Polyhedral Deformations of a Cone Manifold

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Abstract. A single parameter family of polyhedra $P(\psi)$ is constructed in three dimensional spaces of constant curvature $C(\psi)$. Identification of the faces of the polyhedra via isometries results in cone manifolds $M(\psi)$ which are topologically $S^1 \times S^2$, $S^3$ or singular $S^2$. The singular set of $M(\psi)$ can have vertices of degree three for some values of $\psi$ and can also be the Whitehead link or form other configurations. Curvature varies continuously with $\psi$. At $\psi = 0$ spontaneous surgery occurs and the topological type of $M(\psi)$ changes. This phenomenon is described.

0. Introduction

We study continuous families of cone manifolds $M_\psi$ parametrised by cone angle which begin at cone angle zero with the complement of the Whitehead link in $S^3$. We consider the case of equal cone angles on all singular link components. The cone structures for certain non-zero values of cone angles exist in projective models or in $S^3$.

In our Dehn surgery direction the cone manifolds are for certain cone angles, obtained by surgery on the Whitehead link in $S^3$ resulting in a topologically distinct singular set in $S^2 \times S^1$. As cone angle is increased the topological type of the singular set changes and the hyperbolic cone manifold develops two cusps and becomes $S^3$ at cone angle $\frac{2}{3} \pi$. The geometry and topology of $M_\psi$ and its singular set remain unaltered as cone angle increases beyond $\frac{2}{3} \pi$ until we reach a cone angle $\omega$ where $M_\psi$ becomes Euclidean with a topologically different type of singularity. Increasing cone angle past $\omega$ the singular set reverts back to its pre-$\frac{2}{3} \pi$ cone angle topological type and $M_\psi$ becomes spherical in $S^2 \times S^1$. At cone angle $\pi$ the underlying polyhedron becomes a lens in $S^3$ from which the cone manifold is obtained by suitable identifications. For cone angles in the interval $[\pi, \zeta]$, $M_\psi$ is spherical and the topological type of its singular set is unchanged but it is now in $S^3$. At cone angle $\zeta$, $M_\psi$ becomes the suspension of a sphere with four cone points.
It remains the well understood sphere with four singularities for cone angles larger than $\zeta$.

Investigating the deformation on the other side of Dehn surgery we obtain the Whitehead link in $S^3$ for certain non-zero cone angles. We will study this family in our subsequent work.

Here a **Cone Manifold** is a PL manifold with a possibly empty codimension two locally flat submanifold called the **singular set**. In dimensions two and three the singular set consists of isolated points and curves but not graphs respectively. The geometric model is a spherical, Euclidean, or hyperbolic space of constant curvature where the constant is any real number. Points in the complement of the singular set have neighbourhoods homeomorphic to neighbourhoods in the model. Points on the singular set have neighbourhoods homeomorphic to neighbourhoods in the topological space obtained by identifying boundaries of the intersection of two half spaces referred to as a **wedge** in the model. The homeomorphism takes the singular set to the axis of rotation in the topological space and transition functions are isometries. **Orbifolds** are represented by discrete structures where cone angles are of the form $\frac{2\pi}{n}, n \in \mathbb{N}$. cf [1]

The topological space obtained by identifying in pairs faces of a polyhedron in a space of constant curvature via isometries is a cone manifold if:

1. No edge is identified with its inverse in the equivalence class induced by the identifications, and the identifications of wedges along faces are cyclic for each equivalence class of edges.

2. The cone angle at each edge, i.e. the sum of the dihedral angles about the edge is $\leq 2\pi$.

3. The neighbourhood of each vertex is a cone on a sphere. If a vertex neighbourhood is, for example, a cone on a torus the topological space is not a manifold.

4. There are either two or no edges emanating from a vertex where cone angle is not $2\pi$. If two edges then the cone angles must be equal and the edges must be lined up. The vertex neighbourhood is obtained by identifying wedge half planes.

cf [1].
It is reasonable to expect our deformation process to hold for any hyperbolic link complement since the polyhedral description of a link complement utilised here is canonical and we can expect to be able to deform this construction by opening cusps in $M_0$ in the way we have described. Our methods can be viewed as part of an approach to describe all compact connected 3-manifolds through deformations by removing singularities of branched singular covers of $S^3$ along universal links thereby providing an approach to the Poincaré Conjecture in dimension three. The important phenomenon of singular set with vertices of degree three in a cone manifold occurs in our work. Thus our construction may prove useful in studying this phenomenon.

