Y}([\Sigma_1], \ldots, [\Sigma_d]) \mod 2.$$

The proof is given in the following three subsections.

**3.2. Notations and general facts**

For the proof of Theorem 3.1, we will investigate the intersection $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma'} \cap V_\Sigma$ when the neck of $Y \# S^2 \times S^2$ is very long. For the preparation, we define some notations and recall some facts about instantons over the connected sum of 4-manifolds.

Let $Y_1$ and $Y_2$ be a closed, oriented 4-manifold. The connected sum $X = Y_1 \# Y_2$ is constructed in the following way. Fix Riemannian metrics $g_1$ and $g_2$ on $Y_1$ and $Y_2$ which are flat in small neighborhoods of fixed points $y_1 \in Y_1$ and $y_2 \in Y_2$. For $N > 1$ and $\lambda > 0$ with $N\lambda^{\frac{1}{2}} \ll 1$, we put

$$\Omega_i = \Omega_{y_i}(\lambda, N) = \{y \in Y_i|N^{-1}\lambda^{\frac{1}{2}} < d(y, y_i) < N\lambda^{\frac{1}{2}}\} \quad (i = 1, 2).$$

Let

$$\sigma : (TY_1)_{y_1} \xrightarrow{\sim} (TY_2)_{y_2}.$$
be an orientation-reversing linear isometry. For each positive real number \( \lambda > 0 \), we define
\[
  f_\lambda : (TY_1)_{y_1} \setminus \{0\} \to (TY_2)_{y_2} \setminus \{0\}
\]
\[
  \xi \mapsto \frac{\lambda}{|\xi|^2} \sigma(\xi).
\]
This map \( f_\lambda \) induces a diffeomorphism between \( \Omega_1 \) and \( \Omega_2 \). The connected sum \( X \) of \( Y_1 \) and \( Y_2 \) is identified with
\[
X(\lambda) = (Y_1 \setminus B_{y_1}(N^{-1}\lambda^{\frac{1}{2}})) \bigcup_{f_\lambda} (Y_2 \setminus B_{y_2}(N^{-1}\lambda^{\frac{1}{2}}))
\]
where \( B_{y_i}(N^{-1}\lambda^{\frac{1}{2}}) \) is the open ball centered on \( y_i \) with radius \( N^{-1}\lambda^{\frac{1}{2}} \). The metrics \( g_1 \) and \( g_2 \) define a conformal structure on \( X \) since \( g_i \) is flat in a small neighborhood of \( y_i \). We fix a metric \( g_0 \) on \( X \) which represents the conformal structure. Moreover we assume that \( g_\lambda \) is equal to \( g_i \) on \( Y_1 \setminus B((N+1)\lambda^{\frac{1}{2}}) \).

**Definition 3.2.** Fix a real number \( q \) with \( q > 4 \). Let \( [A^{(n)}] \in M_P(g_{\lambda_n}) \) be instantons over \( X = Y_1 \# Y_2 \) for a sequence \( \lambda_n \to 0 \). Let \( z_1, \ldots, z_l \) be points in \( Y_1 \setminus \{y_1\} \), \( z_1', \ldots, z_m' \) be points in \( Y_2 \setminus \{y_2\} \) and \( A_i \) be connections over \( Y_i \). Then we say that \( [A^{(n)}] \) is weakly convergent to \( ([A_1], [A_2]; z_1, \ldots, z_l, z_1', \ldots, z_m') \) when \( [A^{(n)}] \) is \( L^q \)-convergent to \( ([A_1], [A_n]) \) over compact subsets in \( (Y_1 \cup Y_2) \setminus \{y_1, y_2, z_1, \ldots, z_l, z_1', \ldots, z_m'\} \) and \( |F_{A^{(n)}}|^2 \) is convergent as measure to
\[
|F_{A_1}|^2 + |F_{A_2}|^2 + 8\pi^2 \left( \sum_{\nu=1}^l \delta_{z_\nu} + \sum_{\nu=1}^m \delta_{z_\nu}' \right)
\]
over compact subsets in \( (Y_1 \setminus \{y_1\}) \cup (Y_2 \setminus \{y_2\}) \). Here \( \delta_z \) is the delta function supported on \( z \).

We use the following well-known theorem.

**Theorem 3.3 ([D3, DK]).** Let \( P \) be an \( SO(3) \)-bundle over \( X = Y_1 \# Y_2 \). Set \( k = -p_1(P)/4, w = w_2(P), w_i = w|_{Y_i} \). Let \( [A^{(n)}] \in M_{k,w,X}(\lambda_n) \) be instantons over \( X \) for \( \lambda_n \to 0 \). Then there is a subsequence \( \{[A^{(n')}]\}_{n'} \) which is weakly convergent to \( ([A_1], [A_2]; z_1, \ldots, z_l, z_1', \ldots, z_m') \) for some
\[
[A_1] \in M_{k_1,w_1,Y_1}(g_1), \quad [A_2] \in M_{k_2,w_2,Y_2}(g_2), \quad z_1, \ldots, z_l \in Y_1 \setminus \{y_1\}, \quad z_1', \ldots, z_m' \in Y_2 \setminus \{y_2\}.
\]
with \[ k_1 \geq 0, \quad k_2 \geq 0, \quad k_1 + k_2 + l + m \leq k. \]

Next we review gluing of instantons. The theory of gluing of instantons is standard. To fix notations, we recall the theory briefly.

