Hyperfunction Solutions to Fuchsian Hyperbolic Systems

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Abstract. To Fuchsian hyperbolic systems, the unique solvability theorem is proved in two cases: (1) hyperfunction (or microfunction) solutions with a real analytic parameter to Cauchy problem; (2) mild hyperfunction (or mild microfunction) solutions to boundary value problem. These results extend that of H. Tahara to general systems.

Introduction

In this paper, we shall show the unique solvability theorem for hyperfunction solutions with a real analytic parameter of Cauchy problem to Fuchsian hyperbolic systems.

Fuchsian partial differential operator was first defined by Baouendi-Goulaouic [B-G]. This includes non-characteristic type as a special case, and Cauchy-Kovalevskaja type theorem (namely, unique solvability for Cauchy problem) was proved in [B-G] under the conditions of characteristic exponents. After that, Tahara [T] treated a Fuchsian Volevič system and proved Cauchy-Kovalevskaja type theorem in the complex domain under the conditions of characteristic exponents. Further as an application, under the hyperbolicity condition he obtained a Cauchy-Kovalevskaja type theorem for this system in the framework of hyperfunctions. On the other hand, Laurent-Monteiro Fernandes [L-MF 1] extended the notion of Fuchsian type to a general system of differential equations; that is, coherent left $\mathcal{D}_X$-Module, here and in what follows, we shall write a Module with a capital letter, instead of a sheaf of left modules. Their notion includes Fuchsian Volevič system, and they proved a Cauchy-Kovalevskaja type theorem in the complex domain in general settings; that is, without conditions of characteristic exponents. As for the uniqueness of hyperfunction solutions for Cauchy problem, Oaku [O 1] and Oaku-Yamazaki [O-Y] extended

2000 Mathematics Subject Classification. Primary 35A27; Secondary 32C38, 58J15.

191
the uniqueness result to Fuchsian systems. Hence in this paper, we shall prove the solvability theorem for general Fuchsian hyperbolic systems in the framework of hyperfunctions (that is, hyperfunctions with a real analytic parameter, or mild hyperfunctions) without the conditions of characteristic exponents. To this end, in addition to the Cauchy-Kovalevskaja type theorem due to Laurent-Monteiro Fernandes [L-MF 1], we use the theory of microsupports due to Kashiwara-Schapira (see [K-S]). This theory enables us to prove our desired result, in fact, our key theorem (Theorem 2.2) is only an exercise of this theory, and from this, we easily deduce the Cauchy-Kovalevskaja type theorem for general Fuchsian hyperbolic systems in the framework of hyperfunctions.

1. Preliminaries

In this section, we shall fix the notation and recall known results used in later sections. General references are made to Kashiwara-Schapira [K-S].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geq 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{>0} := \{r \in \mathbb{R}; r > 0\}$.

In this paper, all the manifolds are assumed to be paracompact. If $\tau: E \to Z$ is a vector bundle over a manifold $Z$, then we set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of $\tau$ to $\dot{E}$. Let $M$ be an $(n+1)$-dimensional real analytic manifold and $N$ a one-codimensional closed real analytic submanifold of $M$. We denote by $f: N \to M$ the canonical embedding. Let $X$ and $Y$ be complexifications of $M$ and $N$ respectively such that $Y$ is a closed submanifold of $X$ and that $Y \cap M = N$. We also denote by $f: Y \to X$ the canonical embedding with same notation $f$. By local coordinates $(z, \tau) = (x + \sqrt{-1} y, t + \sqrt{-1} s)$ of $X$ around each point of $N$, we have locally the following relation:

\begin{equation}
\begin{array}{cccc}
N & \subset & \mathbb{R}^n_x \times \{0\} & \xrightarrow{f} \ M \supset & \mathbb{R}^n_x \times \mathbb{R}_t \\
\mathbb{C}^n_z \times \{0\} & \xrightarrow{f} \ X & \subset & \mathbb{C}^n_z \times \mathbb{C}_\tau \\
Y & \subset & \mathbb{R}^n_x \times \{0\} & \xrightarrow{f} \ M \supset & \mathbb{R}^n_x \times \mathbb{R}_t
\end{array}
\end{equation}

The embedding $f$ induces a natural embedding $f': T_N Y \hookrightarrow T_M X$ and by this mapping we regard $T_N Y$ as a closed submanifold of $T_M X$. Further,
f induces mappings:

\[
\begin{array}{ccc}
N \times \pi & \to & T^*_N M \\
i_N & \downarrow & \downarrow f \\
 & \downarrow i & \downarrow i_M \\
T^*_N Y \leftarrow & N & \times T^*_M X \\
\downarrow \pi_N & \downarrow \pi & \downarrow \pi_M \\
N & \leftarrow & N \\
M & \leftarrow & M
\end{array}
\]  

(1.2)

Here \(\pi_N, \pi_M\) and \(\pi\) are canonical projections, \(i_N, i_M\) and \(i\) are zero-section embeddings, and \(\square\) means that the square is Cartesian.

We write \(M \setminus N = \Omega_+ \cup \Omega_\), where each \(\Omega_\) is an open subset and \(\partial \Omega_\) = \(N\). We set \(M_+ := \Omega_+ \cup N\). By local coordinates, we can write

\[
\Omega_+ = \{(x, t) \in M; t > 0\} \subset M_+ = \{(x, t) \in M; t \geq 0\}.
\]

We remark that a natural morphism \(C_{M+} \to C_N\) gives natural morphisms

\[
R\text{Hom}_{C_M}(C_N, C_M) = \omega_{N/M} \to R\text{Hom}_{C_M}(C_{M+}, C_M) = C_{\Omega_+} \to C_M.
\]

