Decay of Magnetic Eigenfunctions on Asymptotically Hyperbolic Plane

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Abstract. We prove a decay estimate for eigenfunctions of the magnetic Schrödinger operator on two dimensional Riemannian manifold $M$. We assume that $M$ is simply connected, rotationally symmetric, and asymptotically hyperbolic. The proof is based on a method developed by S. Nakamura in [Nak1].

1. Introduction and Results

1.1. Main result

In this paper we study the decay property of eigenfunctions of the magnetic Schrödinger operators on (the trivial Hermitian line bundle over) a rotationally symmetric, asymptotically hyperbolic two-dimensional Riemannian manifolds with a pole. Here, we mean by pole a point at where the exponential map gives a diffeomorphism.

More precisely, we make the following assumptions (M.1)–(M.4) on manifolds.

(M.1) $(M, g_M)$ is a two-dimensional smooth Riemannian manifold with a pole $p$.

(M.2) In addition, the Riemannian metric $g_M$ is expressed as $dr \otimes dr + g(r)^2 d\theta \otimes d\theta$ in the geodesic polar coordinates $(r, \theta) \in (0, \infty) \times S^1$ at $p$ for some positive $C^2$-function $g$.

(M.3) The radial curvature $K(r) = -g''(r)/g(r)$ is bounded from below on $M$.

(M.4) The radial curvature $K(r)$ is non-positive and has a limit $\lim_{r \to \infty} K(r) = -1$. Moreover, the integral $\int_0^\infty |1 + K(r)| dr$ is finite.

1991 Mathematics Subject Classification. 35P15, 81Q10, 58J50.
To formulate our result, we make the following assumptions on the electro-magnetic fields on $M$.

(A.1) Let $\nabla$ be a compatible ($C^1$-)connection on the trivial Hermitian line bundle over $M$. Moreover, there exists a real-valued, continuous function $B$ on $[0, \infty)$ such that $B$ is positive near infinity, and the curvature 2-form $\omega$ associated with the connection $\nabla$ takes the form $B(r)g(r)dr \wedge d\theta$ in the geodesic polar coordinates at $p$.

(A.2) The scalar potential $V$ is real-valued, continuous function on $M(\cong \{p\} \cup (0, \infty) \times S^1)$ and extends to a bounded function $\tilde{V}$ on $[0, \infty) \times S_{\tau}$ for some $\tau > 0$, where we set

$$S_{\tau} = \{z \in \mathbb{C} \mid |\text{Im } z| < \tau\}.$$

Moreover, $\tilde{V}(r, z)$ is analytic with respect to $z \in S_{\tau}$.

(A.2)$_{\infty}$ The scalar potential $V$ satisfies (A.2)$_{\tau}$ for each $\tau > 0$. Moreover, $\tilde{V}(r, z)$ tends to zero as $r \to \infty$ in each $S_{\tau}$.

We now introduce the magnetic Schrödinger operator with scalar potential

$$H_V = \nabla^* \nabla + V$$

starting from domain $C^\infty_0(M)$, the space of all smooth functions with compact support on $M$. Then $H_V$ is essentially self-adjoint under the conditions (A.1) and (A.2)$_{\tau}$ (see Section 2 below). In what follows we shall identify any essentially self-adjoint operator with its operator closure. We shall denote by $L^2(X, \mu)$ the $L^2$ space on $X$ with measure $\mu$, and denote by $L^2(M)$ the $L^2$ space with the Riemannian measure $dV$ on $M$. We denote by $C^k(X, Y)$ the space of all $C^k$-maps from $X$ to $Y$.

Throughout this paper, we always assume the following:

(E) A function $\psi$ is a (non-zero) $L^2$-eigenfunction of $H_V$ corresponding to an eigenvalue $E$. Moreover, $E$ belongs to the set of discrete spectrum of $H_V$ (i.e., $E$ is an isolated eigenvalue of finite multiplicity).

The main result of this paper is the following:

**Theorem 1.1.** Assume (M.1)–(M.4), (A.1), (A.2)$_{\infty}$, and (E). Assume that there exists a positive constant $B_0$ such that $\lim_{r \to \infty} |B(r) - B_0| = \infty$. Then...
0 and $E < B_0^2 + 1/4$. Set $\rho(r) = (B_0^2 + 1/4 - E)^{1/2} r$. Then we have the following two assertions.

1. For any $\varepsilon > 0$, the function $e^{(1-\varepsilon)\rho}\psi$ belongs to $L^2(M)$ and the estimate $|\psi(r, \theta)| \leq C\varepsilon e^{-(1-\varepsilon)\rho(r)}$ holds on $M$. Moreover, there exists $c > 0$ such that

$$\left(\int_0^{2\pi} |\psi(r, \theta)|^2 d\theta\right)^{1/2} \leq ce^{-(1-\varepsilon)\rho(r)}e^{-r/2}$$

holds for any $r \geq 0$.

2. If we assume further that $V$ is spherically symmetric, then for any $\varepsilon > 0$ there exist $c > 0$ and $r_0 > 0$ such that

$$\left(\int_0^{2\pi} |\psi(r, \theta)|^2 d\theta\right)^{1/2} \geq ce^{-(1+\varepsilon)\rho(r)}e^{-r/2}$$

holds for all $r \geq r_0$.

**Remark 1.2.**

1. We need not to assume the discreteness of the eigenvalue $E$ in the assertion 2 in Theorem 1.1.

2. The positivity of the constant $B_0$ is not crucial. In fact, the arguments in Sections 3–5 below are still valid in the case of $B_0 = 0$, and we can reduce the case $B_0 < 0$ to the case $B_0 > 0$ via the transform $(r, \theta) \mapsto (r, -\theta)$ as in the Euclidean case.

3. A typical example of manifold under consideration is the hyperbolic plane. In this case, the essential spectrum of the Schrödinger operator $H_V$ with smooth, asymptotically constant magnetic field and with scalar potential satisfying (A.2)$_\infty$ consists of two parts; the continuous part $[B_0^2 + 1/4, \infty)$ and the discrete part $\{(2n + 1)B_0 - n(n + 1)\}$ ($0 \leq n < |B_0| - 1/2$), where the latter is empty if $|B_0| < 1/2$ (see Inahama and Shirai [I-S] and see also subsection 1.3 below). Thus, in this case, the number $B_0^2 + 1/4$ is the lower edge of the continuous spectrum, therefore our result is valid for the eigenfunctions in all spectral gaps of the essential spectrum of $H_V$. 


4. The decay rate of the eigenfunctions as in Theorem 1.1 differs from that of the Euclidean case. As is known (see subsection 1.2 below), under conditions similar to (A.1), (A.2), and (E), any eigenfunction of the Schrödinger operator $H_V$ with (asymptotically) constant magnetic field has the Gaussian decay property at infinity (i.e., decays like $Ce^{-cr^2}$) in the Euclidian case. This means that the (asymptotically) constant magnetic field cannot bind strongly a quantum mechanical charged particle in the case of negative curvature. Note that the spectrum of the Landau Hamiltonian $H_0$ ($= H_V$ with $V = 0$) on the hyperbolic plane has the absolutely continuous part, and $H_0$ has the norm-resolvent continuity with respect to the strength $B_0$ of the magnetic field (see Inahama and Shirai [I-S]).

5. In the case of the constant magnetic field on the hyperbolic plane, the presence of non-zero scalar potential $V$ can produce infinitely many discrete spectra in the spectral gaps of $H_0$ (see Shirai [Shi]).

The organization of the paper is as follows. In succeeding subsections 1.2 and 1.3, we recall some related works and some spectral properties of the Schrödinger operator with constant magnetic field on the hyperbolic plane, respectively. Section 2 contains some preliminary results. In Section 3 we derive a weighted $L^2$-estimate of the eigenfunction under a slightly general setting. The proof is based on the method developed by Nakamura [Nak1]. In Section 4 we give a proof of Theorem 1.1. The upper bound estimate for the eigenfunction follows from the result obtained in Section 3, and the lower bound estimate follows from an argument in Donnelly [Don2].

Acknowledgment. The author thank Takefumi Kondo and Takuya Mine for useful discussions.

1.2. Some related results

In this subsection, we recall some related results. The decay properties of the magnetic eigenfunctions have been studied by many authors, in particular, in the case of the Euclidean spaces. Here, we refer to a few number of works very close to ours.

First, we recall some results in the Euclidean spaces. L. Erdős [Erd] shows the Gaussian upper bound $|\psi(r, \theta)| \leq Ce^{-cr^2}$ for eigenfunctions $\psi$ of
the two-dimensional Schrödinger operator $\frac{1}{2}(D - A)^2 + V$ with rotationally symmetric magnetic field $B(r)$ and bounded scalar potential $V$. He assumes that $B(r) \geq B_0 > 0$ and $E < B_0/2$ for some constant $B_0$ and he put some assumptions on the Fourier coefficients $V_m(r) = \int_0^{2\pi} e^{-m \sqrt{\Delta} \theta} V(r, \theta) d\theta$ of scalar potential $V$; in particular, he requires that there exists a sequence \( \{a_m\} \) such that $|V_m| \leq a_m$, $\sum_m ma_m < \infty$, and $a_m \leq C\delta^m$ for some constants $D > 0, \delta$ satisfying $0 < \delta < 1$.

S. Nakamura [Nak1] (see also [Nak2]) proves that any eigenfunctions $\psi$ of $H_V$ have the Gaussian upper bound in the constant magnetic field case with the assumptions (A.2)$_\tau$ and (E) above. In particular, the estimate $|\psi(r, \theta)| \leq Ce^{-(1-\varepsilon)B_0 r^2/4}$ holds for any $\varepsilon > 0$ under (A.2)$_\infty$, where $B_0$ is the strength of constant magnetic field. Nakamura’s result is generalized by V. Sordoni [Sor] to a class of non-constant magnetic fields in higher dimension.