Let \( A_i \) be instantons over \( Y_i \). We denote the \( SO(3) \)-bundles carrying \( A_i \) by \( P_i \). We can construct instantons on \( X = Y_1 \# Y_2 \) close to \( A_i \) on each factor. Outline of the construction is as follows. (See [DK] Chapter 7 for details.)

Let \( b \) be a small positive number with \( b \geq 4N\lambda^1/2 \). By using suitable cut-off functions and trivializations of \( P_i \) on neighborhoods of \( y_i \), we obtain a connections \( A'_i \) which are flat over the annuli \( \Omega_i \) and equal to \( A_i \) outside the balls centered at \( y_i \) with radius \( b \). Take an \( SO(3) \)-isomorphism \( \rho \) between \((P_1)_{y_1}\) and \((P_2)_{y_2}\). We can spread this isomorphism by using flat structures of \( A'_i \), and obtain an isomorphism \( g\rho \) between \( P_1|_{\Omega_1} \) and \( P_2|_{\Omega_2} \) covering \( f\lambda \).

We define an \( SO(3) \)-bundle \( P_\rho \) over \( X \) and a connection \( A'(\rho) = A'_1 \#_\rho A'_2 \) on \( P_\rho \) by gluing \( P_1, A_i \) through \( g\rho \). Then in large region outside the neck of \( X \), \( A'(\rho) \) satisfies the instanton equation, and \( F^{+}_{A'(\rho)} \) is very small near the neck. To obtain a genuine instanton we have to perturb \( A'(\rho) \). We consider the equation

\[
F^{+}_{A'(\rho)+a} = 0
\]

for \( a \in \Omega^1_X(g_{P_\rho}) \). To solve this equation, we take linear maps

\[
\sigma_i : H^2_{A_i} \longrightarrow \Omega^+_i(g_{P_i})
\]

such that \( d_{A_i}^+ \oplus \sigma_i \) are surjective and for each \( h_i \in H^2_{A_i} \), the supports of \( \sigma_i(h_i) \) are in the complement of the ball centered at \( y_i \) with radius \( b \). Then put

\[
\sigma := \sigma_1 + \sigma_2 : H^2_{A_1} \oplus H^2_{A_2} \longrightarrow \Omega^+_X(g_{P_\rho}).
\]

We can construct a right inverse of \( d_{A_i}^+ + \sigma \) starting from right inverses of \( d_{A_i}^+ + \sigma_i \). Decompose the right inverse as \( P \oplus \pi \), where

\[
P : \Omega^+_X(g_{P_\rho}) \longrightarrow \Omega^1(g_{P_\rho}), \quad \pi : \Omega^+_X(g_{P_\rho}) \longrightarrow H^2_{A_1} \oplus H^2_{A_2}.
\]
Instead of (7), we first consider the equation

\[ F_{A'(\rho)+a} + \sigma(h) = 0 \]

for \((a, h) \in \Omega^1_X(g\rho) \times (H^2_{A_1} \oplus H^2_{A_2})\). We find a solution of this equation in the form \(a = P\xi, h = \pi\xi\). In this case, we see that the equation is equivalent to the equation

\[ \xi + (P\xi \wedge P\xi)^+ = -F_{A'(\rho)} \]

by a short calculation. Using the contraction mapping principle, we can show that there is a unique small solution \( \xi_{\rho} \in \Omega^1(g\rho) \) for the equation.

We get a genuine instanton if and only if \( \pi\xi_{\rho} = 0 \). Therefore there is a map \( \Psi : Gl_{y_1, y_2} \rightarrow H^2_{A_1} \times H^2_{A_2} \) such that the solutions of \( \Psi = 0 \) represent instantons over \( X \). Here \( Gl_{y_1, y_2} \) is the space of \( SO(3) \)-equivariant isomorphisms between \((P_1)_{y_1}\) and \((P_2)_{y_2}\). We fix an element \( \rho_0 \in Gl_{y_1, y_2} \) to identify \( Gl_{y_1, y_2} \) with \( SO(3) \).

We can include the deformations of \( [A_i] \) to this construction. For small neighborhoods \( U_{A_i} \) of 0 in \( H^1_{A_i} \), we have a map

\[ \Psi : T := U_{A_1} \times U_{A_2} \times SO(3) \rightarrow H^2_{A_1} \times H^2_{A_2} \]

such that elements of \( \Psi^{-1}(0) \) correspond to instantons.