Here \(\omega_{N/M}\) denotes the relative dualizing complex. As usual, we denote by \(\nu_\bullet(*)\) and \(\mu_\bullet(*)\) the specialization and microlocalization functors respectively. Further, \(\mu_\text{hom}(*, *)\) denotes \(\mu_\text{hom}\) bifunctor. We denote by \(D^b(X)\) the derived category of sheaves of \(\mathbb{C}\)-linear spaces with bounded cohomologies. Let \(F\) be an object of \(D^b(X)\). Then, by Kashiwara-Schapira [K-S, Chapter IV], we have

\[
Rf_d! (f^{-1}_! \mu_M(F) \otimes \omega_{N/M}) \simeq Rf_d! (f^{-1}_! \mu_\text{hom}(C_M, F) \otimes \omega_{N/M})
\]

\[
\to Rf_d! (f^{-1}_! \mu_\text{hom}(C_{\Omega_+}, F) \otimes \omega_{N/M})
\]

\[
\to Rf_d! (f^{-1}_! \mu_\text{hom}(\omega_{N/M}, F) \otimes \omega_{N/M})
\]

\[
= Rf_d! f^{-1}_! \mu_N(F) \to \mu_N(f^{-1}F \otimes \omega_{Y/X}).
\]

Therefore we obtain morphisms:

\[
Rf_d! (f^{-1}_! \mu_M(F) \otimes \omega_{N/M}) \to Rf_d! (f^{-1}_! \mu_\text{hom}(C_{\Omega_+}, F) \otimes \omega_{N/M})
\]

(1.3)

\[
\to \mu_N(f^{-1}F \otimes \omega_{Y/X}).
\]
2. Near-Hyperbolicity Condition

Let $F$ be an object of $\mathbf{D}^b(X)$. We denote by $\text{SS}(F)$ the microsupport of $F$ due to Kashiwara-Schapira (see [K-S]). $\text{SS}(F)$ is a closed conic involutive subset of $T^*X$ and described as follows: Let $(w)$ be local coordinates of $X$ and $(w_0; \zeta_0)$ a point of $T^*X$. Then $(w_0; \zeta_0) \notin \text{SS}(F)$ if and only if the following condition holds: There exist an open neighborhood $U$ of $w_0$ in $X$ and a proper convex closed cone $\gamma \subset X$ satisfying $\zeta_0 \in \text{Int} \gamma^{\circ \alpha} \cup \{0\}$ such that

$$R\Gamma(H_\varepsilon \cap (x + \gamma); F) \sim R\Gamma(L_\varepsilon \cap (x + \gamma); F)$$

holds for any $w \in U$ and any sufficiently small $\varepsilon > 0$. Here $\text{Int} A$ denotes the interior of $A$, $\gamma^{\circ \alpha} := \bigcap_{\zeta \in \gamma} \{w \in X; \text{Re}(w, \zeta) \leq 0\}$ and

$$L_\varepsilon := \{w \in X; \text{Re}(w - w_0, \zeta_0) = -\varepsilon\} \subset H_\varepsilon := \{w \in X; \text{Re}(w - w_0, \zeta_0) \geq -\varepsilon\}.$$

Next, we shall recall the definition of the near-hyperbolicity condition due to Laurent-Monteiro Fernandes [L-MF 2, Definition 1.3.1]:

**Definition 2.1.** Let $F$ be an object of $\mathbf{D}^b(X)$. We say $F$ is near-hyperbolic at $x_0 \in N$ in $\pm dt$-codirection if there exist positive constants $C$ and $\varepsilon_1$ such that

$$\text{SS}(F) \cap \{(z, \tau; z^*, \tau^*) \in T^*X; |z - x_0| < \varepsilon_1, |\tau| < \varepsilon_1, t \neq 0\} \subset \{(z, \tau; z^*, \tau^*) \in T^*X; |t^*| \leq C(|y^*|(y^* + |s|) + |x^*|)\}$$

holds by local coordinates of $X$ in (1.1) and the following associated coordinates of $T^*X$:

$$(z, \tau; z^*, \tau^*) = (x + \sqrt{-1} y, t + \sqrt{-1} s; x^* + \sqrt{-1} y^*, t^* + \sqrt{-1} s^*).$$

Our first main result is as follows:

**Theorem 2.2.** Let $F$ be an object of $\mathbf{D}^b(X)$. Assume that $F$ is near-hyperbolic at $x_0 \in N$ in $\pm dt$-codirection. Then, the morphisms in (1.3)
induce isomorphisms for any \( p^* \in T_N^* Y \cap \pi_N^{-1}(x_0) \):

\[
Rf_d((f^{-1}_* \mu_M(F) \otimes \omega_{N/M})_{p^*}) \simeq Rf_d((f^{-1}_* \mu_{hom}(\mathbb{C}_{\Omega^+}, F) \otimes \omega_{N/M})_{p^*}) \simeq \mu_N(f^{-1} F \otimes \omega_{Y/X})_{p^*}.
\]

**Proof.** (a) First we shall prove that

\[
Rf_d((f^{-1}_* \mu_M(F) \otimes \omega_{N/M})_{p^*} \to \mu_N(f^{-1} F \otimes \omega_{Y/X})_{p^*}
\]

is an isomorphism. Since the Fourier-Sato transformation gives an equivalence, we may prove that

\[
\nu_M(F)|_{T_N Y \cap \tau_N^{-1}(x_0)} \to \nu_N(f^{-1} F)|_{T_N Y \cap \tau_N^{-1}(x_0)}
\]

is an isomorphism. If \( q = x_0 \in T_N N \), then

\[
f^{-1}_* \nu_M(F)_{x_0} = \nu_M(F)_{f(x_0)} = F_f(x_0)
\]

\[
\to \nu_N(f^{-1} F)_{x_0} = (f^{-1} F)_{x_0} = F_f(x_0).
\]

Next consider in \( v = (x_0, \eta_0) \in \dot{T}_N Y \cap \tau_N^{-1}(x_0) \). We may assume \( x_0 = 0 \) under local coordinates of \( X \) in (1.1). We set as in Bony-Schapira [B-S2]

\[
B(0, a) := \{(x, t) \in \mathbb{R}^{n+1}; |x| + |t| < a\},
\]

\[
B'(0, a) := \{x \in \mathbb{R}^n; |x| < a\},
\]

\[
K(a, \delta) := \text{Int} \gamma[B'(0, a) \cup \{(0, \pm a\delta)\}].
\]