Their results require more or less some analyticity of the electro-magnetic fields. In fact, L. Erdős [Erd] also shows the eigenfunctions decay slower than Gaussian in general if we drop the condition $a_m \leq D\delta^m$ above, even in the constant magnetic field case.

In non-analytic case, H. D. Cornean and G. Nenciu [C-N] study the decay property of magnetic eigenfunctions.

Next, we recall some results in the case of (non-compact) Riemannian manifolds. H. Donnelly studies the properties of eigenfunctions of the Laplace-Beltrami(-Schrödinger) operators without magnetic fields in a series of works [Don1]–[Don5]. In [Don1], he studies the decay properties of the eigenfunctions $\phi$ of $-\Delta_D$ on the exterior domain $\{z \in \mathbb{D} | d(z, 0) \geq r_0\}$, where $r_0 > 0$ and $\mathbb{D}$ is the Poincaré disk. He shows that if $E < 1/4$ there exists a real analytic function $A(\theta)$ on $[0, 2\pi)$ such that $g^{1/2}(r) h_0(r)^{-1} \phi(r, \theta)$ tends to $A(\theta)$ as $r \to \infty$ uniformly in $\theta$, where $g(r) = \sinh r$, $h_0$ is a solution to the equation $-h''(r) + (F(r) - E)h(h) = 0$ and $F(r) = \frac{1}{2}g''/g - \frac{1}{4}(g'/g)^2$. Especially, the estimate $|\phi(r, \theta)| \leq C \exp\left( -\frac{1}{4} - E \right)^{1/2} r^{\frac{1}{2}} r^{\frac{1}{2}}$ holds. Recently, A. Vasy and J. Wunsch [V-W] show that no eigenfunction of the Laplace-Beltrami operator (without electro-magnetic fields) decays super-exponentially in the case of certain class of manifolds with pinched negative curvature.

In the magnetic case on (non-compact) Riemannian manifolds, it seems that less results exist. V. Iftimie [Ift] studies the decay of eigenfunctions of
the magnetic Schrödinger operators of the form $H_V = (D_i - a_i)g^{ij}(D_j - a_j) + V$ on manifold $(\mathbb{R}^n, g)$, using the Agmon-type estimates. He assumes that the metric $g^{-1} = (g^{ij})$ satisfies $0 < \lambda(|x|)|\xi|^2 \leq g^{ij}\xi_i \xi_j \leq \Lambda(|x|)|\xi|^2$ for some functions $\lambda, \Lambda$, and $\lim_{r \to \infty} r^{-2}\Lambda(r) = 0$. The assumptions on regularity of scalar and magnetic potentials are rather milder than ours; $a = (a_j) \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, $g^{-1} \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^{n^2})$, and $V \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$.

Under some additional assumptions, he shows the $L^2$-upper bound estimate $\|e^\rho \psi\| < \infty$ and the pointwise estimate $|\psi(x)| \leq Ce^{-\rho(r)}$ for the weight function $\rho(r) = (\Sigma(H_V) - E)^{1/2}r$, where $r$ is the geodesic distance with respect to the metric $g$ and $\Sigma(H_V)$ is the infimum of the essential spectrum of $H_V$ (Theorem 2.4 and Theorem 4.4 in [Ift]). Here, Iftimie assumes that the eigenvalue $E$ is located below $\Sigma(H_V)$, and in the proof of the pointwise upper bound he also assumes that the function $\lambda$ above is bounded from below by some positive constant.

Thus, as we mentioned in Remark 1.2 (see also subsection 1.3 below), the decay estimate for the eigenfunctions corresponding to eigenvalues in the spectral gaps of the essential spectrum of $H_V$ does not follow directly from Iftimie's result in the hyperbolic case, and the weight function in Theorem 1.1 improves Iftimie's one for such eigenfunctions.

### 1.3. The hyperbolic plane

The hyperbolic plane $\mathbb{H}$ is a typical example of manifolds we keep in mind in Theorem 1.1. In this case the Schrödinger operator $H_0$ with constant magnetic field is called the Landau Hamiltonian or the Maass Laplacian (up to gauge transform) and has been extensively studied by many authors (e.g., [Roe], [Els], [Fay], and [Com]). In this subsection we give some comments in this case. However, this subsection contains no new results.

We recall some basic spectral properties of the Landau Hamiltonian $H_0$ (without scalar potential) from Roelcke [Roe], Elstrodt [Els]. The spectrum of $H_0$ is given by

$$\text{Spec}(H_0) = \begin{cases} \bigcup_{n=0}^{N} \{E_n\} \cup [B_0^2 + 1/4, \infty) & \text{if } B_0 > 1/2, \\ [B_0^2 + 1/4, \infty) & \text{if } 0 < B_0 \leq 1/2, \end{cases}$$

where $N$ is the largest integer less than $B_0 - 1/2$ and $E_n = (2n + 1)B_0 - n(n + 1)$. We consider the case of $B_0 > 1/2$. Each $E_n$ is the eigenvalue of infinite multiplicity, and a complete set $\{\psi_{nk}\}_{k=-n}^{\infty}$ of the eigenfunctions
corresponding to $E_n$ is given by

$$\psi_{nk}(r, \theta) = \left(1 - \tanh \frac{r}{2} e^{-\sqrt{-1} \theta}\right)^{-B_0} \times \sqrt{C_{nk}} (1 - \tanh^2 \frac{r}{2})^{B_0 - n} e^{\sqrt{-1} k \theta} \tanh |k| \frac{r}{2}$$

$$\times \, _2F_1 \left( \begin{array}{c} B_0 - n - B_0 \text{sign}(k), B_0 - n + B_0 \text{sign}(k) + |k| \end{array} ; \tanh^2 \frac{r}{2} \right)$$

in the geodesic polar coordinates, where $_2F_1$ is the Gauss hyper-geometric function and $C_{nk}$ is the $L^2$-normalizing constant given by

$$C_{nk} = \frac{\beta_n \Gamma(k + \beta_n + n + 1) \Gamma(k + n + 1)}{4\pi \Gamma(n + 1) \Gamma(k + 1) \Gamma(\beta_n + n + 1)}$$

with $\beta_n = 2B_0 - 2n - 1$.

We note that the Riemannian manifold $(\mathbb{H}, R^2 g_{\mathbb{H}})$ "converges" to $(\mathbb{R}^2, 4g_{\mathbb{R}^2})$ as $R \to \infty$ (see Section IV in Comtet [Com]). The decaying factor $(1 - \tanh^2 (r/2))^{B_0 - n}$ in the expression of $\psi_{nk}$, which has the asymptotics $\exp \left(-\frac{(B_0^2 + 1/4 - E_n)^{1/2} r - r/2}{r}\right)$ as $r \to \infty$, is transformed into $(1 - \tanh^2 (r/2R))^{B_0 R^2 - n}$ on the space $(\mathbb{H}, R^2 g_{\mathbb{H}})$, which converges to the Gaussian factor $e^{-B_0 r^2/4}$ in the flat space limit $R \to \infty$.

In the rest of this subsection, we show that the discreteness assumption in (E) is crucial for Theorem 1.1 in general. More precisely, there exists an eigenfunction corresponding to the ground state energy $E_0$ which has no decay estimate as in Theorem 1.1.

Let $\{\alpha_k\}_{k=0}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{k=0}^{\infty} |\alpha_k|^2 < \infty$ and put $\psi_0 = \sum_{k=0}^{\infty} \alpha_k \psi_{0k}$. In the case of $n = 0$, we have $\psi_{0k}(r, \theta) = \sqrt{C_{0k}} e^{-\sqrt{-1} k \theta} \rho^k (1 - \rho^2)^{B_0}$ and $C_{0k} = \frac{1}{4\pi \Gamma(2B_0 - 1) \Gamma(k + 1)} \Gamma(k + 2B_0)$, where we put $t = \tanh (r/2)$, and the weight function $\rho$ in Theorem 1.1 is given by $(B_0^2 + 1/4 - E_0)^{1/2} r = (B_0 - 1/2) r$. For any $c$ satisfying $0 < c < B_0 - 1/2$, we have

$$e^{cr} \psi_0(r, \theta) = [(1 + t)/(1 - t)]^c \sum_{k=0}^{\infty} \alpha_k \sqrt{C_{0k}} e^{-\sqrt{-1} k \theta} t^k (1 - t^2)^{B_0}$$

$$= \sum_{k=0}^{\infty} \alpha_k \sqrt{C_{0k}} e^{-\sqrt{-1} k \theta} t^k (1 - t^2)^{B_0 - c} (1 + t)^{2c},$$
where we used the relation \( r = 2 \tanh^{-1}(t) = \log [(1 + t) / (1 - t)] \). Then we have

\[
\| e^{\text{cr} \psi_0} \|_{L^2(\mathbb{H})}^2 \\
= \int_0^1 \frac{4t \, dt}{(1 - t^2)^2} \int_0^{2\pi} d\theta \left| \sum_{k=0}^{\infty} \alpha_k \sqrt{C_{0k}} e^{\sqrt{-1}k\theta} t^k (1 - t^2)^{B_0 - c} (1 + t)^{2c} \right|^2 \\
= 4\pi \sum_{k=0}^{\infty} |\alpha_k|^2 C_{0k} \int_0^1 t^{2k} (1 - t^2)^{2(B_0 - c - 1)} (1 + t)^{4c} (2t) \, dt \\
\geq 4\pi \sum_{k=0}^{\infty} |\alpha_k|^2 C_{0k} \int_0^1 t^{2k} (1 - t^2)^{2(B_0 - c - 1)} (2t) \, dt \\
= 4\pi \sum_{k=0}^{\infty} |\alpha_k|^2 C_{0k} \int_0^1 s^{k+1} (1 - s)^{(2B_0 - 2c - 1) - 1} \, ds \\
= 4\pi \sum_{k=0}^{\infty} |\alpha_k|^2 C_{0k} B(k + 1, 2B_0 - 2c - 1) \\
= \frac{\Gamma(2B_0 - 2c - 1)}{\Gamma(2B_0 - 1)} \sum_{k=0}^{\infty} |\alpha_k|^2 (k + 1)^{2c} \left( (k + 1)^{-2c} \frac{\Gamma(k + 2B_0)}{\Gamma(k + 2B_0 - 2c)} \right),
\]