Let \( \Gamma_{A_i} \) be the isotropy group of \( A_i \) in the gauge group and put \( \Gamma = \Gamma_{A_1} \times \Gamma_{A_2} \). We assume that \( U_{A_i} \) is \( \Gamma_{A_i} \)-invariant. Then there are natural actions of \( \Gamma \) on \( T \) and on \( H^2_{A_1} \times H^2_{A_2} \). We can show that \( \Psi \) is \( \Gamma \)-equivariant and instantons corresponding to elements of \( \Psi^{-1}(0) \) are gauge equivalent to each other if and only if they are in the same \( \Gamma \)-orbit. Hence we can regard \( \Psi^{-1}(0)/\Gamma \) as a subspace of \( MP \).

An important feature is that instantons over \( X = Y_1 \# Y_2 \) which is close to \( A_i \) over \( Y_i \) are given in the above description. More precise statement is the following:

Let \( Y''_i \) be the complement of balls centered at \( y_i \) with radius \( \lambda^{1/2}/2 \). Take instantons \( A_i \) over \( Y_i \) and a positive number \( \nu > 0 \). Then put

\[ U_{\lambda}(\nu) := \{ [A] \in B^*_X \mid d_q([A|_{Y''_i}], [A_i|_{Y''_i}]) < \nu, \ i = 1, 2 \} \]
Here \( q \) is the fixed real number with \( q > 4 \) and \( d_q \) is the distance induced by \( L^q \)-norm over \( Y_i'' \). If \( \nu > 0 \) is small, then there is a positive number \( \lambda(\nu) > 0 \) such that for \( \lambda < \lambda(\nu) \) we can take a neighborhood \( T \) of \( \{0\} \times \{0\} \times SO(3) \) in \( H_{A_1}^1 \times H_{A_2}^1 \times SO(3) \) such that \( M_P(g_\lambda) \cap U_\lambda(\nu) \) is homeomorphic to \( \Psi^{-1}(0)/\Gamma \). Summing up these:

**Theorem 3.4.** Let \( A_1, A_2 \) be instantons on \( Y_1, Y_2 \). Then there is a \( \Gamma = \Gamma_{A_1} \times \Gamma_{A_2} \)-invariant neighborhood \( T \) of \( SO(3) \times \{0\} \times \{0\} \) in \( SO(3) \times H_{A_1}^1 \times H_{A_2}^1 \) and \( \Gamma \)-equivariant map

\[
\Psi : T \longrightarrow H_{A_1}^2 \times H_{A_2}^2
\]

such that \( \Psi^{-1}(0)/\Gamma \) is homeomorphic to an open set \( N \) in \( M_P \). Moreover for a small positive number \( \nu > 0 \), there is a \( \lambda(\nu) > 0 \) and \( T \) such that if \( \lambda < \lambda(\nu) \) then \( N = M_P(g_\lambda) \cap U_\lambda(\nu) \).

In particular, when \( Y_2 \) is \( S^4 \) and \( A_2 \) is the fundamental instanton \( J \) with instanton number one, we have:

**Corollary 3.5.** Let \( A_1 \) be an instanton over \( Y_1 \) and \( A_2 \) be the fundamental instanton \( J \) over \( S^4 \). For a small positive number \( \nu > 0 \), there is a positive number \( \lambda_0 > 0 \) and a neighborhood \( U_{A_1} \) of \( 0 \) in \( H_{A_1}^1 \), a neighborhood \( U_0 \) of \( 0 \) in \( S^4 = \mathbb{R}^4 \cup \{\infty\} \) and \( \Gamma = \Gamma_{A_1} \)-equivariant map

\[
\Psi : U_{A_1} \times U_0 \times (0, \lambda_0) \times SO(3) \longrightarrow H_{A_1}^2
\]

such that \( \Psi^{-1}(0)/\Gamma \) is naturally homeomorphic to \( M_P \cap U_{\lambda_0}(\nu) \).

**Remark 3.6.** We can generalize the statements of Theorem 3.4 and Corollary 3.5 to the case of gluing 3 or more instantons.

### 3.3. Shrinking the neck

In the situation of Theorem 3.1, we investigate

\[
M_P(g_\lambda) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}
\]

as \( \lambda \) tends to 0. We use the notations in §3.2.

Let \( Y_1 \) be a closed, oriented, simply connected, non-spin 4-manifold with \( b^+(Y_1) = 2a - 1 \) with \( a > 1 \) and we write \( Y_2 \) for \( S^2 \times S^2 \). Let \( P \) be an
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SO(3)-bundle over $X = Y_1 \# Y_2$ satisfying (2). Assume that the virtual dimension of $M_P$ is $2d + 5$ for a non-negative integer $d$. Take homology classes $[\Sigma_1], \ldots, [\Sigma_d] \in H_2(Y_1; \mathbb{Z})$ with $[\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2$. Set $\Sigma = S^2 \times \{pt\}, \Sigma' = \{pt\} \times S^2 \subset Y_2$. Take instantons