Here \( \gamma[\cdot] \) denotes the **convex hull**. For an open convex cone \( \Gamma \subset \mathbb{R}^{n+1} \) (resp. \( \Gamma' \subset \mathbb{R}^n \)), we set \( \Gamma_{\varepsilon} := \Gamma \cap B(0, \varepsilon) \) (resp. \( \Gamma'_\varepsilon := \Gamma' \cap B'(0, \varepsilon) \)). Then, for any \( k \in \mathbb{Z} \) we have

\[
H^k \nu_M(F)_v = \lim_{V(a, \delta; \Gamma_{\varepsilon})} H^k(V(a, \delta; \Gamma_{\varepsilon}); F)
\]

\[
\to H^k \nu_N(f^{-1} F)_v = \lim_{U(a; \Gamma'_{\varepsilon})} H^k(U(a; \Gamma'_{\varepsilon}); F).
\]

Here \( \Gamma \subset \mathbb{R}^{n+1} \) (resp. \( \Gamma' \subset \mathbb{R}^n \)) ranges through the family of open conic neighborhoods of \((\eta_0, 0)\) (resp. \(\eta_0\)), and \( V(a, \delta; \Gamma_{\varepsilon}) \) (resp. \( U(a; \Gamma'_{\varepsilon}) \)) ranges
through the family of open neighborhoods of $K(a, \delta) + \Gamma_\kappa$ (resp. $B'(0, a) + \Gamma'_\kappa$). Then the proof of (a) is reduced to Proposition 2.3 below.

**Proposition 2.3.** Let $\Gamma' \subset \mathbb{R}^n$ be a conic neighborhood of $\eta_0$. If $a$ and $\varepsilon$ are sufficiently small positive constants, then for any $k \in \mathbb{Z}$ there exist $\varepsilon'$, $\delta > 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^{n+1}$ of $(\eta_0, 0)$ such that

$$\lim_{V(a, \delta; \Gamma_\varepsilon)} H^k(V(a, \delta; \Gamma_\varepsilon); F) \simeq \lim_{U(a; \Gamma'_\varepsilon)} H^k(U(a; \Gamma'_\varepsilon); F).$$

**Proof.** The proof is same as [B-S 2, Lemme 3.2]. We use the following lemma instead of [B-S 2, Théorème 1.1]:

**Lemma 2.4 (cf. [B-S 1, Théorème 2.1]).** Let $Z \subset \Omega \subset X$ be convex sets such that $\Omega$ is an open set and $Z$ is closed in $\Omega$. Let $G$ be an object of $\mathbb{D}^b(X)$. Set

$$A := \{ \zeta; (w; \zeta) \in \text{SS}(G) \text{ for some } w \in \Omega \}.$$

Suppose that if a hyperplane with normal vector in $A$ crosses $\Omega$, then this hyperplane always crosses $Z$. Then it follows that

$$R \Gamma(\Omega; G) \simeq R \Gamma(Z; G).$$

**Proof of Lemma.** This lemma is proved in Yamazaki [Y]. For the sake of completeness of this paper, we reproduce the proof. Fix any $k \in \mathbb{Z}$. Then it is known that

$$\lim_{V} H^k(V; G) \simeq H^k(Z; G),$$

where $V$ ranges through the family of open neighborhoods of $Z$ in $\Omega$ (cf. [K-S, Proposition 2.6.9]). Take every $u \in H^k(Z; G)$. Then there exists a neighborhood $\Omega' \subset \Omega$ of $Z$ such that $u$ is regarded as a section of $H^k(\Omega'; G)$. Take any $p \in \Omega$. Set $U(p, \delta) := \{ q \in \mathbb{R}^{n+1}; |p - q| < \delta \}$. Then there exists a compact convex set $K \subset Z$ such that if a hyperplane with normal vector
in $A$ crosses the closure $\text{Cl}U(p, \delta)$, then this hyperplane always crosses $K$. Take $p_0 \in K$ and choose sufficiently small $0 < \varepsilon < \delta$. We set

$$p_t := (1 - t)p_0 + tp,$$

$$K_\varepsilon := \{ q \in \mathbb{R}^{n+1}; \text{dist}(q, K) < \varepsilon \} \subset \Omega';$$

$$S_t(p) := \text{Int} \gamma[U(p_t, \varepsilon) \cup K_\varepsilon] \subset \Omega.$$

We remark that $S_0(p) \subset \Omega'$. Further for any $q \in Z$, there exists a $p \in \Omega \setminus Z$ such that $q \in S_t(p)$ for any $0 \leq t \leq 1$ (we may choose that $p = q + c\zeta$ for suitable $c \in \mathbb{R}$ and $\zeta \in X \setminus Z$). Hence there exists a subset $\{p_i\}_{i \in I_0} \subset \Omega$ such that $\{S_t(p_i)\}_{i \in I_0}$ is an open covering of $Z$ in $\Omega$ for any $0 \leq t \leq 1$. Now, instead of Zerner’s theorem, we can use the theory of microsupports to prove:

$$\mathcal{R}\Gamma(S_1(p_1); G) \simeq \mathcal{R}\Gamma(S_0(p_i); G)$$

(see the proof of [B-S 1, Théorème 2.1] and [K-S, Lemma 5.2.2]). We refer this as a sweeping out method. Hence there exists a unique $u_i \in H^k(S_1(p_i); G)$ such that $u_i|_{S_0(p_i)} = u|_{S_0(p_i)}$. We shall prove

$$u_i|_{S_i(p_i) \cap S_i(p_j)} = u_j|_{S_i(p_i) \cap S_i(p_j)}.$$

We may show $u_i = u_j$ at any $p \in S_1(p_i) \cap S_1(p_j)$.

1. Assume that $p \in S_0(p_i) \cap S_0(p_j)$. Then we see $u_i = u_j$ at $p$ since

$$u_i|_{S_0(p_i) \cap S_0(p_j)} = u|_{S_0(p_i) \cap S_0(p_j)} = u_j|_{S_0(p_i) \cap S_0(p_j)}.$$

2. Assume that $p \in S_0(p_i) \cap (S_1(p_j) \setminus S_0(p_j)) \subset \Omega'$. Then we can construct $S_t(p)$ such that

$$\begin{cases}
S_t(p) \subset S_1(p_j) \cap \Omega' \text{ for } 0 \leq t \leq 1; \\
S_0(p) \subset S_0(p_j); \\
p \in S_1(p) \cap S_0(p_i).
\end{cases}$$

Since $u_i|_{S_0(p_i)} = u|_{S_0(p_i)}$, we have $u_i|_{S_1(p) \cap S_0(p_i)} = u_i|_{S_1(p) \cap S_0(p_i)}$. On the other hand, we can find a unique $v \in H^k(S_1(p); G)$ such that $v|_{S_0(p)} = u|_{S_0(p)}$ by the sweeping out method. Using the uniqueness twice, we have $u_j|_{S_1(p)} = v = u_i|_{S_1(p)}$. Therefore we have $u_i = u = u_j$ at $p$. 