where we used the Plancherel formula with respect to \( L^2((0, 2\pi), d\theta) \) in the first equality and changed the variable \( s = t^2 \) in the fourth equality, and \( B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \) is the beta function. Given \( c > 0 \), we can find a sequence \( \{\alpha_k\}_{k=0}^{\infty} \) for which \( \sum_{k=0}^{\infty} |\alpha_k|^2 \) converges but \( \sum_{k=0}^{\infty} |\alpha_k|^{2k} \) diverges. Then, for such \( \{\alpha_k\} \), the rhs of (1.1) diverges since the Stirling formula for the gamma function yields that \( \lim_{k \to \infty} k^{-2c} \Gamma(k + 2B_0) / \Gamma(k + 2B_0 - 2c) = 1 \). This shows that \( e^{\text{cr} \psi} \) does not belong to \( L^2(\mathbb{H}) \) for any \( c > 0 \).

2. Preliminaries

2.1. Essential self-adjointness of \( H_0 \)

We start with the essential self-adjointness of \( H_0 = \nabla^* \nabla \) on \( C_0^\infty(M) \). Throughout this paper, we always assume (M.1)–(M.3) and we identify the pole \( p \) in (M.1) with the origin \( 0 \) in the tangent space at \( p \).

Let \( g \) and \( B \) be as in (M) and (A.1), respectively. We introduce

\[
a(r) = \int_0^r B(t) g(t) \, dt.
\]

(2.1)
Lemma 2.1. Assume (A.1). Let $(x_1, x_2) = (r \cos \theta, r \sin \theta)$ be the geodesic coordinates around the pole $p$ and let $a$ be as in (2.1). Put $\tilde{a} = a_1 dx_1 + a_2 dx_2 = -x_2 a(r)/r^2 \, dx_1 + x_1 a(r)/r^2 \, dx_2$. Then $\tilde{a}$ defines a $C^1$-section of $T^* M$ and $d\tilde{a} = (Bg/r) dx_1 \wedge dx_2 = Bg dr \wedge d\theta$ in this coordinates.

Proof. The assertion in the lemma is obvious unless $r = 0$, since $d\tilde{a}(x) = (Bg/r) dx_1 \wedge dx_2$ holds if $r \neq 0$. We need to consider the behavior of $\tilde{a}$ at the origin. Note that $g(0) = 0$ and $g'(0) = 1$ in the geodesic polar coordinate. The continuity of $\tilde{a}$ at the origin follows from the estimate

$$|\tilde{a}_j(x) - 0| \leq \frac{|x_j|}{r^2} |a(r)| \leq \frac{1}{r} \int_0^r B(t) g(t) dt \leq \sup_{0 \leq t \leq r} |B(t) g(t)| \to 0$$

as $r \to +0$. Next we show the existence of the partial derivative $\partial_{x_i} \tilde{a}_2(0) = -B(0)/2$. This follows from the definition $\lim_{h \to 0} (\tilde{a}_2(h, 0) - \tilde{a}_2(0, 0))/h = \lim_{h \to 0} h^{-2} \int_0^h B(t) g(t) dt$ combined with the Taylor expansion of $Bg$ at the origin, because of the fact that $g(0) = 0$. Similarly we have $\partial_{x_2} \tilde{a}_1(0) = -B(0)/2$, $\partial_{x_1} \tilde{a}_1(0) = \partial_{x_2} \tilde{a}_2(0) = 0$. Thus we have $d\tilde{a}(0) = B(0) dx_1 \wedge dx_2$, from which the lemma follows, because of the fact that $g'(0) = 1$. \\

Throughout this paper, we denote $-\sqrt{-1} \partial/\partial r$ by $D_r$, etc.

Lemma 2.2. Under the assumption (A.1), we can find a $C^1$-section $\tilde{a}$ of $\Lambda^1 T^* M$ satisfying the following properties (i)–(iii).

(i) If we denote by $\tilde{\nabla}$ the connection on $M$ defined by $D - \tilde{a}$, the operator $\tilde{\nabla}^* \tilde{\nabla}$ is expressed as

$$\frac{1}{g} D_r g D_r + \frac{1}{g^2} (D_\theta - a(r))^2$$

in the geodesic polar coordinate at $p$, where $a$ is as in (2.1).

(ii) The operator $\tilde{\nabla}^* \tilde{\nabla}$ is essentially self-adjoint on $C^\infty_0(M)$.

(iii) The original Bochner Laplacian $\nabla^* \nabla$ is also essentially self-adjoint on $C^\infty_0(M)$ and its operator closure is unitarily equivalent to that of $\tilde{\nabla}^* \tilde{\nabla}$ by some gauge transform.

Proof. We take $\tilde{a}$ as the one in the previous lemma. Then we find the expression (2.2) by a simple calculation. The essentially self-adjointness of the operator $\tilde{\nabla}^* \tilde{\nabla}$ follows from Theorem 1.1 (with $V = 0$) in Shubin
[Shu] since $\tilde{a} \in C^1$ by Lemma 2.1. The assertion (iii) follows from the same argument as in the Euclidean case (see, e.g., the proof of Theorem 1.3 in Leinfelder [Lei]) since both $\nabla$ and $\tilde{\nabla}$ give the same magnetic field $Bgdr \wedge d\theta$ by (A.1) and Lemma 2.1. □

In what follows we identify $\nabla$ with $\tilde{\nabla}$ and adopt the expression (2.2), for the assertions in the main theorems are independent of the choice of gauge. (The eigenfunction $\psi$ is transformed into $e^{\sqrt{-1}\lambda}\psi$.) Needless to say, the perturbed operator $H_V$ also has the unique self-adjoint extension under the condition (V.2).$_\tau$.

### 2.2. Diamagnetic inequality

In this subsection we show that the scalar potential $V$ decaying at infinity is relatively compact with respect to $H_0$ (see Lemma 2.5 below). The so-called diamagnetic inequality for $H_0$ is crucial. All the results in this subsection are well-known (see, e.g., Theorem KI in Brüning, Geyler, and Pankrashkin [B-G-P]). We give, however, a proof for the sake of completeness.

First we recall a basic property of the heat kernel on $M$. Let $\Delta_M$ be the (negative) Laplace-Beltrami operator on $M$ and let $p(t, x, y)$ is the heat kernel on $M$.

**Lemma 2.3.** Assume (M.1)–(M.4). Then, for each $t > 0$, there exist positive constants $C_1$ and $C_2$, which may depend on $t$, such that the estimate

$$0 \leq p(t, x, y) \leq C_1 e^{-C_2d(x,y)^2}$$

holds for all $(x, y) \in M \times M$, where $d(x, y)$ is the Riemannian distance on $M$. Moreover, $\int_M p(t, x, y)dV(y) = 1$ holds for any $x \in M$. Here, $B_r(x)$ stands for the geodesic ball centred at $x$ of radius $r$.

**Proof.** This is a well-known fact in Riemannian geometry. However, we give a proof for the sake of completeness. Under the assumptions (M.1)–(M.4), the manifold $M$ is complete by the Hopf-Rinow theorem, and the Ricci curvature $\text{Ric}_M$ of $M$ is bounded below. In fact, $\text{Ric}_M$ is given by $-(g''/g)g_M$. Then $M$ is a Cartan-Hadamard manifold, i.e., a complete, simply connected manifold with non-positive sectional curvature. In particular, this implies that the injectivity radius of $M$ is $\pm\infty$. The completeness
of $M$ implies that the essential self-adjointness of $-\Delta_M$ on $C_0^\infty(M)$ by the classical result of Chernoff [Che], and in this case, the heat kernel $p(t, x, y)$ of $e^{t\Delta_M}$ exists.

It follows from a result at the end of Section 5 in Davies [Dav] that for any $\delta > 0$ there exists a positive constant $c_\delta$ such that

$$0 \leq p(t, x, y) \leq c_\delta [\text{Vol}(B_{t^{1/2}}(x))\text{Vol}(B_{t^{1/2}}(y))]^{-1/2} e^{(\delta - \lambda)t} e^{-d(x, y)^2/(4 + \delta)t}$$

holds for any $x, y \in M$ and all $t > 0$, where $\lambda$ is the infimum of the spectrum of $-\Delta_M$. It is enough to show that for any $t > 0$ there exists a positive constant $C_t$ such that $\text{Vol}(B_{t^{1/2}}(x)) \geq C_t$ holds for all $x \in M$.

The Ricci curvature of the two dimensional manifold $M$ coincides with the sectional curvature of $M$, so we can use Bishop’s volume comparison theorem (see e.g., [Cha], Theorem 3 in Section III). Since $K(r) \leq 0$, it follows that for fixed $t > 0$ the quantity $\text{Vol}(B_{t^{1/2}}(x))$ is bounded from below by the volume of a ball of radius $t^{1/2}$ in $\mathbb{R}^2$, which is given by $\pi t$ and this does not depend on the location of the ball.