$$[A^{(n)}] \in M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'}$$

for a sequence $\lambda_n \to 0$. By Theorem 3.3, a subsequence of $\{[A^{(n)}]\}_n$ is weakly convergent to some

$$([A_1], [A_2]; z_1, \ldots, z_l, z_1', \ldots, z_m'),$$

where

$$[A_1] \in M_{k_1, w, Y_1}(g_1), \quad [A_2] \in M_{k_2, Y_2}(g_2),$$

$$z_1, \ldots, z_l \in Y_1 \{y_1\}, \quad z_1', \ldots, z_m' \in Y_2 \{y_2\}.$$

**Lemma 3.7.** In the above situation, we have

$$k_1 = k - 1, \quad l = 0, \quad [A_1] \in M_{k-1, w, Y_1}(g_1) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d},$$

$$m = 1, \quad z_1' \in \nu(\Sigma) \cap \nu(\Sigma'), \quad [A_2] = [\Theta_{Y_2}].$$

Here $\Theta_{Y_2}$ is the trivial connection on $Y_2$.

**Proof.** From Theorem 3.3, we have

$$k_1 + k_2 + l + m \leq k. \quad (9)$$

Let $p$ be the number of $\nu(\Sigma_i)$ which contain some point $z_\alpha$ and $q$ be the number of $\nu(\Sigma), \nu(\Sigma')$ which contain some point $z'_\alpha$. Then by the transversality condition (4), we have

$$0 \leq p \leq 2l, \quad 0 \leq q \leq 2m. \quad (10)$$

Without loss of generality, we may assume

$$[A_1] \in M_{k_1, w, Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-p}}$$

if we change the order of surfaces. Since $w_2(P)|_{Y_1}$ is non-trivial, we can show $k_1 > 0$ in the same way as the proof of Lemma 2.11. For generic sections,
the intersection $M_{k_1,w,Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-p}}$ is transverse by Lemma 2.10. Hence we have

\[(11) \quad 2(d - p) \leq \dim M_{k_1,w,Y_1}.\]

We would like to show $k_2 = 0$. Suppose that $k_2$ is positive. Then we also obtain

\[(12) \quad 2(2 - q) \leq \dim M_{k_2,Y_2}.\]

By index theorem, there is the formula

\[(13) \quad \dim M_{k_1,w,Y_1} + \dim M_{k_2,Y_2} + 3 = \dim M_{k_1+k_2,w,X}.\]

From (9), (11), (12) and (13), we have

\[2(d - p) + 2(2 - q) + 3 \leq \dim M_{k_1+k_2,w,X} \leq \dim M_{k,w,X} - 8(l + m) = 2d + 5 - 8(l + m).\]

This inequality and (10) imply

\[8(l + m) + 2 \leq 2p + 2q \leq 4(l + m).\]

We have a contradiction. Hence $k_2$ is 0 which implies that $[A_2]$ is the class of trivial flat connection $[\Theta_{Y_2}]$.

Since $k_2$ is 0, the virtual dimension of $M_{0,Y_2}$ is $-6$. From (13), we have

\[(14) \quad \dim M_{k_1,w,Y_1} - 3 = \dim M_{k_1,w,X}.\]

By (9), (10), (11) and (14), we have

\[2(d - 2l) - 3 \leq 2(d - p) - 3 \leq \dim M_{k_1,w,Y_1} - 3 = \dim M_{k_1,w,X} \leq \dim M_{k,w,X} - 8(l + m).\]

Therefore we obtain

\[4l + 8m \leq 8.\]

In particular, we have $m \leq 1$. We show $m = 1$. Suppose $m = 0$, then we have $[\Theta_{Y_2}] \in V_\Sigma, [\Theta_{Y_2}] \in V_{\Sigma'}$. To obtain a contradiction, we need to choose $V_\Sigma$ and $V_{\Sigma'}$ in a specific way. As mentioned in Remark 2.9, we can choose...
V_Σ and V_{Σ'} do not include [Θ_{Y_2}]. If we choose such V_Σ and V_{Σ'}, we have a contradiction. We obtain l = 0, m = 1 and z'_1 \in \nu(Σ) \cap \nu(Σ'). Hence

\[ [A_1] \in M_{k_1, w, Y_1} \cap V_{Σ_1} \cap \cdots \cap V_{Σ_d}. \]

Lastly we show \( k_1 = k - 1 \). From (9), we have \( k_1 \leq k - 1 \). On the other hand, from (11) we have

\[ 2d \leq \dim M_{k_1, w, Y_1} = \dim M_{k-1, w, Y_1} - 8(k - 1 - k_1) = 2d - 8(k - 1 - k_1). \]

This implies \( k_1 \geq k - 1 \). Therefore \( k_1 \) is equal to \( k - 1 \). We complete the proof. \( \square \)

Let \( w'_0 \) be the unique intersection point of \( Σ \) and \( Σ' \). Fix a small neighborhood \( U_{w'_0} \) of \( w'_0 \) with \( \nu(Σ) \cap \nu(Σ') \subset U_{w'_0} \). We suppose that the metric \( g_2 \) on \( Y_2 \) is flat on \( U_{w'_0} \) for simplicity.

