Gauge Theory and the A-Polynomial

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Abstract. In this article, we explain how to use instanton Floer homology of various Dehn surgeries along knots in integer homology spheres to prove that their $A$-polynomial is non-trivial. In particular, we show that all non-trivial knots in $S^3$ have non-trivial $A$-polynomial.

Not long ago, Kronheimer and Mrowka gave a gauge theoretic proof of Property P for knots in $S^3$. In fact, the proof found in [11] establishes much more: all $(1/n)$–surgeries along a non-trivial knot in $S^3$ admit a representation of their fundamental group in $SU(2)$ with non-abelian image. Their work has been used in [2] by Boyer and Zhang to show that the $A$–polynomial of any non-trivial knot in $S^3$ is non-trivial. In this short note we give a gauge-theoretic proof of this fact and various generalizations, the approach being closer in spirit to the results of Kronheimer and Mrowka contained in [11] and [12] since we use holonomy perturbations that naturally arise in gauge theory.

Let us begin by recalling a crucial step in the proof of Property P: one must establish that the 0–surgery of $S^3$ along $K$ has non-vanishing Floer homology $HF^*(Y_0(K))$. These Floer groups are generated by non-degenerate flat connections on the $SO(3)$–bundle $P$ over $Y_0(K)$ with non-trivial second Steifel-Whitney class. If the moduli space of flat connections $M(Y_0(K))$ is degenerate, holonomy perturbations of the flatness equation are used to define the Floer chain groups. The explicit construction of Floer homology for $Y_0(K)$ is not important for our purpose, only matters the fact that if the moduli space $M(Y_0(K))$ is empty, then $HF^*(Y_0(K))$ is trivial. The 0–surgery being obtained by Dehn filling the knot complement $Y_K$ along the longitude $\lambda_K$, the moduli space $M(Y_0(K))$ on $P$ such that $w_2(P) \neq 0$ can be obtained from the moduli space of irreducible flat $SU(2)$–connections over the knot complement $M^*(Y_K)$ by taking flat connections
with holonomy $-I \in SU(2)$ along $\lambda_K$, see for example [3]. The result of Kronheimer and Mrowka implies the following:

**Theorem 1.** Let $K$ be a non-trivial knot in $S^3$. The moduli space of irreducible flat $SU(2)$–connections $M^*(Y_{K})$ is non-empty.

We want to use this to say something about the $A$–polynomial of $K$. We first recall that $SU(2)$ being a subgroup of $SL(2, \mathbb{C})$, an $SU(2)$–representation naturally gives an $SL(2, \mathbb{C})$–representation. Moreover, it is well known that different $SU(2)$–characters yield different $SL(2, \mathbb{C})$–characters. For the construction of the $A$–polynomial, we refer the reader to [5]. The definition uses characters of $SL(2, \mathbb{C})$–representations of $\pi_1(Y_{K})$ and a restriction map to $\mathbb{C}^* \times \mathbb{C}^*$ corresponding to characters of $SL(2, \mathbb{C})$–representations of $\pi_1(\partial Y_{K})$. Showing that the $A$-polynomial is non-trivial amounts to showing that the image of the restriction map is of complex dimension 1 in $\mathbb{C}^* \times \mathbb{C}^*$ for some component in the character variety which contains an irreducible character, and that on such a curve the character of the knot longitude is not identically equal to 2. On the gauge-theoretic side, there is a corresponding restriction map for flat $SU(2)$–connections: $r: M(Y_{K}) \rightarrow M(\partial Y_{K})$, commonly referred to as the pillow-case restriction. We prove the following about this restriction map:

**Theorem 2.** Let $K$ be a non-trivial knot in $S^3$. Then $M^*(Y_{K})$ contains an arc of irreducible connections $\{A_t\}$ that can be locally parametrized by the holonomy along the longitude $\lambda_K$.