(3) Assume that \( p \in (S_1(p_t) \setminus S_0(p_t)) \cap (S_1(p_j) \setminus S_0(p_j)) \). Then we can construct \( S_i(p) \) as \( S_i(p) \subset S_1(p_t) \cap S_1(p_j) \) for any \( 0 \leq t \leq 1 \). By the uniqueness, we have \( u_i|_{S_1(p)} = u_j|_{S_1(p)} \). Hence \( u_i = u_j \) at \( p \).

Thus there exists a unique \( u' \in H^k(\bigcup_{i \in I_0} S_1(p_i); G) \) such that \( u'|_{S_1(p_i)} = u_i \). In particular, \( u' = u \) holds in \( \bigcup_{p \in I_0} S_0(p_i) \). We set:

\[
\Phi := \{ \bigcup_{j \in J} S_1(p_j); p_j \in \Omega, Z \subset \bigcup_{j \in J} S_1(p_j), \text{ there exists a unique} \}
\]

\[
v \in H^k(\bigcup_{j \in J} S_1(p_j); G) \text{ such that } v = u \text{ in } \bigcup_{j \in J} S_0(p_j) \cap \Omega' \}.
\]

Then \( \bigcup_{i \in I_0} S_1(p_i) \in \Phi \). Let \( \{ V_\alpha \}_{\alpha \in A} \) be a totally ordered set in \( \Phi \). There exists a unique \( v_\alpha \in H^k(V_\alpha; G) \) such that \( v|_{V_\alpha \cap \Omega'} = u|_{V_\alpha \cap \Omega'} \). Set \( V := \bigcup_{\alpha \in A} V_\alpha \).

Then we can write \( V = \bigcup_{j \in J} S_1(p_j) \) for some \( \{ p_j \}_{j \in J} \subset \Omega \). Then by the uniqueness we can define a unique \( v \in H^k(V; G) \) such that \( v|_{V_\alpha} = v_\alpha \), hence \( v = u \) holds in \( \bigcup_{j \in J} S_0(p_j) \). Thus \( V \in \Phi \).

By Zorn’s lemma, there exists a maximal element \( V = \bigcup_{i \in I} S_1(p_i) \in \Phi \). Thus there exists a unique \( v \in H^k(V; G) \) such that \( v = u \) holds in \( \bigcup_{i \in I} S_0(p_i) \).

Suppose that there exists a \( p \in \Omega \setminus V \). By the sweeping out method, we can find a unique \( u' \in H^k(S_1(p); G) \) such that \( u'|_{S_0(p)} = u|_{S_0(p)} \). By the preceding argument, we have \( u'|_{S_i(p) \cap S_i(p)} = v|_{S_i(p) \cap S_i(p)} \) for any \( i \in I \).

Therefore there exists a unique \( v' \in H^k(V \cup S_1(p); G) \) such that \( v'|_{V} = v \) and \( v'|_{S_1(p)} = u' \). Therefore \( V \subsetneq V \cup S_1(p) \in \Phi \), which is a contradiction. Thus \( \Omega \in \Phi \). Hence there exists a unique \( v \in H^k(\Omega; G) \) such that \( v|_{Z} = u \in H^k(Z; G) \). Summing up, we obtain an isomorphism

\[
H^k(\Omega; G) \simeq H^k(Z; G).
\]

We shall prove Proposition 2.3. We have only to follow the argument in the proof of [B-S 1, Lemme 3.2] (cf. Tahara [T, Lemmata 2.1.1 and 2.1.2]).
to obtain
\[ H^k(M_{\eta,\varepsilon}; F) \cong \lim_{U(a; \Gamma^\varepsilon)} H^k(U(a; \Gamma^\varepsilon); F). \]

Here
\[ M_{\eta,\varepsilon} := \text{Int} \gamma [(B'(0, a) + \sqrt{-1} \Gamma^\varepsilon_{\varepsilon/2}) \cup \{(0, \pm\alpha \delta) + \sqrt{-1} \eta\}] \]
for an \( \eta \in \Gamma^\varepsilon_{\varepsilon/4} \) and an independent constant \( \alpha > 0 \). By the same argument as in the proof of Lemma 2.4, we have
\[ H^k(\bigcup_{\eta \in \Gamma^\varepsilon_{\varepsilon/4}} M_{\eta,\varepsilon}; F) \cong \lim_{U(a; \Gamma^\varepsilon)} H^k(U(a; \Gamma^\varepsilon); F). \]

We can find \( \varepsilon', \delta > 0 \) and a conic neighborhood \( \Gamma \subset \mathbb{R}^{n+1} \) of \((y_0, 0)\) such that
\[ K(a, \delta') + \sqrt{-1} \Gamma^\varepsilon_{\varepsilon/4} \subset \bigcup_{\eta \in \Gamma^\varepsilon_{\varepsilon/4}} M_{\eta,\varepsilon}. \]
The proof is complete. \( \square \)

(b) Next, we shall prove that
\[ Rf_! d_! f^{-1} \mu_M(F)_{p^*} \to Rf_! d_! f^{-1} \mu_{\text{hom}}(\mathbb{C}_{\Omega^+_x}, F)_{p^*} \]
is an isomorphism.