Finally, it is well-known that $\int_M p(t, x, y)dV(y) = 1$ holds if $M$ is complete and the Ricci curvature is bounded from below (see, e.g., [Cha], Theorem 5 in Section VIII). □

Let $L^2_0(M)$ be the space of all $L^2$-functions on $M$ with compact support. We say $f$ belongs to $L^2_0(M) + L^\infty_c(M)$ if for any $\delta > 0$ there exist $f_1 \in L^2_0(M)$ and $f_2 \in L^\infty(M)$ such that $f = f_1 + f_2$ and $\|f_2\|_{L^\infty} \leq \delta$.

**Lemma 2.4.** Let $\Delta_M$ be the (negative) Laplace-Beltrami operator on $M$. The operator $Ve^{t\Delta_M}$ is compact if $V \in L^2_0(M) + L^\infty_c(M)$ and $0 < t < 1$. In particular, $Ve^{t\Delta_M}$ is a Hilbert-Schmidt operator if $V \in L^2_0(M)$.

**Proof.** It follows from Lemma 2.3 that

$$\int_M p(t, x, y)^2dV(y) = \int_M p(t, x, y)p(t, y, x)dV(y) = p(2t, x, x) \leq Ct^{-1}(\text{Vol}(B_1(x)))^{-1},$$

where we used, in the first equality, the symmetricity of the heat kernel, which follows from the self-adjointness of the Laplacian, used the semi-group...
property in the second inequality and the estimate mentioned above in the last inequality. Thus the integral kernel of \( Ve^{t\Delta_M} \) satisfies the estimate

\[
\iint_{M \times M} |V(x)p(t, x, y)|^2 dV(x) dV(y) \leq C t^{-1} \int_M dV(x) |V(x)|^2 (\text{Vol}(B_1(x)))^{-1},
\]

which is finite if \( V \in L^2_0(M) \) and \( 0 < t < 1 \). Thus, \( Ve^{t\Delta_M} \) is a Hilbert-Schmidt operator.

In the case of general \( V = V_1 + V_2 \in L^2_0(M) + L^\infty(M) \), the lemma follows from the uniform estimate \( \|Ve^{t\Delta_M} - V_1e^{t\Delta_M}\|_{op} \leq \|V_2\|_{L^\infty} \).

**Lemma 2.5.** Let \( z \) belong to the resolvent set of \( H_0 \). Let \( V \in L^2_0(M) + L^\infty(M) \). Then the operator \( V(H_0 - z)^{-1} \) is compact on \( L^2(M) \).

**Proof.** It is enough to show that \( Ve^{-tH_0} \) is compact for any \( t > 0 \) because of the formula \( (A - z)^{-1} = \int_0^\infty e^{-tA} dt \).

We first show the so-called diamagnetic inequality for \( H_0 \), following the line of argument in the proof of Theorem 1 in Simon [Sim]. In order that, we use Kato’s inequality \( -\Delta_M |f| \leq \text{Re}(\text{sgn} f) \nabla^\ast \nabla \) for any \( f \) satisfying \( \nabla^\ast \nabla f \in L^1_{\text{loc}}(M) \), which is a special case of Theorem 5.7 in Braverman, Milatovic and Shubin [B-M-S]. For any \( \phi \in D(H_0) \) satisfying \( \phi \geq 0 \) a.e. and any \( u \in C^\infty_0(M) \), we have

\[
(\phi, (-\Delta_M + 1)|u|)_{L^2} \leq \text{Re}(\phi, (\text{sgn} u)(\nabla^\ast \nabla + 1)u)_{L^2} \leq (\phi, |(\nabla^\ast \nabla + 1)u|)_{L^2}
\]

by Kato’s inequality. If we set \( \phi = (-\Delta_M + 1)\psi \) for any \( \psi \in C^\infty_0(M) \) satisfying \( \psi \geq 0 \), then \( \phi \geq 0 \) since \( (-\Delta_M + 1)^{-1} \) is positivity preserving, which follows from the non-negativity of the heat kernel of \( -\Delta_M \). Set \( f = (\nabla^\ast \nabla + 1)u \) for any \( u \in C^\infty_0(M) \). Then (2.3) implies

\[
(\psi, |(\nabla^\ast \nabla + 1)^{-1}f|)_{L^2} \leq (\psi, (-\Delta_M + 1)^{-1}|f|)_{L^2}.
\]

Since \( \nabla^\ast \nabla \) is essentially self-adjoint on \( C^\infty_0(M) \) by Lemma 2.2, the range \( (\nabla^\ast \nabla + 1)C^\infty_0(M) \) is dense in \( L^2(M) \). Then, by a simple density argument, we obtain \( |(\nabla^\ast \nabla + 1)^{-1}f| \leq (-\Delta_M + 1)^{-1}|f| \) for any \( f \in L^2(M) \). Using
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the positivity preserving property of \((-\Delta_M + 1)^{-1}\) repeatedly, we have

\[ |(\nabla^* \nabla + 1)^{-n} f| \leq (-\Delta_M + 1)^{-1} |f| \]

for any \( f \in L^2(M) \) and any \( n \in \mathbb{N} \). Then we obtain the inequality

\[ |e^{-tH_0} f| \leq e^{t\Delta_M} |f| \]

by the formula \( e^{-tA} = \text{s-lim}_{n \to \infty} (A + n/t)^{-n} \).

Then it follows from Proposition 3.1 in Doi, Iwatsuka, and Mine [D-I-M] and Lemma 2.4 that \( V e^{-tH_0} \) is a Hilbert-Schmidt operator if \( 0 < t < 1 \) and \( V \in L^2_0(M) \) since \( |V e^{-tH_0} f| \leq |V| e^{t\Delta_M} |f| \). For general \( V \in L^2_0(M) + L^\infty(M) \), by the same argument as that at the end of the proof of Lemma 2.4, we conclude that \( V e^{-tH_0} \) is compact if \( 0 < t < 1 \). In fact, the conclusion is true for all \( t > 0 \) because of the semi-group property of \( e^{-tH_0} \). □

2.3. Fourier transform

Define the Fourier transform \( \mathcal{F} \) from \( L^2(S^1, d\theta) \) to \( l^2(\mathbb{Z}) \) by

\[ \mathcal{F} f(n) = (2\pi)^{-1/2} \int_0^{2\pi} e^{-\sqrt{-1}m\theta} f(\theta) d\theta. \]

Naturally, \( \mathcal{F} \) extends to a unitary operator \( \text{Id} \otimes \mathcal{F} \) from \( L^2(M) \) to \( L^2((0, \infty)) \otimes l^2(\mathbb{Z}) \), etc. In the sequel we write \( \mathcal{F} \) also for such extended operators for simplicity. (Note that \( \mathcal{F} \) commutes with any “radial” operator.) We find that

\[ \mathcal{F} H_V \mathcal{F}^{-1} = \sum_{n \in \mathbb{Z}} \mathcal{F} V(r, n) f(r, m). \]

Here, \( \sum \oplus \) stands for the direct sum with respect to the direct sum decomposition of Hilbert space \( L^2((0, \infty)) \otimes l^2(\mathbb{Z}) \cong \sum_{n \in \mathbb{N}} L^2((0, \infty)). \)

As in Nakamura [Nak1], we can show the following two lemmata.

**Lemma 2.6.** Assume (A.1), (A.2), and (E). Then there exists \( \sigma_0 > 0 \) such that

\[ e^{\sigma|n|} |\mathcal{F} \psi| \] belongs to \( L^2((0, \infty), gdr) \otimes l^2(\mathbb{Z}) \)
if $0 < \sigma \leq \sigma_0$.

If we assume further $(A.2)_\infty$, it is true that (2.6) for any $\sigma > 0$.

**Proof.** This follows from the standard translation-analytic argument as in the proof of Lemma 3.1 and Lemma 3.2 in Nakamura [Nak1]: For any $\varepsilon > 0$, define a unitary operator $T_\varepsilon$ on $L^2(M)$ by $T_\varepsilon f(r, \theta) = f(r, \theta - \varepsilon)$, where we regard $\theta - \varepsilon$ as an element of $S^1 \cong \mathbb{R}/(2\pi \mathbb{Z})$. We write $H_V(\varepsilon)$ for $T_\varepsilon H_V T_\varepsilon^{-1}$, then it follows that $H_V(\varepsilon) = H_V + (V(r, \theta - \varepsilon) - V(r, \theta))$. Under $(A.2)_\tau$, the family $\{H_V(\varepsilon)\}_{\varepsilon}$ extends to an analytic family of type (A) on $\{\varepsilon \in \mathbb{C} | \Im \varepsilon < \tau\}$ (see, e.g., [R-S4]), and the eigenvalue $E$ is stable for all $\varepsilon$ in a small region $S_{\sigma_0}$. Moreover, Lemma 2.5 and $(A.2)_\infty$ ensure the stability of the essential spectrum of $H_V(\varepsilon)$ for any $\varepsilon \in \mathbb{C}$. The argument is still valid for any $\varepsilon$ under $(A.2)_\infty$. Note that there exists a dense set of translation-analytic vectors, which is in fact obtained by the pull back of translation-analytic vectors in $L^2(\mathbb{R}^2)$, via the unitary operator $U_g$. □

**Lemma 2.7.** Let $0 < \sigma < \tau$. Assume $(A.2)_\tau$. Then $e^{\sigma|n|}|(FV)(r, n)|$ is uniformly bounded with respect to $(r, n)$. If we assume further $(A.2)_\infty$, then $\lim_{r \to \infty} \sup_n e^{\sigma|n|}|(FV)(r, n)| = 0$ holds.