Take

\[ [A^{(n)}] \in M_P(g_{λ_n}) \cap V_{Σ_1} \cap \cdots \cap V_{Σ_d} \cap V_Σ \cap V_{Σ'}, \]

for \( λ_n \to 0 \) and assume that \( \{[A^{(n)}]\}_{n \in \mathbb{N}} \) weakly converges to \( ([A_1], [Θ_{Y_2}]; z'_1) \) for some \([A_1] \in M_{k-1, w, Y_1} \cap V_{Σ_1} \cap \cdots \cap V_{Σ_d}, z'_1 \in \nu(Σ) \cap \nu(Σ')\). We can define the local center of mass \( c_n \in U_{w'_0} \) and scale \( λ'_n > 0 \) of \( [A^{(n)}] \) around \( z'_1 \) when \( n \) is sufficiently large. If \( n \) is large enough, then we obtain

\[ \int_{U_{w'_0}} |F_{A^{(n)}}|^2 dμ_{g_2} > 4π^2 \]

since \( |F_{A^{(n)}}|^2 \) converges to \( 8π^2δ_{z'_1} \) on \( U_{w'_0} \). We define the center of mass \( c_n \) to be the center of the smallest ball in \( U_{w'_0} \) where the integral of \( |F_{A^{(n)}}|^2 \) is equal to \( 4π^2 \) and the scale \( λ'_n \) to be the radius of the ball. The center of mass and scale is determined uniquely ([D1]). The center \( c_n \) converges to \( z'_1 \) and the scale \( λ'_n \) converges to 0.

Let \( m : \mathbb{R}^4 \to S^4 = \mathbb{R}^4 \cup \{∞\} \) be the stereographic map and \( d_λ : \mathbb{R}^4 \to \mathbb{R}^4 \) be the map \( d_λ(y) = λ^{-1}y \). Put \( χ_n := m \circ d_{λ'_n} \). Then \( χ_n \) induces a conformal isomorphism between \( X \) and the connected sum

\[ X \# S^4 = (X \setminus B_{c_n}(N^{-1}λ'_n)) \cup_{f_{λ'_n}} (S^4 \setminus B_∞(N^{-1}λ'_n)). \]
since the metric $g_2$ is flat on $U_{w_0'}$. Here $f_{\lambda'_n}$ is defined in the following way: Using the geodesic coordinate near $c_n$ and the stereographic map, we identify $(TX)_{c_n}$ with $(TS^4)_0$. Let $\sigma'$ be the natural, orientation reversing isometry between $(TS^4)_0$ and $(TS^4)_\infty$, then $f_{\lambda'_n}$ is given by

$$f_{\lambda'_n} : (TX)_{c_n} \setminus \{0\} \longrightarrow (TS^4)_\infty \setminus \{0\}, \quad \xi \longmapsto \frac{\lambda'_n}{|\xi|^2} \sigma'(\xi).$$

We can regard $A^{(n)}$ as an instanton on $X \# S^4$ such that $A^{(n)}$ is close to $A_1$, $\Theta_{Y_2}$ on $Y_1, Y_2$ and close to the standard instanton $J$ on $S^4$.

Fix a small positive number $\lambda_0$ and a small neighborhood $U'_{\{A_1\}}$ of $[A_1]$ in $M_Q$. Let $O_{\{A_1\}} \subset B'_p$ be a small open neighborhood of

$$\{ [B' \# y, \lambda, \rho] \Theta_{Y_2} \# z_1, \lambda', \rho', J'' ] \mid B \in U'_{\{A_1\}}, \lambda, \lambda' \in (0, \lambda_0), \rho, \rho' \in SO(3), z_1 \in \nu(\Sigma) \cap \nu(\Sigma') \},$$

Here $B', J'$ are connections which are flat near $y_1, \infty$ and equal to $B, J$ outside $b$-balls. (The real number $b$ is a small positive number fixed in §3.2). The notation $\# z_1, \lambda', \rho'$ means gluing of connections at $z'_1$ using the identification $f_{\lambda'}$ twisted by $\rho'$, and similarly for $\# y, \lambda, \rho$. The instanton $[A^{(n)}]$ is in $O_{\{A_1\}}$ when $n$ is large. We can define the local centers for elements of $O_{\{A_1\}}$ and we have a map $p : O_{\{A_1\}} \rightarrow U_{w_0'}$ which maps connections to their centers. By Donaldson [D2] Proposition (3.18), we can take sections $s_{\Sigma}, s_{\Sigma'}$ such that $O_{\{A_1\}} \cap V_{\Sigma}, O_{\{A_1\}} \cap V_{\Sigma'}$ are equal to $p^{-1}(U'_{z_1} \cap \Sigma), p^{-1}(U'_{z_1} \cap \Sigma')$.

Hence we may suppose that the center $c_n$ of $[A^{(n)}]$ is $w_0'$ for large $n$.