Let \( \nu_{\Omega^+_x}(F) \) be the inverse Fourier-Sato transform of \( \mu_{\text{hom}}(\mathbb{C}_{\Omega^+_x}, F) \). By the same argument as in the proof of (a), we have only to prove that a natural morphism
\[ \nu_{\Omega^+_x}(F)|_{T_N Y \cap \tau_N^{-1}(x_0)} \to \nu_{\Omega^+_x}(F)|_{T_N Y \cap \tau_N^{-1}(x_0)} \]
is an isomorphism. To this end, for any \( v \in TX \) and \( k \in \mathbb{Z}, \) we calculate
\[ H^k(\nu_{\Omega^+_x}(F))_v = \lim_{V} H^k(V; \nu_{\Omega^+_x}(F)), \]
where \( V \) ranges through the family of open conic neighborhoods of \( v \) such that \( V^{oo} = \text{Int} V. \) Here \( V^o \) denotes the dual cone. By Kashiwara-Schapira [K-S, Proposition 3.7.12], we have
\[ H^k(V; \nu_{\Omega^+_x}(F)) = H^{k+n+1}(T^* X; \mu_{\text{hom}}(\mathbb{C}_{\Omega^+_x}, F) \otimes or_{M/X}). \]
Here $V^{\text{oa}} := \{\xi \in T^*X; -\xi \in V^o\}$, and $or_{M/X}$ denotes the relative orientation sheaf. By Schapira-Zampieri [Sc-Z, Theorem 1.1], it follows that

$$H^k(V; \nu_{\Omega_+}(F)) = \lim_U H^k(U; F),$$

where $U$ ranges through the family of open subsets of $X$ such that

$C(X \setminus U, M_+) \cap V = \emptyset.$

Here $C(\ast, \ast)$ denotes the normal cone (see [K-S]). Therefore we have

$$H^k\nu_{\Omega_+}(F)_{\nu} = \lim_W H^k(W; F),$$

where $W$ ranges through the family of open subsets of $X$ such that $v \notin C(X \setminus W, M_+)$. Take $v = (0, \eta_0) \in T_NY \subset T_MX \subset TX$ arbitrary. Hence we have

$$\mathcal{H}^k\nu_{\Omega_+}(F)_{\nu} = \lim_{a, \delta, \Gamma^{'\varepsilon}} H^k(D_+(a, \delta; \Gamma^{'\varepsilon}); F).$$

Here $\Gamma^{'\varepsilon} \subset \mathbb{R}^n$ ranges through the family of open conic neighborhoods of $\eta_0$, and

$$D_+(a, \delta; \Gamma^{'\varepsilon}) := \{(z, \tau); (x, t) \in K(a, \delta), y \in \Gamma^{'\varepsilon}, \max\{-t, 0\} + |s| < \varepsilon|y|\}.$$  

For the same reasoning as in the proof of (a), we have

$$\mathcal{H}^k\nu_{M}(F)_{\nu} \simeq \mathcal{H}^k\nu_{\Omega_+}(F)_{\nu}. \quad \square$$

### 3. Microfunction with a Real Analytic Parameter

Recall the diagram (1.2). As usual, we denote by $\mathcal{O}_X, \mathcal{A}_M := \mathcal{O}_X|_M, \mathcal{B}_M$ and $\mathcal{E}_M$ the sheaves of holomorphic functions on $X$, of real analytic functions on $M$, of hyperfunctions on $M$ and of microfunctions on $T^*_MX$ respectively. Although it is well-known that $f_\ast f^{-1}_\pi \mathcal{E}_M$ is the sheaf of microfunction with a real analytic parameter $t$, and $f_\ast f^{-1}_\pi \mathcal{E}_M|_N$ is the sheaf of hyperfunction with a real analytic parameter $t$ (see Sato [S] and Sato-Kawai-Kashiwara [S-K-K]), we give a detailed exposition about these sheaves for the convenience of the reader.
Definition 3.1. We set
\[ \mathcal{A}_N^A := Rf_d! f^{-1}_\pi \mathcal{E}_M, \quad \mathcal{B}_N^A := i_N^{-1} \mathcal{E}_N^A = R\pi_N* \mathcal{E}_N^A. \]

Proposition 3.2 (cf. [S-K-K, Chapter I, Theorem 2.2.6]).

(1) \( \mathcal{E}_N^A \) is concentrated in degree zero and conically soft; that is, the direct image of \( \mathcal{E}_N^A \) on \( \hat{T}_N^*Y/\mathbb{R}_{>0} \) is a soft sheaf.

(2) There exists the following exact sequence:
\[ 0 \to \mathcal{A}_M|_N \to \mathcal{B}_N^A \to \hat{\pi}_N* \mathcal{E}_N^A \to 0. \]

(3) There exists the following exact sequence:
\[ 0 \to \mathcal{B}_N^A \to \mathcal{B}_M|_N \to \hat{\pi}_N* (\mathcal{E}_M|_{\sqrt{\hat{T}_N^*M}}) \to 0. \]

Proof. (1) Consider a distinguished triangle:
\[ R\Gamma_M(\mathcal{E}_M) \to \mathcal{E}_M \to R\Gamma_{\hat{T}_M^*X}(\mathcal{E}_M) \to \]
and apply the functor \( Rf_d! f^{-1}_\pi \). Since \( i_M : M \to T_M^*X \) and \( i_N : N \to \hat{T}_N^*Y \) are closed embeddings, we have
\[ Rf_d! f^{-1}_\pi R\Gamma_M(\mathcal{E}_M) = Rf_d! f^{-1}_\pi Ri_M! \mathcal{A}_M \simeq Ri_N! f^{-1}\mathcal{A}_M = Ri_N* f^{-1}\mathcal{A}_M. \]

(2) On the other hand, since \( R\Gamma_{\hat{T}_M^*X}(\mathcal{E}_M) = \hat{T}_M^*X(\mathcal{E}_M) \) is a conically flabby sheaf, \( Rf_d! f^{-1}_\pi R\Gamma_{\hat{T}_M^*X}(\mathcal{E}_M) = f_d! f^{-1}_\pi \mathcal{E}_M \) and \( f_d! f^{-1}_\pi \mathcal{E}_M|_{\hat{T}_N^*Y} \) is conically soft. Here we set \( \hat{f}_\pi : N \times \hat{T}_M^*X \to \hat{T}_M^*X \). Summing up, we obtain a distinguished triangle:
\[ i_N* f^{-1}_\pi \mathcal{A}_M \to \mathcal{E}_N^A \to f_d! f^{-1}_\pi \mathcal{E}_M \to. \]

Hence \( \mathcal{E}_N^A \) is concentrated by degree zero, and
\[ \mathcal{E}_N^A \mid \hat{T}_N^*Y = f_d! f^{-1}_\pi \mathcal{E}_M \mid \hat{T}_N^*Y. \]
is conically soft.