**Proof.** This follows from the standard Payley–Wiener type argument: $FV(r, n)$ has the expression $(2\pi)^{-1/2} e^{-\sigma|n|} \int_0^{2\pi} e^{\sqrt{-1}\sigma t} \tilde{V}(r, t + \sqrt{-1}\sigma) dt$ if $\pm n \leq 0$, respectively. By Cauchy’s integral formula, we can show that

$$e^{\sigma|n|}|(FV)(r, n)| \leq (2\pi)^{-1/2} \int_0^{2\pi} \max\{|\tilde{V}(r, t + \sqrt{-1}\sigma)|, |\tilde{V}(r, t - \sqrt{-1}\sigma)|\} dt,$$

using the fact that $\tilde{V}(r, z + 2\pi) = \tilde{V}(r, z)$ holds for any $(r, z) \in (0, \infty) \times S_\tau$ by the analytic continuation theorem. Then the lemma follows immediately. □

**Lemma 2.8.** Let $r_0 > 0$. Assume that the radial curvature $K(r) = -g''(r)/g(r)$ is bounded from below on some interval $[r_0, \infty)$ and assume that

$$k > \max\{|g'(r_0)/g(r_0)|, \sup_{r \geq r_0} K_-(r)^{1/2}\}.$$

Here, we set $K_- = -\min\{0, K\}$. Then we have the estimate $|g'(r)/g(r)| \leq k$ for any $r \geq r_0$. 

$\Box$
Proof. Let \( \epsilon \) denote either +1 or −1 and let \( k \) be as above. Put \( G_\epsilon(r) = \epsilon e^{\epsilon k(r-r_0)}. \) We have

\[
(g'G_\epsilon - gG_\epsilon')' = -(K + k^2)gG_\epsilon. 
\]

(2.7)

In the case of \( \epsilon = +1 \), we have \((g'G_\epsilon - gG_\epsilon')(r) \leq (g'G_\epsilon - gG_\epsilon')(r_0)\) for any \( r \geq r_0 \), since the rhs of (2.7) is non-positive. Then we have

\[
g'(r)/g(r) \leq G_\epsilon'(r)/G_\epsilon(r) + \frac{g(r_0)}{g(r)}(g'(r_0)/g(r_0) - k)
\]

\[
\leq G_\epsilon'(r)/G_\epsilon(r) = k,
\]

where we used the fact that \( k > g'(r_0)/g(r_0) \) in the second inequality.

Similarly, in the case of \( \epsilon = -1 \), it follows from (2.7) that

\[
g'(r)/g(r) \geq G_\epsilon'(r)/G_\epsilon(r) + \frac{g(r_0)}{g(r)}(g'(r_0)/g(r_0) + k)
\]

\[
\geq G_\epsilon'(r)/G_\epsilon(r) = -k.
\]

This completes the proof. \( \square \)

3. \( L^2 \)-Exponential Estimate of the Eigenfunction

In this section, following the line of argument as in Nakamura [Nak1], we derive an exponential estimate for the eigenfunctions, assuming (A.1), (A.2)\( \tau \), (E), and

(A.3) Positive, monotone increasing, continuous functions \( a_1 \) and \( a_2 \) on \([0, \infty)\) satisfy the following conditions (i) and (ii):

(i) \( 0 < a_1(r) < a_2(r) < a(r) \) holds for all \( r > 0 \), where \( a \) is as in (2.1).

(ii) There exist \( R > 0 \) and \( C_0 > 0 \) such that

\[
\int_{a_2^{-1}(y)}^{a_2^{-1}(x)} \left[ \frac{(a(t) - a_2(t))^2}{g(t)^2} + F(t) - E \right]^{1/2} dt \leq C_0 |x - y|
\]

holds if \( x \geq y \geq R \), where \([x]_+\) stands for the non-negative part of \( x \) and

\[
F(r) = \frac{1}{2} \frac{g''(r)}{g(r)} - \frac{1}{4} \left( \frac{g'(r)}{g(r)} \right)^2.
\]

(3.1)
Given $a_1, a_2$ and $a$ as in (A.3), we take and fix a monotone increasing, continuous function $a_3$ on $[0, \infty)$ satisfying $0 < a_1 < a_2 < a_3 < a$ on $[0, \infty)$. Note that each of $a_j$s has the inverse function $a_j^{-1}$ on $[0, \infty)$. In what follows we set $a_j^{-1}(n) = 0$ for all $n \leq 0$ for notational convenience.

For any real-valued function $\tilde{a}$ on $[0, \infty)$, we introduce the set

$$\Omega(\tilde{a}) = \{(r, n) \in [0, \infty) \times \mathbb{Z} | n < \tilde{a}(r)\}.$$ 

Then the inequality $(n - a)^2 \geq (a - a_j)^2$ holds on $\Omega(a_j)$ for any $j = 1, 2, 3$, since the condition $n < a_j$ implies that $n - a < -(a - a_j)$, i.e., $|n - a| \geq |a - a_j|$.

Under the conventions above, we define a weight function

$$\rho(r, n) = \begin{cases} \int_{a_j^{-1}(n)}^{\tilde{a}^{-1}(n)} [(a(t) - a_2(t))^2 / g(t)^2 + F(t) - E_+^{1/2}] dt & \text{if } (r, n) \in \Omega(a_2), \\ 0 & \text{if } (r, n) \notin \Omega(a_2). \end{cases}$$

**Lemma 3.1.** Let $\mathcal{V}$ be as in (2.5). Then the operator $e^{\delta \rho(r, n)} \mathcal{V} e^{-\delta \rho(r, n)}$ is bounded on $L^2((0, \infty)) \otimes l^2(\mathbb{Z})$ if $\delta C_0 < \tau$, where $C_0$ is the constant as in (A.3) (ii). In addition, if we assume (A.2)$_{\infty}$, then $\lim_{R \to \infty} \| \chi_{\{r \geq R\}} e^{\delta \rho} \mathcal{V} e^{-\delta \rho} \|_{op} = 0$, where $\chi_{\{r \geq R\}}$ stands for the characteristic function of $\{(r, n)| r > R, n \in \mathbb{Z}\}$ and $\| \cdot \|_{op}$ stands for the operator norm.

**Proof.** We prove this lemma as in the proof of Lemma 4.1 in Naka-mura [Nak1]. First, we claim that

$$(3.2) \quad |\rho(r, n) - \rho(r, m)| \leq \int_{a_j^{-1}(m)}^{a_j^{-1}(n)} [(a(t) - a_2(t))^2 / g(t)^2 + F(t) - E_+^{1/2}] dt$$

holds if $n \geq m$. Indeed, $\rho(r, n) - \rho(r, m) = 0$ if $a_2(r) \leq m \leq n$, and $|\rho(r, n) - \rho(r, m)| = |0 - \int_{a_j^{-1}(m)}^{a_j^{-1}(n)}| \leq |\int_{a_j^{-1}(m)}^{a_j^{-1}(n)}| \text{ if } m \leq a_2(r) \leq n$, and $\rho(r, n) - \rho(r, m) = \int_{a_j^{-1}(m)}^{a_j^{-1}(n)}$ if $m \leq n \leq a_2(r)$.

Next, for any $\sigma$ satisfying $\delta C_0 < \sigma < \tau$, we have

$$|e^{\delta \rho} \mathcal{V} e^{-\delta \rho} f(r, n)|^2 = (2\pi)^{-1} \left( \sum_{m \in \mathbb{Z}} e^{\delta (\rho(r, n) - \rho(r, m))} \left| (\mathcal{F}V)(r, n - m) f(r, m) \right| \right)^2.$$
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\[ \leq (2\pi)^{-1} \left( \sum_{m \in \mathbb{Z}} e^{\delta C_0 |n-m|} |(\mathcal{F}V)(r, n-m) f(r, m)| \right)^2 \]

\[ \leq (2\pi)^{-1} \left( \sup_{m \in \mathbb{Z}} |e^{\sigma|m|} (\mathcal{F}V)(r, m)| \sum_{m \in \mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|} |f(r, m)| \right)^2 \]

\[ \leq (2\pi)^{-1} \sup_{m \in \mathbb{Z}} |e^{\sigma|m|} (\mathcal{F}V)(r, m)|^2 \]

\[ \times \left( \sum_{m \in \mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|} \right) \left( \sum_{m \in \mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|} |f(r, m)|^2 \right) \]

\[ = C \sup_{m \in \mathbb{Z}} |e^{\sigma|m|} (\mathcal{F}V)(r, m)|^2 \sum_{m \in \mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|} |f(r, m)|^2, \]

where we used (A.3) (ii) and (3.2) in the first inequality and the Schwarz inequality in the third inequality. Then it follows that

\[ \| \chi_{\{ r > R \}} e^{\delta \rho V} e^{-\delta \rho} f \| \leq C \sup_{r > R} \sup_{m \in \mathbb{Z}} |e^{\sigma|m|} (\mathcal{F}V)(r, m)| \| f \|, \]

from which we have the result because of Lemma 2.7. □

Define a unitary operator \( U_g \) from \( L^2(M, dV) \) to \( L^2(M, drd\theta) \) by \( U_g f = g^{1/2} f \). Then a simple calculation shows that

\[ (3.3) \quad U_g H_V U_g^{-1} = D_r^2 + \frac{1}{g^{2}} (D_\theta - a)^2 + F + V. \]