1. Cone Angle Interval $[0, \omega]$  

As defined in the abstract let $M_\psi$ denote the cone manifold obtained by identifying the faces of the polyhedron $P_\psi$.

**Proposition 1.1.** $M_\psi$ is $S^2 \times S^1$ with two singular components and is hyperbolic for cone angles $\psi \in (0, \frac{2}{3} \pi)$. $M_{\frac{2}{3} \pi}$ is cusped hyperbolic and has a singular set with degree three vertices. There exists $\omega \in \mathbb{R}$ such that $M_\psi$ is hyperbolic with a degree three vertexed singular set when $\frac{2}{3} \pi < \psi < \omega$ and $M_\omega$ is $S^3$.

**Proof.** $M_0$ the complement of the Whitehead link in $S^3$ is obtained from a two cusped octahedron $P_0$ with identifications described in the projective Klein model of $\mathbb{H}^3$ in figure 2. We refer the reader to [2], [3] and [4].

![Fig. 1. The Whitehead link](image-url)
Fig. 2. $P_0$ with identifications

for details of this construction.

To find out more about $P_\psi$ we view it relative to a coordinate system and inside a reference box in the Klein model as in figure 4. The length, width and height of the reference box are denoted by $a, b$ and $c$ respectively. Opening cusps in $M_0$ we obtain $P_\psi$ for $\psi \in (0, \frac{2}{3} \pi)$ as in figure 3.

From figure 4 we have the following identifications:

Face identifications:

1. $A \leftrightarrow A'$ i.e. $\triangle (omp) \leftrightarrow \triangle (sqr)$
2. $C \leftrightarrow C'$ i.e. $\triangle (jhi) \leftrightarrow \triangle (klm)$
3. $D \leftrightarrow D'$ i.e. $\text{hexagon}(nokmi) \leftrightarrow \text{hexagon}(poktrq)$
4. $B \leftrightarrow B'$ i.e. $\text{hexagon}(inpqsj) \leftrightarrow \text{hexagon}(hmilrsj)$

Edge pairings:

5. $ok \leftrightarrow ok$
6. $sj \leftrightarrow sj$
7. $on \leftrightarrow op \leftrightarrow sq \leftrightarrow sr$
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Fig. 3. opening cusps

Fig. 4. opening cusps
Edge pairings 5, 6 and 10 represent singular components of $M_\psi$.

We observe:

The dihedral angle between planes incident in a member of (5) or (6) is the cone angle $\psi$.

Moreover (7), (8) and (9) are not part of the singular set. Therefore

The dihedral angle between planes incident in a member of (7), (8) or (9) is $\frac{\pi}{2}$.

Information in (10) represents segments $\beta_1, \beta_2, \beta_3$ and $\beta_4$ which are identified to give a singular component $\beta$ of the singular set of $M_\psi$. Considering (10) with equal cone angles on all singular components we deduce:

Planes incident in a member of (10) intersect at dihedral angle $\frac{\psi}{4}$.

We deduce from (10) that $a = b$. Box coordinates can therefore be normalised to give $a = 1 = b$ and $c$ where $c$ is measured in the $z$–direction. The reference box is therefore a cube with square base and height $2c$ as in figure 4.

Topologically $M_\psi$ is $S^1 \times S^2$ with two unlinked singular components as in figure 5.

To find out about the geometry of $M_\psi$ we look at $P_\psi$ inside the Klein model of $\mathbb{H}^3$. Let $R$ be the radius of the Klein ball $B_R$ and let $c$ represent the height of the main box. We will show that $P_\psi$ lives inside $B_R$ and can be specified by $R$ and $c$. We also have

$$\text{curvature of } \mathbb{H}^3 = -\frac{1}{R^2}$$

The geometry of $M_\psi$ can therefore be described in terms of $\psi$. We now obtain $R$ and $c$ as functions of $\psi$.

Figure 6 depicts $P_\psi$ in the Klein model.
Fig. 5. Topology of $M_\psi$.