(2) Consider Sato’s distinguished triangle:

$$
R\pi_N! \mathcal{E}^A_{N|M} \to R\pi_{N*} \mathcal{E}^A_{N|M} \to R\hat{\pi}_{N*} \mathcal{E}^A_{N|M} \xrightarrow{+1}.
$$

Then we have

$$
R\pi_{N*} \mathcal{E}^A_{N|M} \mid N = \mathcal{E}^A_{N|M} | N,
$$

and

$$
R\pi_{N!} \mathcal{E}^A_{N|M} = R\pi_{N!} Rf_* \mathcal{E}^A_M \simeq R\pi_{!} \mathcal{E}^A_M \simeq \mathcal{A}_M | N.
$$

(3) Set $V := N \times \dot{T}^*_M X \setminus \sqrt{-1} \dot{T}^*_N M$ for short. Consider a distinguished triangle:

$$(3.3) \quad (\dot{f}^{-1}_\pi \mathcal{E}_M)_V \to \dot{f}^{-1}_\pi \mathcal{E}_M \to (\dot{f}^{-1}_\pi \mathcal{E}_M) \sqrt{-1} \dot{T}^*_N M \xrightarrow{+1}.$$

Applying the functor $Rf_{d!}$ to (3.3) we have

$$
Rf_{d!}((\dot{f}^{-1}_\pi \mathcal{E}_M)_V) \to Rf_{d!} \dot{f}^{-1}_\pi \mathcal{E}_M \to Rf_{d!} (\dot{f}^{-1}_\pi \mathcal{E}_M) \sqrt{-1} \dot{T}^*_N M \xrightarrow{+1}.
$$

Since

$$
Rf_{d!} (\dot{f}^{-1}_\pi \mathcal{E}_M) \sqrt{-1} \dot{T}^*_N M \mid \dot{T}^*_N Y = Ri_{N!} R\hat{\pi}_* i^{-1} \dot{f}^{-1}_\pi \mathcal{E}_M \mid \dot{T}^*_N Y = 0,
$$

we have

$$
(3.4) \quad Rf_{d!}((\dot{f}^{-1}_\pi \mathcal{E}_M)_V) | \dot{T}^*_N Y \simeq Rf_{d!} \dot{f}^{-1}_\pi \mathcal{E}_M | \dot{T}^*_N Y \simeq \mathcal{E}^A_{N|M} | \dot{T}^*_N Y.
$$

Next, applying the functor $Rf_{d*}$ to (3.3) we have

$$(3.5) \quad Rf_{d*}((\dot{f}^{-1}_\pi \mathcal{E}_M)_V) \to Rf_{d*} \dot{f}^{-1}_\pi \mathcal{E}_M \to Ri_{N*} R\hat{\pi}_* \mathcal{E}_M | \sqrt{-1} \dot{T}^*_N M \xrightarrow{+1}.$$

Since $f_d: V \to \dot{T}^*_N Y$ is proper, by (3.4) we have

$$
Rf_{d*}((\dot{f}^{-1}_\pi \mathcal{E}_M)_V) | \dot{T}^*_N Y \simeq Rf_{d!}((\dot{f}^{-1}_\pi \mathcal{E}_M)_V) | \dot{T}^*_N Y \simeq \mathcal{E}^A_{N|M} | \dot{T}^*_N Y.
$$
On the other hand, from (3.1), we have

\[
\begin{array}{c}
\xymatrix{
R\pi_{M*}R\Gamma_M(\mathcal{E}_M) \ar[r] \ar[d] & R\pi_{N*}Rf_{d*} f^{-1}_\pi R\Gamma_M(\mathcal{E}_M) \ar[d] \\
\pi f^{-1}_R\Gamma_M(\mathcal{E}_M) \ar[d] & R\pi_{N*}Rf_{d*} f^{-1}_\pi \mathcal{E}_M \ar[d] \\
\pi f^{-1}_R\Gamma_M\mathcal{T} M X(\mathcal{E}_M) \ar[r] \ar[d]^{+1} & R\pi_{N*}Rf_{d*} f^{-1}_\pi \mathcal{T} M X(\mathcal{E}_M) \ar[d]^{+1} \\
& & &
}\end{array}
\]

As in (3.2) we have

\[
R\pi_{N*}Rf_{d*} f^{-1}_\pi R\Gamma_M(\mathcal{E}_M) = \mathcal{A}_M|_N
\]

and by [K-S, Proposition 3.7.5]

\[
R\pi_{N*}Rf_{d*} f^{-1}_\pi \mathcal{E}_M = R\pi_{N*} f^{-1}_\pi \mathcal{E}_M = f^{-1}_\pi \mathcal{E}_M|_N = \mathcal{B}_M|_N.
\]

Hence we have

\[
\begin{array}{c}
\xymatrix{
0 \ar[r] & \mathcal{A}_M|_N \ar[r] \ar@{=}[d] & \mathcal{B}_M|_N \ar[r] & \hat{\pi}_M^{*}\mathcal{E}_M|_N \ar[r] & 0 \\
0 \ar[r] & \mathcal{A}_M|_N \ar[r] \ar[d] & \mathcal{B}_M|_N \ar[r] & \hat{\pi}_M^{*}f^{-1}_\pi \mathcal{E}_M \ar[r] & 0
}\end{array}
\]

thus \(\hat{\pi}_M^{*}\mathcal{E}_M|_N = \hat{\pi}_M^{*} f^{-1}_\pi \mathcal{E}_M\). Applying the functor \(R\hat{\pi}_{N*}\) to (3.5) we have

\[
\begin{array}{c}
\xymatrix{
0 \ar[r] & \mathcal{A}_M|_N \ar[r] \ar@{=}[d] & \mathcal{B}_M^A|_N \ar[r] & \hat{\pi}_{N*}^{A}\mathcal{E}_N|_M \ar[r] & 0 \\
0 \ar[r] & \mathcal{A}_M|_N \ar[r] \ar[d]^{+1} & \mathcal{B}_M|_N \ar[r] & \hat{\pi}_M^{*}f^{-1}_\pi \mathcal{E}_M \ar[r] & 0 \\
& \hat{\pi}_M^{*}(\mathcal{E}_M|_{\sqrt{-1}\mathcal{T} N M}) \ar[d] & &
\end{array}
\]