**Lemma 3.2.** Let \( \chi_{\Omega} \) be the characteristic function of \( \Omega \) and let \( a_j \)'s be as above. Assume that \( 0 < \delta < 1 \). Then there exists \( R > 0 \) such that

\[ (3.4) \quad \text{Re} \chi_{\{ r > R \}} \chi_{\Omega(a)} e^{\delta \rho(r, n)} \mathcal{F}(H_V - E) \mathcal{F}^{-1} e^{-\delta \rho(r, n)} \chi_{\Omega(a)} \chi_{\{ r > R \}} \geq \min \left\{ (1 - \delta^2) \left( \frac{(a - a_2)^2}{g^2} + F - E \right), \right. \]

\[ \left. \times (a - a_3)^2 / g^2 + F - E \right\} \chi_{\Omega(a)} \chi_{\{ r > R \}} \]

\[ - \text{Re} \chi_{\{ r > R \}} \chi_{\Omega(a)} e^{\delta \rho V} e^{-\delta \rho} \chi_{\Omega(a)} \chi_{\{ r > R \}}, \]

where \( \text{Re} \) stands for the real part.
Proof. We have
\[
\text{Re} \, e^{\delta \rho} \mathcal{F}(H_V - E)\mathcal{F}^{-1} e^{-\delta \rho} = \text{Re} \, U_g^{-1} e^{\delta \rho} \left\{ \sum_{n \in \mathbb{Z}} \oplus \left( D_n^2 + (n-a)^2/g^2 + F - E \right) + \mathcal{V} \right\} e^{-\delta \rho} U_g.
\]
Since \( \rho(r, n) \) vanishes on \( \Omega(a_3) \setminus \Omega(a_2) \), the rhs of (3.5) is bounded from below by
\[
\sum_n \oplus \left( (a-a_3)^2/g^2 + F - E \right) + \text{Re} \, e^{\delta \rho} \mathcal{V} e^{-\delta \rho},
\]
where we used the fact that \( (n-a)^2 \geq (a-a_3)^2 \) holds on \( \Omega(a_3) \). On \( \Omega(a_2) \cap \{(r,n)|r > R\} \), the rhs of (3.5) is bounded from below by
\[
\text{Re} \, U_g^{-1} \sum_n \oplus \left( (n-a)^2/g^2 + F - E - \delta^2 (\partial_r \rho)^2 \right) U_g + \text{Re} \, U_g^{-1} e^{\delta \rho} \mathcal{V} e^{-\delta \rho} U_g \geq \sum_n \oplus (1 - \delta^2) \left( (a-a_2)^2/g^2 + F - E \right) - \text{Re} \, e^{\delta \rho} \mathcal{V} e^{-\delta \rho}
\]
if we take \( R > 0 \) so large that \( ((a-a_2)/g)^2 + F - E > 0 \) holds for all \( r > R \). Then the result follows. \(\square\)

Let \( R > 0 \) and let \( f_R \) be a smooth function on \( (0, \infty) \times \mathbb{Z} \) satisfying the following conditions: \( |\partial_r f_R| + |\partial_r^2 f_R| \) is bounded, \( 0 \leq f_R(r, n) \leq 1 \) holds for all \( (r, n) \) and
\[
f_R(r, n) = \begin{cases} 
1 & \text{if } (r, n) \in \Omega(a_2) \text{ and } r \geq 2R, \\
0 & \text{if } (r, n) \notin \Omega(a_3) \text{ or } r \leq R.
\end{cases}
\]

Lemma 3.3. Let \( 0 < \delta < 1 \) and let \( f_R \) be as above. Then there exists a constant \( C = C(R, \rho, \delta, g, V, E) > 0 \) such that
\[
C_R \| e^{\delta \rho} f_R \mathcal{F} \psi \| \leq C \| \psi \|
\]
holds for any large \( R > 0 \), where we define \( C_R \) by
\[
\inf_{r > R} \min \left\{ (1 - \delta^2) \left( \frac{(a(r) - a_2(r))^2}{g(r)^2} + F(r) - E \right), \right. \\
\left. \frac{(a(r) - a_3(r))^2}{g(r)^2} + F(r) - E \right\} - 3 \| \chi_{\{r > R\}} e^{\delta \rho} \mathcal{V} e^{-\delta \rho} \|_{op}.
\]
Proof. In this proof we put
\[ Q = \text{Re} \left( e^{\delta \rho} f_R F \psi, e^{\delta \rho} F(HV - E) F^{-1} e^{-\delta \rho} f_R F \psi \right). \]

Then it follows from Lemma 3.2 that \( Q \) is bounded from below by
\[ \left( \min \left\{ (1 - \delta^2) \left( (a - a_2)^2 / g^2 + F - E \right), \frac{(a - a_3)^2}{g^2} + F - E \right\} \right) e^{\delta \rho} f_R F \psi, e^{\delta \rho} f_R F \psi \]
\[ - \| \chi_{\{ r > R \}} e^{\delta \rho} V e^{-\delta \rho} \|_{op} \| e^{\delta \rho} f_R F \psi \|^2, \]
where we used the fact that \( \text{supp} f_R \subset \Omega(a_3) \) and \( f_R \cdot \chi_{\Omega(a_3)} = f_R \) hold.

On the other hand, \( Q \) is bounded from above by
\[ \left| \left( e^{2 \delta \rho} f_R F \psi \right), [F(HV - E) F^{-1}, f_R F \psi] \right| \]
\[ \leq \left| \left( e^{2 \delta \rho} f_R F \psi, \left[ \sum_n \oplus \frac{1}{g} D_r g D_r, f_R F \psi \right] \right) \right| + \left| \left( e^{2 \delta \rho} f_R F \psi, [V, f_R F \psi] \right) \right| \]
\[ = \left| \left( e^{\delta \rho} f_R F \psi, e^{\delta \rho} \left[ \sum_n \oplus \frac{1}{g} D_r g D_r, f_R F \psi \right] \right) \right| \]
\[ + \left| \left( e^{\delta \rho} f_R F \psi, [e^{\delta \rho} V e^{-\delta \rho}, f_R] e^{\delta \rho} F \psi \right) \right|, \]
where we used the eigen-equation. Note that
\[ \rho(r, n) \leq \int_0^{2R} \sqrt{[(a - a_2)^2 / g^2 + F - E] + dt} \]
holds on the support of \( \partial_r f_R \) or on the support of \( 1 - f_R \). The first term on the rhs of (3.7) is bounded from above by
\[ C(\| \chi_{\text{supp} \partial_r f_R} F D_r \psi \| + \| \psi \|) \| e^{\delta \rho} f_R F \psi \| \]
\[ \leq C(\| H_0 \psi \| + \| \psi \|) \| e^{\delta \rho} f_R F \psi \|, \]
where we used the fact that \( \| D_r \psi \| \leq C(\| H_0 \psi \| + \| \psi \|) \) and used Lemma 2.8 and the boundedness of \( \partial_r f_R \) and \( \partial_r^2 f_R \). The second term on the rhs of (3.7) is bounded from above by
\[ 2\| \chi_{\{ r > R \}} e^{\delta \rho} V e^{-\delta \rho} \| \| e^{\delta \rho} f_R F \psi \| (\| e^{\delta \rho} F \psi \| + \| e^{\delta \rho} (1 - f_R) F \psi \|) \]
\[ \leq 2\| \chi_{\{ r > R \}} e^{\delta \rho} V e^{-\delta \rho} \| \| e^{\delta \rho} f_R F \psi \| (\| e^{\delta \rho} f_R F \psi \| + (\sup_{0 \leq r \leq 2R} e^{\delta \rho(r, 0)}) \| \psi \|). \]
Then the result follows from (3.6)–(3.9).

In fact, we need to regularize the weight function \( \rho \) appropriately, since \( \rho \) is not bounded. However, the argument above remains valid for such regularized weights. (For instance, we can use \( \rho_N(r,n) = \min\{\rho(r,n), N\} \) as in [Nak1]. See also Section 4 in [Sor].) □

Let \( 0 < \sigma < \tau \) and \( 0 < \delta < 1 \) and put

\[
(3.10) \quad \rho_1(r) = \min\{\sigma a_1(r), \delta \int_{a_2^{-1}(a_1(r))}^{r} [(a(t) - a_2(t))^2/g(t)^2 + F(t) - E]^1/2 \, dt\}.
\]

The main result in this section is the following:

**Proposition 3.4.** Assume (M.1)–(M.3), (A.1), (A.2), (A.3), and (E). Let \( \delta, \sigma, \) and \( \rho_1 \) be as above. Assume that the constant \( C_R \) as in Lemma 3.3 is positive for some \( R > 0 \). Then \( e^{\rho_1} \psi \) belongs to \( L^2(M) \).

**Proof.** Since \( a_2^{-1}(n) \leq a_2^{-1}(a_1(r)) \) holds on \( \Omega(a_1) = \{(r,n)|n < a_1(r)\} \), we have

\[
\rho(r,n) = \int_{a_2^{-1}(n)}^{r} [(a - a_2)^2/g^2 + F - E]^1/2 \, dt 
\geq \int_{a_2^{-1}(a_1(r))}^{r} [(a - a_2)^2/g^2 + F - E]^1/2 
=: \tilde{\rho}(r).
\]

Let \( f_R \) be as before and take \( R > 0 \) sufficiently large. Then we obtain

\[
(3.11) \quad \left\| \chi_{\Omega(a_1)} e^{\delta \tilde{\rho}} \mathcal{F} \psi \right\| 
\leq \left\| \chi_{\Omega(a_1)} e^{\delta \rho(r,n)} \mathcal{F} \psi \right\| 
\leq \left\| f_R \chi_{\Omega(a_1)} e^{\delta \rho(r,n)} \mathcal{F} \psi \right\| 
+ \left\| (1 - f_R) \chi_{\Omega(a_1)} e^{\delta \rho(r,n)} \mathcal{F} \psi \right\| 
\leq \left\| f_R e^{\delta \rho(r,n)} \mathcal{F} \psi \right\| + \sup_{0 \leq r \leq 2R} e^{\delta \rho(r,0)} \| \psi \|,
\]

which is finite because of Lemma 3.3 and the positivity assumption on \( C_R \).

On the other hand, on \( \Omega(a_1)^c \), we have

\[
(3.12) \quad \left\| \chi_{\Omega(a_1)^c} e^{\sigma_1(r)} \mathcal{F} \psi \right\| 
\leq \left\| \chi_{\Omega(a_1)^c} e^{\sigma(n)} \mathcal{F} \psi \right\|,
\]
which is finite because of Lemma 2.6. Then the results follows from (3.11) and (3.12). □

The following result is a straightforward generalization of the $L^2$-estimate of eigenfunctions obtained by Nakamura [Nak1] in the Euclidean case.