Fig. 6. $P_\psi$ in the Klein model.
In homogeneous coordinates the equations of planes of $P_\psi$ and their poles are:

<table>
<thead>
<tr>
<th>Plane</th>
<th>Equation</th>
<th>Pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$cx + cy + z = c$</td>
<td>$(cR, cR, R, c)$</td>
</tr>
<tr>
<td>$B$</td>
<td>$cx - cy - z = c$</td>
<td>$(cR, -cR, -R, c)$</td>
</tr>
<tr>
<td>$C$</td>
<td>$-cx - cy + z = c$</td>
<td>$(-cR, -cR, R, c)$</td>
</tr>
<tr>
<td>$D$</td>
<td>$-cx + cy - z = c$</td>
<td>$(-cR, cR, -R, c)$</td>
</tr>
</tbody>
</table>

Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$, $\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. Then the hyperbolic bilinear form is given by

$$
\langle \mathbf{v}, \mathbf{w} \rangle_H = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4
$$

Let $\theta$ be the dihedral angle of intersection between planes $P$ and $Q$ with poles $\mathbf{v}$ and $\mathbf{w}$ respectively. Then

$$
\cos \theta = -\frac{\langle \mathbf{v}, \mathbf{w} \rangle_H}{\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_H \langle \mathbf{w}, \mathbf{w} \rangle_H}}
$$

Combining (10), (13), (15) and (17) we can derive expressions for $\cos \psi$ and $\cos \frac{\psi}{4}$ in terms of $R$ and $c$ to obtain:

$$
c^2 = \frac{1 + \cos \psi}{2 \cos \frac{\psi}{4} - \cos \psi + 1}
$$

$$
R^2 = \frac{1 + \cos \psi}{2 \cos \frac{\psi}{4} + \cos \psi - 1}
$$

Substituting $(R, c)$—expressions of $\cos \frac{\psi}{4}$ and $\cos \psi$ in (18) and (19) we can verify these identities.

We note $\omega = 4 \cos^{-1}\left(\frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \cos^{-1}\left(-\frac{3\sqrt{3}}{8}\right)\right)\right) \approx 2.311984...$ is the smallest positive value of $\psi$ for which the denominator of (19) is zero. Hence $\omega$ is the smallest positive value of $\psi$ for which $R^2$ is infinite. This combined with (14) implies that curvature is zero and $M_\psi$ is therefore Euclidean when $\psi = \omega$. 
Combining (12), (15), (16) and (17) the equation of plane $N$ in figure 6 is:

$$N : -x + y + cz = R^2$$

Let $p = (-1, 1, c, R)$ denote the pole of $N$ with $\mathbb{H}^3$ embedded in $\mathbb{R}P^3$. We have

$$\langle p, p \rangle = 2 + c^2 - R^2$$

$$= \frac{3 - 4 \cos^2 \frac{\psi}{4}}{(4 \cos^3 \frac{\psi}{4} - 4 \cos \frac{\psi}{4}) (4 \cos^3 \frac{\psi}{4} - 4 \cos \frac{\psi}{4} + 1)}$$

Therefore

$$\langle p, p \rangle \begin{cases} > 0 & \text{when } 0 < \psi < \frac{2}{3} \pi, \\ = 0 & \text{when } \psi = \frac{2}{3} \pi, \\ < 0 & \text{when } \frac{2}{3} \pi < \psi < \omega. \end{cases}$$

Information contained in (22) corresponds to the configurations depicted in figure 7 in the case of $\mathbb{H}^2$ embedded in $\mathbb{R}P^2$ and is true for $n$ hyperplanes in $\mathbb{H}^n$ embedded in $\mathbb{R}P^n$.

Referring to (20), figure 6 and figure 7, when $\langle p, p \rangle > 0$ there is a plane $N$ inside $\mathbb{H}^3$. When $\langle p, p \rangle = 0$ the plane $N$ is the point of intersection of the planes $A, C$ and $D$ on $\partial \mathbb{H}^3$. When $\langle p, p \rangle < 0$ the plane $N$ is the point of intersection of the planes $A, C$ and $D$ inside $\mathbb{H}^3$.

We now verify that $P_\psi \subset \mathbb{H}^3 \cup \partial \mathbb{H}^3$ for $\psi \in [0, \frac{2}{3} \pi]$. Since $P_\psi$ is symmetric with respect to the origin it is sufficient to show that the vertices $(A \cap N \cap C)$ and $(A \cap N \cap D)$ of figure 6 live inside $\mathbb{H}^3 \cup \partial \mathbb{H}^3$. Let

$$f(\psi) = R^2 - (A \cap N \cap C)^2 = \frac{1}{2}(R^2 - c^2)(2 + c^2 - R^2)$$

where $(A \cap N \cap C)^2$ denotes square of the distance of vertex $(A \cap N \cap C)$ from the origin. We have $f > 0$ for $\psi \in (0, \frac{2}{3} \pi)$ and $f = 0$ when $\psi = 0$ or $\frac{2}{3} \pi$. 
Fig. 7. (a) \( \langle p, p \rangle > 0 \): lines meet outside \( \mathbb{H}^2 \). (b) \( \langle p, p \rangle = 0 \): lines meet on \( \partial \mathbb{H}^2 \). (c) \( \langle p, p \rangle < 0 \): lines meet inside \( \mathbb{H}^2 \).