0
Thus by Snake Lemma we obtain
$$0 \to \mathcal{B}_N^A \mid M \to \mathcal{B}_M \mid N \to \tilde{\pi}_*(\mathcal{E}_M \mid \sqrt{-1} T^*_N M) \to 0. \square$$

We recall that the morphism in (1.3) induces restriction morphisms:
$$\mathcal{E}_N^A \mid M \to \mathcal{E}_N, \quad \mathcal{B}_N^A \mid M \to \mathcal{B}_N.$$ 

In order to study microlocal boundary value problems, Kataoka [Kt] defined the sheaf $\mathcal{E}_N^0 \mid M_+$ of mild microfunctions on $T^*_N Y$, and $\mathcal{B}_N^0 \mid M_+ := \mathcal{E}_N^0 \mid M_+ \mid N$ is called the sheaf of mild hyperfunctions. Note that
$$\mathcal{E}_N^0 \mid M_+ = Rf! \lf_{\tau}^{-1} \mu_{\text{hom}}(\mathcal{O}_{P_+}, \mathcal{O}_X) \otimes \text{or}^0_{M/X}[n + 1]$$
holds by Schapira-Zampieri [Sc-Z]. $\mathcal{E}_N^0 \mid M_+$ is conically soft, and there exists an exact sequence:
$$0 \to \mathcal{A}_N \mid N \to \mathcal{B}_N^0 \mid M_+ \to \hat{\pi}_N^* \mathcal{E}_N^0 \mid M_+ \to 0.$$ 

Further by (1.3), the restriction morphism $\mathcal{E}_N^0 \mid M \to \mathcal{E}_N$ factorizes through the boundary value morphism
$$\mathcal{E}_N^0 \mid M_+ \to \mathcal{E}_N:$$

4. Cauchy and Boundary Value Problems for Fuchsian Hyperbolic Systems

First, we recall the definition of Fuchsian differential operators in the sense of Baouendi-Goulaouic [B-G].

Definition 4.1. Let us take local coordinates in (1.1). Then we say that $P$ is a Fuchsian differential operator of weight $(k, m)$ in the sense of Baouendi-Goulaouic [B-G] if $P$ can be written in the following form

$$P(z, \tau, \partial_z, \partial_\tau) = \tau^k \partial_\tau^m + \sum_{j=1}^{k} P_j(z, \tau, \partial_z) \tau^{k-j} \partial_\tau^{m-j} + \sum_{j=k+1}^{m} P_j(z, \tau, \partial_z) \partial_\tau^{m-j}.$$
Here \( \text{ord} \ P_j \leq j \ (0 \leq j \leq m) \), and \( P_j(z,0,\partial_z) \in \mathcal{O}_Y \ (1 \leq j \leq k) \).

Note that a Fuchsian differential operator of weight \((m,m)\) is nothing but an operator with regular singularity along \( Y \) in a weak sense due to Kashiwara-Oshima [K-O].

Let \( \mathcal{M} \) be a \( \mathcal{D}_X \)-Module. The inverse image in the sense of \( \mathcal{D} \)-Module is defined by

\[
Df^*\mathcal{M} := \mathcal{D}_Y \rightarrow X \bigotimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M} \in \text{Ob} \mathcal{D}^b(\mathcal{D}_Y).
\]

Here \( \mathcal{D}_Y \rightarrow X \bigotimes_{f^{-1}\mathcal{D}_X} \) is the transfer bi-Module as usual. Further we set \( \mathcal{M}_Y := \mathcal{H}^0 Df^*\mathcal{M} = \mathcal{O}_Y \bigotimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M} \).

Next, let \( \mathcal{M} \) be a Fuchsian system along \( Y \) in the sense of Laurent-Monteiro Fernandes [L-MF 1]. Since the precise definition of Fuchsian system is complicated, we do not recall it here. We remark that \( \mathcal{M} \) is Fuchsian along \( Y \) if and only if there exists locally an epimorphism

\[
\bigoplus_{i=1}^m \mathcal{D}_X/\mathcal{D}_X P_i \twoheadrightarrow \mathcal{M},
\]

where each differential operator \( P_i \) is an operator with regular singularity along \( Y \) in a weak sense.

**Remark 4.2.** (1) Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X|_Y \)-Module for which \( Y \) is non-characteristic. Then \( \mathcal{M} \) is Fuchsian. More generally, any regular-specializable system is Fuchsian.

(2) Let \( \mathcal{M} \) be a Fuchsian system along \( Y \). Then:

(i) By Laurent-Schapira [L-S, Théorème 3.3], all the cohomologies of \( Df^*\mathcal{M} \) are coherent \( \mathcal{D}_Y \)-Modules.

(ii) Laurent-Monteiro Fernandes [L-MF 1, Théorème 3.2.2] proved that there exists the following isomorphism (that is, Cauchy-Kovalevskaja type theorem):

\[
f^{-1}R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\text{Hom}_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y).
\]

**Definition 4.3.** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X|_Y \)-Module. Then we say \( \mathcal{M} \) is near-hyperbolic at \( x_0 \in N \) in \( \pm dt \) codirection if \( R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \) is near-hyperbolic in the sense of Definition 2.1. We remark that

\[
\text{SS}(R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \text{char}(\mathcal{M}).
\]
Here char(\(\mathcal{M}\)) is the characteristic variety of \(\mathcal{M}\).

Example 4.4. (1) Let \(P\) be a Fuchsian differential operator of weight \((k, m)\) in the sense of Baouendi-Goulaouic [B-G]. Then \(\mathcal{D}_X/\mathcal{D}_X P\) is Fuchsian along \(Y\). Moreover, assume that \(P\) is Fuchsian hyperbolic in the sense of Tahara [T]; that is, the principal symbol is written as \(\sigma_m(P)(z, \tau; z^*, \tau) = \tau^k p(z, \tau; z^*, \tau^*)\). Here \(p(z, \tau; z^*, \tau^*)\) satisfies the following condition:

\[
\begin{cases}
\text{If } (x, t; x^*) \text{ are real, then all the roots } \tau_j^*(x, t; x^*) \text{ of the equation } p(x, t; x^*, \tau^*) = 0 \text{ with respect to } \tau^* \text{ are real.}
\end{cases}
\]

Then \(\mathcal{D}_X/\mathcal{D}_X P\) is near-hyperbolic (see [L-MF 2, Lemma 1.3.2]).