**Corollary 3.5.** Assume (A.1), (A.2)$_\infty$, (A.3) and (E). Moreover, assume that

\[
\lim_{r \to \infty} \frac{(a(r) - a_2(r))}{g(r)} = \infty.
\]

Then $e^{\rho_1 \psi}$ belongs to $L^2(M)$.

Furthermore, if we assume (A.2)$_\infty$, then we have the same conclusion replaced $\rho_1$ by

\[
\rho_2(r) = \delta \int_{a_2^{-1}(a_1(r))}^{r} \frac{[(a(t) - a_2(t))^2 + F(t) - E]^{1/2}}{g(t)} dt.
\]

**Proof.** The assumption (3.13) implies that the constant $C_R$ is positive for any large $R > 0$. Under (A.2)$_\infty$, the assertion follows since we can take $\tau$ (and so $\sigma$) arbitrary large by Lemma 2.6. □

**Remark 3.6.** In the case of spherically symmetric scalar potential $V$, the conclusions in Theorem 3.5 are still valid if we replace the function $F$ by $F + V$ and the condition (3.13) by

\[
E < \liminf_{r \to \infty} \frac{((a(r) - a_2(r))^2 + F(r) + V(r))}{g(r)}.
\]

Indeed, in the proof of Lemma 3.3 above, we can replace $F$ by $F + V$, and the last term in (3.6) and the commutator in (3.7) vanish.

4. **Proof of Theorem 1.1**

4.1. **Upper bound**

In this section we assume (M.1)–(M.4), (A.1), (A.2)$_\infty$, (E), and assume further that the magnetic field $B = B(r)$ approaches to a positive constant $B_0$ satisfying $E < B_0^2 + 1/4$ at infinity. It is known that the assumptions
on the radial curvature $K = -g''/g$ in Theorem 1.1 ensures the existence of the two limits

$$
(4.1) \quad c_1 = \lim_{r \to \infty} e^{-r}g(r) \quad \text{and} \quad \lim_{r \to \infty} F(r) = 1/4
$$

(see Lemma 3.5 in [Don2]).

By the assumption on $B$, for any $\varepsilon > 0$, there exists $R_0 > 0$ such that $|B(r) - B_0| < \varepsilon$ for all $r \geq R_0$, so we have

$$
|a(r) - B_0 \int_0^r g(t)dt| \leq (\int_0^{R_0} + \int_{R_0}^r)|B(t) - B_0|g(t)dt
\leq C_\varepsilon + \varepsilon \int_0^r g(t)dt
$$

for some constant $C_\varepsilon$ and for all $r \geq R_0$. By (4.1), for any $\varepsilon > 0$, there exists $R_1 > 0$ such that $(1 - \varepsilon)c_1 e^r \leq g(r) \leq (1 + \varepsilon)c_1 e^r$ holds for all $r \geq R_1$. Hence, for any $\varepsilon > 0$, we find that

$$
(1 - \varepsilon)B_0c_1 e^r + C_-(\varepsilon) \leq a(r) \leq (1 + \varepsilon)B_0c_1 e^r + C_+(\varepsilon)
$$

holds for all $r \geq R_1$, where $C_\pm(\varepsilon)$ are independent of $r$.

Let $R_2 > 0$ and $0 < \varepsilon_1 < \varepsilon_2 < 1$, which are appropriately chosen below. We choose $a_1, a_2$ in (A.3) so that

$$
(4.2) \quad a_j(r) = \begin{cases} 
B_0c_1 e^{\varepsilon_j r} & \text{if } r \geq 2R_2, \\
(j/3)a(r) & \text{if } 0 \leq r \leq R_2,
\end{cases}
$$

and $a_j$'s are continuous and monotone increasing. Then it follows that

$$
(4.3) \quad \lim_{r \to \infty} \left(\frac{(a(r) - a_j(r))^2}{g(t)^2} + F(r)\right) = B_0^2 + 1/4
$$

for $j = 1, 2$. Note that $a_2^{-1}(x) = \varepsilon_2^{-1}(\log x - \log B_0c_1)$ for large $x$.

**Lemma 4.1.** For any $\varepsilon > 0$, there exists $N > 0$ such that

$$
(4.4) \quad \int_{a_2^{-1}(m)}^{a_2^{-1}(n)} \left[\frac{(a(t) - a_2(t))^2}{g(t)^2} + F(t) - E\right]^{1/2} dt
\leq \frac{1 + \varepsilon}{\varepsilon_2} (B_0^2 + 1/4 - E)^{1/2}|n - m|
$$
holds if $n \geq m \geq N$.

**Proof.** It follows from (4.3) that, for any $\varepsilon > 0$, there exists $N > 0$ such that, if $n \geq m \geq N$, the lhs of (4.4) is bounded from above by

$$
(1 + \varepsilon)(B_0^2 + 1/4 - E)^{1/2}(a_2^{-1}(n) - a_2^{-1}(m))
$$

$$
= (1 + \varepsilon)(B_0^2 + 1/4 - E)^{1/2}(\varepsilon_2^{-1} \log n - \varepsilon_2^{-1} \log m)
$$

$$
\leq \left(1 + \varepsilon\right)(B_0^2 + 1/4 - E)^{1/2} \log (n/m),
$$

since $\log (X/x) = \log (1 + (X - x)/x) \leq (X - x)/x \leq X - x$ holds if $X \geq x > 1$. □

All the conditions in (A.3) are now satisfied for our choice of $a_j$s, therefore, we can apply Proposition 3.4. In this case, there exists $R_3 > 0$ such that

$$(a(t) - a_2(t))^2/g(t)^2 + F(t) - E > 0$$

and therefore

$$
\rho_2(r) = \delta \int_{(\varepsilon_1/\varepsilon_2)r}^r \left((a(t) - a_2(t))^2/g(t)^2 + F(t) - E\right)^{1/2} dt
$$

for all $r \geq R_3$, because of (4.3), (4.2) and the assumption $E < B_0^2 + 1/4$.

Moreover, there exists $R_4 > 0$ such that

$$
((a(t) - a_2(t))^2/g(t)^2 + F(t) - E)^{1/2} \geq (1 - \varepsilon_1)(B_0^2 + 1/4 - E)^{1/2}
$$

holds for all $t \geq R_4$. Finally, it follows that

$$
\rho_2(r) \geq \delta(1 - \varepsilon_1)(1 - (\varepsilon_1/\varepsilon_2))(B_0^2 + 1/4 - E)^{1/2} r
$$

holds for large $r \geq \max\{R_1, R_2, R_3, R_4\}$, from which we obtain the upper estimate of $\psi$ in Theorem 1.1 since we can choose positive numbers $\varepsilon_1$ and $1 - \delta$ arbitrarily small.

### 4.2. Upper bound of the $L^2$-spherical average

In this subsection we obtain the upper bound of the $L^2$-spherical average of the eigenfunction $\psi$ as in Theorem 1.1 and we obtain also the pointwise upper bound of $\psi$. 
Lemma 4.2. Under the same assumption as in Theorem 1.1, the function $\Delta_M(e^{(1-\varepsilon)\rho}\psi)$ belongs to $L^2(M)$ for any $\varepsilon > 0$.

Proof. In this proof we denote the weight function $(1-\varepsilon)\rho$ by $\rho$ for simplicity. By Kato’s inequality, we have

$$
-\Delta_M(e^{\rho}\psi) \leq \text{Re}[(\text{sgn}(\psi))\nabla^*\nabla(e^{\rho}\psi)]
$$

$$
= \text{Re}[(\text{sgn}(\psi))(e^{\rho}\nabla^*\nabla\psi + [\nabla^*\nabla, e^{\rho}]\psi)]
$$

$$
= \text{Re}[(\text{sgn}(\psi))(e^{\rho}(E - V)\psi + [\nabla^*\nabla, e^{\rho}]\psi)],
$$

where we used the eigen-equation in the last equality. We can find that

$$
[\nabla^*\nabla, e^{\rho}] = [g^{-1}D_r g D_r, e^{\rho}]
$$

$$
= 2(D_r \rho)e^{\rho} D_r + [(D_r^2 \rho) + (D_r \rho)^2]e^{\rho}
$$

$$
+ g^{-1}(D_r g)(D_r \rho)e^{\rho}.
$$

Note that $g'/g$ is bounded on $M$ by Lemma 2.8, and also $D_r \rho$, $D_r^2 \rho$ and $V$ are bounded. Then the rhs of (4.5) is bounded from above by $Ce^{\rho}(|D_r \psi| + |\psi|)$ for some constant $C > 0$.