Let

\[
g(\psi) = R^2 - (A \cap N \cap D)^2 = \frac{(R^2 - 1)(2 + c^2 - R^2)}{c^2 + 1}
\]

We note \( g > 0 \) when \( \psi \in (0, \frac{2}{3}\pi) \) and \( g = 0 \) if \( \psi = 0 \) or \( \frac{2}{3}\pi \).

Therefore \( P_\psi \subset \mathbb{H}^3 \cup \partial \mathbb{H}^3 \) when \( \psi \in [0, \frac{2}{3}\pi] \). Hence \( M_\psi \) is hyperbolic when \( \psi \in [0, \frac{2}{3}\pi] \).

We note from (22) and figure 7 that \( M_{\frac{2}{3}\pi} \) is cusped. From figure 8 we observe that \( M_{\frac{2}{3}\pi} \) has two cusps, it is topologically \( S^3 \) with singular set as shown in figure 9 and its cusp neighborhoods are Euclidean turnovers as in figure 10.

From (22) and figure 7 we observe that the bounding planes of \( P_\psi \) meet inside \( \mathbb{H}^3 \) when \( \psi \in \left(\frac{2}{3}\pi, \omega\right) \). Therefore, \( P_\psi \) is in \( \mathbb{H}^3 \). Hence \( M_\psi \) is hyperbolic when \( \psi \in \left(\frac{2}{3}\pi, \omega\right) \). The singular set of \( M_\psi \) for \( \psi \in \left(\frac{2}{3}\pi, \omega\right) \) is shown in figure 9.

We note from (19) that \( R(\omega) = \infty \) hence \( M_\omega \) is Euclidean. \( P_\omega \) is a tetrahedron from which \( M_\omega \) is obtained using identifications. We observe
Fig. 8. $P_{\frac{2}{3}\pi}$ inside $H^3 \cup \partial H^3$

Fig. 9. Singular set of $M_\psi$ when $\psi \in \left(\frac{2}{3}\pi, \omega\right)$ and its cusp neighbourhoods when $\psi = \frac{2}{3}\pi$.

Fig. 10. A Euclidean turnover
that $M_\omega$ is $S^3$. The singular set of $M_\omega$ is depicted by figure 9 with one cusp. □

2. Cone Angles Larger than $\omega$

As defined in the abstract let $M_\psi$ denote the cone manifold obtained by identifying the faces of the polyhedron $P_\psi$.

**Proposition 2.1.** $M_\psi$ is spherical when cone angle $\psi$ is larger than $\omega$. There exists $\zeta \in \mathbb{R}$ such that $M_\psi$ is topologically $S^3$ with a singular set which has degree three vertices when $\omega < \psi < \zeta$. $M_\psi$ is the suspension of a sphere with four cone points when $\zeta \leq \psi$.

**Proof.** When $\omega < \psi < \pi$ we note $R^2$ the square radius of the Klein model becomes negative so that the model has imaginary radius. This leads us to use the spherical bilinear form for $v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$:

$$\langle v, w \rangle_S = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4 \quad (25)$$

when $\psi \in (\omega, \pi)$.

This is the usual scalar product on $\mathbb{R}^4$. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Since $H^3_R = \{ x \mid \langle x, x \rangle_H = -R^2 \}$ we define $S^3_R = \{ x \mid \langle x, x \rangle_S = R^2 \}$. Thus $S^3_R$ is the sphere of radius $R$. The “Klein model” $K^3_R$ for the sphere of radius $R$ is the hyperplane $K^3_R = \{ x \mid t = R \}$. In contrast to the hyperbolic case we don’t need to verify that the polytope lies inside the sphere of radius $R$ in $K^3_R$ since projection from the origin which is not conformal defines a 1-1 correspondence between $K^3_R$ and the upper hemisphere of $S^3_R$. The metric on $K^3_R$ is then the pull back of the metric on $S^3_R$. As in the hyperbolic case reflections in planes through the origin and rotation about axes through the origin are both Euclidean and spherical isometries.