(2) Let \(P = \vartheta - A(z, \tau, \partial_z)\) be a Fuchsian Volević system of size \(m\) due to Tahara [T]; that is,

(i) \(A(z, \tau, \partial_z) = (A_{ij}(z, \tau, \partial_z))_{i,j=1}^m\) is a matrix of size \(m\) whose components are in \(\mathcal{D}_X\) with \([A_{ij}, \tau] = 0\);

(ii) There exists \(\{n_i\}_{i=1}^m \subset \mathbb{Z}\) such that ord \(A_{ij}(z, \tau, \partial_z) \leq n_i - n_j + 1\) and \(A_{ij}(z, 0, \partial_z) \in \mathcal{O}_Y\) for any \(1 \leq i, j \leq m\).

Set \(\sigma(A)(z, \tau; z^*) := (\sigma_{n_i-n_j+1}(A_{ij})(z, \tau; z^*))_{i,j=1}^m\). Then

\[
\text{char}(\mathcal{D}_X^m/\mathcal{D}_X^m P) = \{(z, \tau; z^*, \tau^*) \in T^*X; \det(\tau \tau^* - \sigma(A)(z, \tau; z^*)) = 0\},
\]

and we can prove that \(\mathcal{D}_X^m/\mathcal{D}_X^m P\) is Fuchsian along \(Y\). Moreover assume that \(P\) is Fuchsian hyperbolic in the sense of Tahara [T]; that is,

\[
\det(\tau \tau^* - \sigma(A)(z, \tau; z^*)) = \tau^m p(z, \tau; z^*, \tau^*),
\]

and \(p(z, \tau; z^*, \tau^*)\) satisfies the condition (4.1). Then \(\mathcal{D}_X^m/\mathcal{D}_X^m P\) satisfies the near-hyperbolicity condition.

Our main Theorem is:

**Theorem 4.5.** Let \(\mathcal{M}\) be a Fuchsian system along \(Y\). Assume that \(\mathcal{M}\) is near-hyperbolic at \(x_0 \in N\) in \(\pm dt\)-codirection. Then for any \(p^* \in \mathcal{O}_Y\)
$T_N^*Y \cap \pi_N^{-1}(x_0)$, the morphisms in (1.3) induce isomorphisms

$$R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^A|_{\mathcal{M}})_p^* \simeq R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^o|_{\mathcal{M}+})_p^* \simeq R\text{Hom}_{\mathcal{D}Y}(Df^*\mathcal{M}, \mathcal{C}_N)_p^*. $$

In particular, the morphisms in (1.3) induce isomorphisms

$$R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{B}_N^A|_{\mathcal{M}})_{x_0}^* \simeq R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{B}_N^o|_{\mathcal{M}+})_{x_0}^* \simeq R\text{Hom}_{\mathcal{D}Y}(Df^*\mathcal{M}, \mathcal{B}_N)_{x_0}^*. $$

**Proof.** Set $F := R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{O}_X)$ for short. Then, we have

$$R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^A|_{\mathcal{M}}) = Rf_!(f^{-1}\mu_M(F) \otimes \mathcal{O}_{M/X})[n + 1],$$
$$R\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^o|_{\mathcal{M}+}) = Rf_!(f^{-1}\mu_{\text{hom}}(\mathcal{C}_\Omega^+, F) \otimes \mathcal{O}_{M/X})[n + 1],$$

and (see Remark 4.2):

$$R\text{Hom}_{\mathcal{D}Y}(Df^*\mathcal{M}, \mathcal{C}_N) = \mu_N(R\text{Hom}_{\mathcal{D}X}(Df^*\mathcal{M}, \mathcal{O}_Y) \otimes \mathcal{O}_{N/Y})[n]$$
$$= \mu_N(f^{-1}F \otimes \mathcal{O}_{N/Y})[n].$$

Therefore by Theorem 2.2, we obtain the theorem. $\square$

**Remark 4.6.** Oaku-Yamazaki [O-Y] showed that for any Fuchsian system $\mathcal{M}$ along $Y$, two morphisms

$$\text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^A|_{\mathcal{M}})_p^* \rightarrow \text{Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{C}_N^o|_{\mathcal{M}+})_p^* \rightarrow \text{Hom}_{\mathcal{D}Y}(\mathcal{M}_Y, \mathcal{C}_N)_p^*$$

are always injective without the near-hyperbolicity condition. Precisely speaking, we always assumed that $\text{codim}_M N \geq 2$ in [O-Y]. However, the same proof also works even in the case where $\text{codim}_M N = 1$; Oaku ([O 1], [O 2]) defined the sheaf $\mathcal{C}_N^F$ of $F$-mild microfunctions on $T_N^*Y$, and $\mathcal{B}_N^F := \mathcal{C}_N^F|_{\mathcal{M}+}$ is called the sheaf of $F$-mild hyperfunctions ([O 1], [O 2], cf. [O-Y]). As is mentioned above, we can apply the methods in [O-Y]
of the higher-codimensional case to the one-codimensional case to prove the following: there exist natural morphisms

\[ \mathcal{C}^A_{N|M} \to \mathcal{C}^\circ_{N|M_+} \to \mathcal{C}^F_{N|M_+} \to \mathcal{C}_N \]

such that the composite coincides with \( \mathcal{C}^A_{N|M} \to \mathcal{C}_N \), and that these induce monomorphisms:

\[
\begin{array}{ccc}
\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}^A_{N|M}) & \to & \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}^\circ_{N|M_+}) \\
\downarrow & & \downarrow \\
\mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{C}_N) & \leftarrow & \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}^F_{N|M_+})
\end{array}
\]

for any Fuchsian system \( M \) along \( Y \). Hence, under the near-hyperbolic condition, we obtain isomorphisms:

\[
\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}^F_{N|M_+}) \simeq \mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{C}_N),
\]

\[
\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{B}^F_{N|M_+}) \simeq \mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{B}_N).
\]

Our conjecture is: if \( M \) is a Fuchsian system along \( Y \) and satisfies near-hyperbolicity condition, then the following holds:

\[
R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}^F_{N|M_+}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{C}_N).
\]

References


(Received December 14, 2004)

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