We denote $S^4$ by $Y_3$ and denote $\Theta_{Y_2}, J$ by $A_2, A_3$ and put

$$Y_{1,n}'' = Y_1 \setminus B_{y_1}(\lambda_n/2), \quad Y_{2,n}'' = Y_2 \setminus (B_{y_2}(\lambda_n/2) \cup B_{w_0'}(\lambda_n/2)), \quad Y_{3,n}'' = Y_3 \setminus B_{\infty}(\lambda_n/2).$$

For $\nu > 0$, put

$$U_{\{A_1\}}(\nu) = \{ [A] \in B_{X \# S^4}^* \mid d_q([A]_{Y_{i,n}''}, [A_i]_{Y_{i,n}'}) < \nu, \ i = 1, 2, 3 \}.$$

We have proved the following:

**Lemma 3.8.** Fix a positive number $\nu > 0$. Take instantons $[A^{(n)}] \in M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'}$ for a sequence $\lambda_n \rightarrow 0$. Then $[A^{(n)}]$ is
in $U_{[A_1],\lambda_n}(\nu)$ for some $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ when $n$ is sufficiently large.

Fix $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ and a small positive number $\nu$. By Theorem 3.4, Corollary 3.5 and Remark 3.6, there is a small neighborhood $U_{A_1}$ of 0 in $H^1_{A_1}$, a positive real number $\lambda_0$ and a $\Gamma_{\Theta_Y}$-equivariant map

$$\Psi : T = U_{A_1} \times SO(3) \times U_{w_0'} \times (0, \lambda_0) \times SO(3) \longrightarrow H^2_{\Theta_Y}$$

such that $\Psi^{-1}(0)/\Gamma_{\Theta_Y}$ is homeomorphic to $M_P(g_{\lambda_n}) \cap U_{[A_1],\lambda_n}(\nu)$. Note that $H^2_{A_1} = 0$ and $\dim H^1_{A_1} = 2d$ (for generic metrics on $Y_1$). Since the action of $\Gamma_{\Theta_Y} = SO(3)$ on $SO(3) \times SO(3)$ is the diagonal action, $\Psi^{-1}(0)/SO(3)$ is naturally identified with

$$\Psi^{-1}(0) \cap (U_{A_1} \times \{1\} \times U_{w_0'} \times (0, \lambda_0) \times SO(3)).$$

We write $T'$ for $U_{A_1} \times \{1\} \times U_{w_0'} \times (0, \lambda_0) \times SO(3)$. Since $T'$ parametrizes connections on $X$, it makes sense to take the intersection $T' \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'}$. We can suppose

$$T' \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'} = \{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3).$$

Hence $M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'} \cap U_{[A_1],\lambda_n}(\nu)$ is homeomorphic to

$$\Psi^{-1}(0) \cap (\{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3)) \subset H^1_{A_1} \times SO(3) \times U_{w_0'} \times (0, \lambda_0) \times SO(3).$$

Donaldson calculated the leading term of $\Psi$ in [D2] explicitly. By the explicit expression of the leading term of $\Psi$ and calculations similar to those in [D2] V, we can show the following:

**Lemma 3.9.** For generic metrics $g_1$ and $g_2$, points $y_1$, $y_2$ and $w_0'$ and the metric $g_{\lambda_n}$, the intersection

$$\Psi^{-1}(0) \cap (\{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3))$$

is homeomorphic to

$$\{c\lambda_n\} \times \gamma \subset (0, \lambda_0) \times SO(3)$$
where $\gamma$ is a loop in $SO(3)$ which represent the generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and $c > 0$ is a constant number independent of $n$.

Define $N_{[A_1]}$ by

\[ N_{[A_1]} = \{ [A'_1 \#_{\lambda_n} \Theta Y_2 \#_{w'_0, c\lambda_n, \rho} J'] | \rho \in \gamma \} \]  

(15)

Here $\#_{\lambda_n}$ is an abbreviation for $\#_{y_1, \lambda_n, 1}$. We have obtained the following:

**Corollary 3.10.** Let $Y$ be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(Y) = 2a - 1$ for $a > 1$ and $P$ be an $SO(3)$-bundle over $X = Y \# S^2 \times S^2$ which satisfies the condition (2). Suppose that the virtual dimension of $M_P$ is $2d + 5$ for a non-negative integer $d$. Take $d$ homology classes $[\Sigma_i]$ in $H_2(Y; \mathbb{Z})$ with self-intersection numbers even. Then for a small $\lambda > 0$, generic metrics $g_1$ and $g_2$, and generic points $y_1, y_2$ and $w'_0$, the intersection

\[ M_P(g_\lambda) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'} \]

is homeomorphic to

\[ \prod_{[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}} N_{[A_1]} \]

**3.4. End of the proof**

From Corollary 3.10, we have

\[ q_{k, w, Y \# S^2 \times S^2}([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma']) = \sum_{[A_1]} \langle u_1, N_{[A_1]} \rangle \in \mathbb{Z}_2, \]

where $[A_1]$ runs in $M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$. Therefore it is sufficient to show that the pairing $\langle u_1, N_{[A_1]} \rangle$ is non-trivial for the proof of Theorem 3.1. The last step is carried out by making use of the following Proposition due to Akbulut, Mrowka and Ruan.