Then it is enough to show that $e^{\rho} D_r \psi$ belongs to $L^2(M)$ since we have already shown that $e^{\rho} \psi$ belongs to $L^2(M)$. To see this, we first consider the case where $\rho$ is smooth and bounded. Then we have $\|e^{\rho} D_r \psi\| \leq \|D_r(e^{\rho} \psi)\| + C\|e^{\rho} \psi\|$ and

$$
\|D_r(e^{\rho} \psi)\|^2 \leq \|D_r(e^{\rho} \psi)\|^2 + \|g^{-1}(D_\theta - a)(e^{\rho} \psi)\|^2
$$

$$
= (e^{\rho} \psi, \nabla^*\nabla(e^{\rho} \psi))
$$

$$
= (e^{\rho} \psi, e^{\rho}\nabla^*\nabla\psi) + (e^{\rho} \psi, [\nabla^*\nabla, e^{\rho}]\psi)
$$

$$
\leq C\|e^{\rho} D_r \psi\|\|e^{\rho} \psi\| + C\|e^{\rho} \psi\|^2
$$

$$
\leq \frac{1}{2}\|e^{\rho} D_r \psi\|^2 + C\|e^{\rho} \psi\|^2,
$$

where we used (4.6) in the second inequality and used the elementary inequality $2XY \leq X^2 + Y^2$ in the last inequality. Thus we have shown that $\|e^{\rho} D_r \psi\| \leq C\|e^{\rho} \psi\|$. Finally, we have the same conclusion by approximating the original weight function $\rho$ by smooth, bounded ones. $\Box$

Lemma 4.3. Assume that $(M, g)$ satisfies (M.1)–(M.4). Assume that both $f$ and $\Delta_M f$ belong to $L^2(M)$. Then there exists a constant $C > 0$ such that $\int_0^{2\pi} |f(r, \theta)|^2 d\theta \leq C e^{-r/2}$ for all large $r \geq 0$. 
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Proof. By (4.1), there exists $R > 0$ such that $g(r) \geq e^r/2$ holds for any $r \geq R$. Let $n$ be a positive integer. Fix $\varepsilon > 0$. Let $\phi \in C_0^\infty(\mathbb{R})$ satisfy the following conditions: $1 \leq \phi \leq 1$, $\phi(t) = 1$ if $n \leq t \leq n + 1$, $\phi(t) = 0$ outside $[n - \varepsilon, n + 1 + \varepsilon]$. For any $r \in [n, n + 1]$, we have

$$|f(r, \theta)| \leq \left( \int_{n-\varepsilon}^r \left| \partial_t [f(t, \theta)\phi(t)] \right| dt \right)^{1/2} \leq \left( \int_{n-\varepsilon}^r g(t) dt \right)^{1/2} \left( \int_{n-\varepsilon}^r \left| \partial_t [f(t, \theta)\phi(t)] \right|^2 g(t) dt \right)^{1/2} \leq \left( \int_{n-\varepsilon}^r e^{-t} dt \right)^{1/2} \left( \int_{n-\varepsilon}^r \left| \partial_t [f(t, \theta)\phi(t)] \right|^2 g(t) dt \right)^{1/2} \leq (2(\varepsilon + 1)e^{\varepsilon + 1})^{1/2} e^{-r/2} \left( \int_0^\infty \left| \partial_t [f(t, \theta)\phi(t)] \right|^2 g(t) dt \right)^{1/2},$$

where we used Schwarz’ inequality in the second inequality and used the fact that $n \leq r \leq n + 1$ in the last inequality. Then we have

$$\left( \int_0^{2\pi} |f(r, \theta)| d\theta \right)^{1/2} \leq C_\varepsilon e^{-r/2} \|D_r (f\phi)\|_{L^2(M)} \leq C_\varepsilon' e^{-r/2}(\|\Delta_M f\|_{L^2(M)} + \|f\|_{L^2(M)}),$$

where we used the facts that $\|D_r f\|^2 \leq \|\Delta_M f\|\|f\|$ and that each of the derivatives of $\phi$ are bounded in the last inequality. The estimate (4.7) holds for any $r \geq R + \varepsilon$ since the constant $C_\varepsilon'$ is independent of $n$. Then the result obeys. □

Then the upper bound of the $L^2$-spherical average of the eigenfunction $\psi$ follows from Lemma 4.2 and Lemma 4.3 with $f = e^{(1-\varepsilon)\rho}\psi$. Note that a local elliptic argument shows that the eigenfunction $\psi$ is continuous, hence bounded on each compact subset of $M$.

Finally, we show the pointwise estimate of $|\psi(r, \theta)|$. We recall Sobolev’s embedding theorem from Hebey [Heb], Theorem 3.4: Let $(M, g)$ be a smooth, complete $n$-dimensional Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. Assume that $q \geq 1$, $0 \leq m < k$, integers, and $1/q < (k - m)/n$. Then $H^q_k(M) \subset C^m_B(M)$. Here,
the spaces $H^q_k(M)$, $C^m_B(M)$ are defined by the norms
\[ \|u\|_{k,q} = \left( \sum_{j=0}^k \int_M |\nabla^j u|^q dV \right)^{1/q}, \]
\[ \|u\|_{C^m_B(M)} = \sum_{j=0}^m \sup_{x \in M} |(\nabla^j u)(x)|, \]
respectively.

We use this with $u = e^{(1-\varepsilon)\rho} \psi$, $q = 2$, $k = 2$, $n = 2$, and $m = 0$. Then the conclusion is that $|e^{(1-\varepsilon)\rho} \psi|$ is bounded on $M$. (This estimate can be also obtained as in the proof of Theorem 5.3 in Donnelly [Don5] using Lemma 2.3.) This completes the proof of the assertion 1 in Theorem 1.1.

4.3. Lower bound of the $L^2$-spherical average

In this subsection we obtain a lower bound of the $L^2$-spherical average of the eigenfunction $\psi$ as in Theorem 1.1, following the line of argument as in the proof of Theorem 3.6 in Donnelly [Don2]. We may assume that the eigenfunction $\psi$ is real-valued; otherwise we consider the real and complex parts of $\psi$. We note that the discreteness assumption on the eigenvalue $E$ is not needed in the proof below.

**Lemma 4.4.** Let $r_0 \in \mathbb{R}$. Let $u \in C^2([r_0, \infty), \mathbb{R}) \cap L^2((r_0, \infty), dr)$ and $u(r_0) > 0$. Let $f, q \in C([r_0, \infty))$, $f \geq 0$ and $q > 0$. Assume that $u$ satisfies the ODE: $u'' = qu - f$ on $[r_0, \infty)$. Then $u$ is positive on $[r_0, \infty)$.

**Proof.** This lemma is elementary and well-known in the theory of ODE. However, we give a proof for the sake of completeness. We show this by contradiction. Assume that there exists the first zero $r_1 (> r_0)$ of $u$. Then we can deduce that $u'(r_1) \leq 0$ and $u$ is concave at $r_1$, which implies that $u$ is concave and strictly negative on $(r_1, \infty)$ by the ODE. This contradicts the fact that $u \in L^2$. $\square$

In what follows we assume that $V$ is spherically symmetric in addition to (A.2)$_\infty$. The Plancherel formula with respect to $L^2(S^1)$ yields that
\[
\int_0^{2\pi} |\psi(r, \theta)|^2 d\theta = \sum_{n \in \mathbb{Z}} |(\mathcal{F}\psi)(r, n)|^2 \geq |(\mathcal{F}\psi)(r, j)|^2.
\]
for any \( j \in \mathbb{Z} \). Put \( \psi_j(r) = g^{1/2}(r)(\mathcal{F}\psi)(r, j) \). Then, by the assumption on \( V \), each \( \psi_j \) satisfies the equation

\[
(D_r^2 + (j - a(r))^2/g(r)^2 + F(r) + V(r) - E)\psi_j(r) = 0
\]
on \([0, \infty)\) by (3.3). We rewrite this equation as \( (D_r^2 + \mu - E + q_1)\psi_j = 0 \), where we put \( \mu = B_0^2 + 1/4, q_1 = (j - a)^2/g^2 + F + V - \mu \). We can find and fix an integer \( j \) for which \( (\mathcal{F}\psi)(\cdot, j) \) is not identically zero, since \( \psi \neq 0 \). Then \( \psi_j \) is real-valued, smooth on \([0, \infty)\) and belongs to \( L^2((0, \infty), dr) \).

We claim that there exists \( r_0 > 0 \) such that \( \psi_j(r) > 0 \) holds for all \( r \geq r_0 \). To see this we take \( \tilde{r}_0 > 0 \) such that \( \mu - E + q_1 > 0 \) holds for all \( r \geq \tilde{r}_0 \), since \( \mu - E > 0 \), \( q_1 \to 0 \) as \( r \to \infty \). Moreover, we can find \( r_0 \) (\( \geq \tilde{r}_0 \)) so that \( \psi_j(r_0) > 0 \) because of the choice of \( j \); otherwise, we take \( -\psi_j \) instead of \( \psi_j \). Then the claim follows from Lemma 4.4 with \( u = \psi_j, f = 0, q = \mu - E + q_1 \).

Let \( r_0 \) be as above and let \( \varepsilon > 0 \) be fixed small enough. We introduce the auxiliary function \( v(r) = (1/2)\psi_j(r_0)\exp\left(-\left(\mu - E + \varepsilon\right)^{1/2}(r-r_0)\right) \), which solves the equation \( (D_r^2 + \mu - E + \varepsilon)v = 0 \) on \([r_0, \infty)\).

We put \( \gamma = \psi_j - v \), then \( \gamma \) satisfies the equation \( \gamma'' = (\mu - E + q_1)\gamma - (\varepsilon - q_1)v \) with the boundary condition \( \gamma(r_0) = \psi_j(r_0)/2 > 0 \). Again, we apply Lemma 4.4 with \( u = \gamma, q = \mu - E + q_1, \) and \( f = (\varepsilon - q_1)v \). All the assumptions in Lemma 4.4 are satisfied for large \( r_0 > 0 \), and so Lemma 4.4 implies that \( \gamma(r) > 0 \) for all large \( r \). This means that \( \psi_j(r) > (1/2)\psi_j(r_0)\exp\left(-\left(\mu - E + \varepsilon\right)^{1/2}(r-r_0)\right) \) if \( r \) is large enough. Then, by (4.8), there exists \( c = c(r_0) > 0 \) such that

\[
\int_0^{2\pi} |\psi(r, \theta)|^2d\theta \geq c \exp\left(-\left(\mu - E + \varepsilon\right)^{1/2}r\right) \exp(-r/2)
\]

holds for any large \( r > 0 \), from which the lower bound estimate in Theorem 1.1 follows. Now we complete the proof of Theorem 1.1.

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(Received January 31, 2005)

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