If a plane in $K^3_R$ has equation $\alpha x + \beta y + \gamma z = \delta$ and $t = R$ then in homogeneous coordinates the equation is $\alpha x + \beta y + \gamma z - (\delta/R)t = 0$. In the spherical case the pole has homogeneous coordinates $(\alpha, \beta, \gamma, -\delta/R)$ whereas in the hyperbolic case the pole has coordinates $(\alpha, \beta, \gamma, \delta/R)$. If a pair of planes with poles $v$ and $w$ in the spherical case intersect with
dihedral angle $\theta$ then

$$
\pm \cos \theta = \frac{\langle v, w \rangle_S}{\sqrt{\langle v, v \rangle_S \langle w, w \rangle_S}}
$$

Using (25), (26) and the spherical version of (15) we obtain

$$
0 \leq R^2 = -\frac{1 + \cos \psi}{2 \cos \frac{\psi}{4} - 1 + \cos \psi}
$$

when $\omega \leq \psi \leq \zeta$.

When $\psi \in (\omega, \pi)$ we observe that $P_\psi$ is a tetrahedron with identification as in figure 11.

![Diagram of $P_\psi$ inside the reference box](image)

Therefore $M_\psi$ is $S^3$ with a singular set which has degree three vertices as in figure 9 with spherical turnover neighbourhoods.

Since $R^2(\pi) = 0 = e^2(\pi)$ the three dimensional model has collapsed into a two dimensional disc at cone angle $\pi$. By projection onto $S^3_1 = \{x | x^2 + y^2 + z^2 + t^2 = 1\}$ the sphere of radius 1 we observe that $P_\pi$ is a lens with angle $\psi/4$ as in figure 13 from which $M_\pi$ is obtained by identifications.
Fig. 12. A spherical turnover

Fig. 13. \( P_\pi \) is a lens with angle \( \frac{\pi}{4} \)

Fig. 14. Singular set of \( M_\pi \)
We note $\zeta = 4 \cos^{-1} \left( \frac{2}{\sqrt{3}} \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{-3\sqrt{3}}{8} + \frac{4}{3}\pi \right) \right) \right) \approx 5.191298...$ is the second smallest value of $\psi$ for which $R$ is infinite. The equation of $c^2$ is the same as in the hyperbolic case. The model continues to be $S^3$ for $\pi < \psi < \zeta$ therefore $M_\psi$ is spherical with a degree three vertexed singular set as in figure 14 when $\psi \in (\pi, \zeta)$.

As cone angle $\psi$ approaches $\zeta$ the segments $h_1$ and $h_2$ decrease in length towards 0 as shown in figure 15 so that we get the singular set shown in figure 16 when $\psi = \zeta$.

We therefore obtain $M_\psi$ as the suspension of a sphere with four cone points as in figure 16 when $\zeta < \psi$. □

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**Fig. 15.** Segments $h_1$ and $h_2$ decrease in length as $\psi$ increases towards $\zeta$

**Fig. 16.** $M_\zeta$ is the suspension of a sphere with four cone points
3. Spontaneous Surgery

We now look at how we get spontaneous surgery and the Whitehead link as the singular set. $P_\psi$ and its identifications “before” spontaneous surgery are shown in figure 17. Face pairings are $A \leftrightarrow A', B \leftrightarrow B', C \leftrightarrow C'$ and $D \leftrightarrow D'$ and edges with the same label are identified.

We note that $M_\psi$ for $\psi \in (0, \frac{2}{3}\pi)$ “after” spontaneous surgery is the result of performing $(0, 1)$-Dehn surgery on one component of the Whitehead link in $S^3$. This can be observed by considering the boundary of a tubular neighbourhood of the $\beta$-component of the singular set. In Figure 18 we have shown a tubular neighbourhood of the $\beta_1$-component of the singular locus as we move from “before” to “after” spontaneous surgery from left to right in Figure 18. The singular component shrinks in length as cone angle decreases and becomes a point at cone angle zero. As cone angle increases the singular component increases in length in a transverse direction. We observe that a meridian of $\beta_1$ “before” spontaneous surgery is mapped to a longitude of $\beta_1$ “after” spontaneous surgery. The manifold “after” spontaneous surgery is therefore the result of performing $(0, 1)$-Dehn surgery on the $\beta$-component of the singular set.

Fig. 17. $P_\psi$ “before” spontaneous surgery gives $S^3$ with the Whitehead link as its singular set
Equations of planes are unaltered “after” spontaneous surgery and can be obtained from (20) and the symmetry of $P_\psi$. Dihedral angles at the $\beta$ edges of $P_\psi$ are the cone angle $\psi$ and other incidence angles between the planes of $P_\psi$ are the same as “after” spontaneous surgery. Calculations to show the behaviour of $M_\psi$ before surgery remain to be done.

References


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