**Proposition 3.11 ([AMR]).** Let $X_i$ be closed, oriented, simply connected 4-manifolds for $i = 1, 2$ and $x_i$ be points of $X_i$. Take $SO(3)$-bundles $P_i$ over $X_i$ with $w_2(P_i)$ equal to $w_2(X_i)$. Choose $U(2)$-lifts $\tilde{P}_i$ of $P_i$ and
assume that the second Chern numbers of $\bar{P}_i$ are odd. (In this case, $P_1 \# P_2$ satisfies the condition (2). See Remark 2.6.) We fix trivializations of $P_i$ on small neighborhoods $U_{x_i}$ of $x_i$. For irreducible connections $B_i$ on $P_i$ with trivial on $U_{x_i}$ with respect to fixed trivializations, we have a family of connections

$$G := \{ [B_1 \# \rho B_2] \mid \rho \in SO(3) \} \cong SO(3) \subset B_{P_1 \# P_2}^*.$$

Then the restriction $u_1|_G$ is non-trivial in $H^1(G; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

In our case,

$$X_1 = Y \# S^2 \times S^2, \quad P_1 = Q \# P_{S^2 \times S^2}, \quad B_1 = A'_1 \# \Theta_{S^2 \times S^2},$$

$$X_2 = S^4, \quad P_2 = P_{S^4}/\{\pm 1\}, \quad B_2 = J'.$$

Here $Q$ is an $SO(3)$-bundle over $Y$ with

$$w_2(Q) = w_2(Y), \quad p_1(Q) \equiv \sigma(Y) + 4 \mod 8,$$

$P_{S^2 \times S^2}$ is the trivial $SO(3)$-bundle over $S^2 \times S^2$ and $P_{S^4}$ is an $SU(2)$-bundle with second Chern number equal to 1. By the formulas

$$p_1(Q) = -4c_2(\bar{Q}) + c_1(\bar{Q})^2, \quad w_2(Y)^2 \equiv \sigma(Y) \mod 8$$

and (16), we have

$$c_2(\bar{Q}) \equiv 1 \mod 2.$$

Hence the assumptions of Proposition 3.11 is satisfied. Since $N_{[A_1]}$ is a loop in $G$ which represent the generator of $\pi_1(G) \cong \mathbb{Z}_2$, we obtain:

**Corollary 3.12.** For each $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$, the pairing $\langle u_1, N_{[A_1]} \rangle$ is non-trivial in $\mathbb{Z}_2$.

This completes the proof of Theorem 3.1.
4. Example

4.1. Non-triviality of $q_{2\mathbb{C}P^2 \# \mathbb{C}P^2}^{u_1}$

We see that the $SO(3)$-torsion invariant for $X = 2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is non-trivial.

To distinguish two $\mathbb{C}P^2$'s, we write $X = \mathbb{C}P^2_1 \# \mathbb{C}P^2_2 \# \overline{\mathbb{C}P^2}$.

**Theorem 4.1.** Let $H_i$ be the canonical generator of $H_2(\mathbb{C}P^2_i; \mathbb{Z})$ for $i = 1, 2$ and $E$ be the canonical generator of $H_2(\overline{\mathbb{C}P^2}; \mathbb{Z})$. Then we have

$$q_{\mathbb{C}P^2_1 \# \mathbb{C}P^2_2 \# \overline{\mathbb{C}P^2}}^{u_1}(-H_1 + E, H_2 - E) \equiv 1 \mod 2.$$

**Proof.** Let $Q$ be an $SO(3)$-bundle on $\mathbb{C}P^2$ with

$$w_2(Q) = w_2(\mathbb{C}P^2), \quad p_1(Q) = -3.$$  

Then the dimension of $M_Q$ is 0. Kotschick showed that the Donaldson invariant associated with $Q$ is

$$q^{u_1}_{\mathbb{C}P^2, \mathbb{C}P^2} = -1$$

if we choose a suitable orientation on $M_Q$ ([K1, K2]). Note that there is no wall since $b^-(\mathbb{C}P^2)$ is 0. The signature of $\mathbb{C}P^2$ is 1, hence we have

$$p_1(Q) \equiv \sigma(\mathbb{C}P^2) + 4 \mod 8$$

and $q^{u_1}_{\mathbb{C}P^2, \mathbb{C}P^2 \# S^2 \times S^2}([\Sigma], [\Sigma'])$ is defined. From Theorem 3.1, we have

$$q^{u_1}_{\mathbb{C}P^2, \mathbb{C}P^2 \# S^2 \times S^2}([\Sigma], [\Sigma']) \equiv 1 \mod 2.$$  

On the other hand, $\mathbb{C}P^2 \# S^2 \times S^2$ is diffeomorphic to $\mathbb{C}P^2_1 \# \mathbb{C}P^2_2 \# \overline{\mathbb{C}P^2}$ ([Wal]). The induced isomorphism between the 2-dimensional homology groups is given by

$$
\begin{align*}
H_2(\mathbb{C}P^2 \# S^2 \times S^2; \mathbb{Z}) & \xrightarrow{\cong} H_2(\mathbb{C}P^2_1 \# \mathbb{C}P^2_2 \# \overline{\mathbb{C}P^2}; \mathbb{Z}) \\
H & \mapsto H_1 + H_2 - E \\
[\Sigma] & \mapsto -H_1 + E \\
[\Sigma'] & \mapsto H_2 - E.
\end{align*}
$$
The torsion cohomology class $w$ is $w_2(\mathbb{CP}^2 \# S^2 \times S^2)$, and the image of $w$ under the isomorphism is $w_2(2\mathbb{CP}^2 \# \mathbb{CP}^2)$. We also denote this class by $w$. The images of $[\Sigma]$ and $[\Sigma']$ under the isomorphism are $-H_1 + E$ and $H_2 - E$ respectively. Hence we obtain

$$q^{w_1}_{\mathbb{CP}^2 \# \mathbb{CP}^2}(-H_1 + E, H_2 - E) \equiv 1 \mod 2. \quad \Box$$

### 4.2. A vanishing theorem

Let $X$ be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(X) = 2a$ for some $a > 0$. Moreover assume that $X$ can be written as the connected sum $Y_1 \# Y_2$ of non-spin 4-manifolds $Y_i$ with $b^+(Y_i) \geq 1$. In this situation, we can show a vanishing theorem similar to the usual Donaldson invariant. However we must require a certain condition for homology classes in $X$. The condition is that each homology class lies in $H_2(Y_1; \mathbb{Z})$ or $H_2(Y_2; \mathbb{Z})$.

Suppose that $P$ is an $SO(3)$-bundle over $X$ satisfying (2) and that $\dim M_P$ is $2d + 1$ for some non-negative integer $d$. Moreover suppose that $d = d_1 + d_2$ for some $d_1 \geq 0, d_2 \geq 0$. Take homology classes $[\Sigma_1], \ldots, [\Sigma_{d_1}] \in H_2(Y_1; \mathbb{Z}), [\Sigma'_1], \ldots, [\Sigma'_{d_2}] \in H_2(Y_2; \mathbb{Z})$ with self-intersection numbers even. Then by the standard dimension-count argument [MM], we can show

$$M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d_1}} \cap V_{\Sigma'_1} \cap \cdots \cap V_{\Sigma'_{d_2}} = \emptyset$$

when the neck is sufficiently long. Hence we have:

**Theorem 4.2.** Let $Y_1, Y_2$ be closed, oriented, simply connected, non-spin 4-manifolds with $b^+(Y_i) > 0$ and $b^+(Y_1) \equiv b^+(Y_2) \mod 2$. Then for homology classes $[\Sigma_1], \ldots, [\Sigma_{d_1}] \in H_2(Y_1; \mathbb{Z}), [\Sigma'_1], \ldots, [\Sigma'_{d_2}] \in H_2(Y_2; \mathbb{Z})$ with self-intersection numbers even, we have

$$q^{w_1}_{Y_1 \# Y_2}([\Sigma_1], \ldots, [\Sigma_{d_1}], [\Sigma'_1], \ldots, [\Sigma'_{d_2}]) \equiv 0 \mod 2.$$

**Remark 4.3.** We regard $X = 2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ as the connected sum of $Y_1 = \mathbb{CP}^2$ and $Y_2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Then $w$ is non-trivial on $Y_i$ for $i = 1, 2$. By
Theorem 3.1, \( q_{Y_1 \# Y_2}^{u_1}(-H_1 + E, H_2 - E) \) is non-trivial in contrast to Theorem 4.2. If there were a formula like

\[
q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(-H_1 + E, H_2 - E) \equiv \left( “q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(-H_1, H_2 - E)” + “q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(E, H_2 - E)” \right) \mod 2,
\]

then we would be able to apply Theorem 4.2 to showing the vanishing of \( q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(-H_1 + E, H_2 - E) \). However “\( q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(-H_1, H_2 - E)” \) nor “\( q_{\frac{u_1}{2},w,Y_1 \# Y_2}^{u_1}(E, H_2 - E)” \) are not defined because

\[
(-H_1) \cdot (-H_1) \equiv E \cdot E \equiv 1 \mod 2.
\]

References


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[K2] Kotschick, D., Moduli of vector bundles with odd $c_1$ on surfaces with $q = p_g = 0$, Amer. J. Math. 114 (1992), 297–313.


[PT] Pidstrigach, V. and A. Tyurin, Localisation of the Donaldson’s invariants along Seiberg-Witten classes, dg-ga/9507004


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