A short proof that the $A$-polynomial of $K$ is non-trivial follows. The arc $\{A_t\} \subset M^*(Y_{K})$ of Theorem 2 yields, via the holonomy correspondence, an arc of irreducible $SU(2)$–characters with non-constant trace along the longitude of $K$. This arc therefore lies on a component $X_0$ of $SL(2, \mathbb{C})$–characters whose restriction to $\mathbb{C}^* \times \mathbb{C}^*$ is 1-dimensional and can be locally parametrized by the trace along the longitude. It follows directly that $X_0$ cannot contribute trivially to the $A$-polynomial of $K$.

The proof of Theorem 2 relies on the use of holonomy perturbations. These were first used by Floer to define his invariant in [8] and, later on, to prove the existence of a surgery exact sequence in [9]. An extension to knot complements was defined by Herald, and this is the version we use here. We
give a brief summary and refer to [10] and [4] for details. Take a collection \( \{\gamma_i: S^1 \times D^2 \to Y_K\}_{1 \leq i \leq n} \) of embeddings of solid tori in \( Y_K \) whose images are disjoint and away from the boundary torus of \( Y_K \). Let \( \eta: D^2 \to \mathbb{R} \) be a bump function on \( D^2 \) and define a function \( h \) on the moduli space of \( SU(2) \)-connections by

\[
h(A) = \sum_{i=1}^{n} \int_{D^2} h_i(\text{tr} \text{hol}_A(\gamma_i(S^1 \times \{x\}))) \eta(x) \, dx^2,
\]

where \( \{h_i: \mathbb{R} \to \mathbb{R}\}_{1 \leq i \leq n} \) is a collection of smooth functions. The function \( h \) is called an admissible perturbation function. We shall make use of the perturbed moduli space \( \mathcal{M}_h(Y_K) \) of flat \( SU(2) \)-connections satisfying the equation \( *F_A + \nabla h(A) = 0 \). Herald proved in [10] that a generic holonomy perturbation \( h \) makes \( \mathcal{M}^*(Y_K) \) into a smooth 1-manifold \( \mathcal{M}_h^*(Y_K) \). This allows us to explicitly construct a perturbed moduli space \( \mathcal{M}_h(Y_0(K)) \), by considering elements in \( \mathcal{M}_h(Y_K) \) that have holonomy equal to \(-I\) along \( \lambda_K \). Alternatively, we could construct, as in [12], the space \( \mathcal{M}_h(Y_0(K)) \) by considering elements in \( \mathcal{M}^*(Y_K) \) satisfying a perturbed holonomy condition along the longitude \( \lambda_K \). The two approaches are equivalent. Also note that, while large scale perturbations are needed to obtain a result like the Floer surgery exact sequence, here we only need (small) holonomy perturbations that change the Floer chain groups but not the Floer homology groups.

**Proof of Theorem 2.** Consider \( \mathcal{M}(Y_0(K)) \) as a subset of \( \mathcal{M}^*(Y_K) \). We first claim that some element in \( \mathcal{M}(Y_0(K)) \) lies on an arc \( \{A_t\} \) in \( \mathcal{M}^*(Y_K) \). Otherwise, since \( \mathcal{M}^*(Y_K) \) is a compact real algebraic set, any element \( A \in \mathcal{M}(Y_0(K)) \cap \mathcal{M}^*(Y_K) \) is isolated. Because \( A \) is isolated we can choose our generic holonomy perturbation \( h \) such that for any \( A_h \in \mathcal{M}_h^*(Y_K) \), the holonomy of \( A_h \) along the longitude \( \lambda_K \) is different from \(-I\). This means that we have a generic holonomy perturbation for the 3-manifold \( Y_0(K) \) for which the perturbed Floer chain complex is empty, ie we have pushed off \( A \) from the top line in the pillow-case. It follows that \( HF_*(Y_0(K)) \) is trivial, since it is invariant under admissible perturbations, which contradicts Kronheimer and Mrowka [11]. The rest of the proof is very similar to the above. Suppose now that no arc \( \{A_t\} \) constructed above in \( \mathcal{M}^*(Y_K) \) admits a local parametrization by holonomy along \( \lambda_K \). In particular, this means that \( \{A_t\} \subset \mathcal{M}(Y_0(K)) \). Now perturb exactly as above to
exhibit an empty perturbed Floer chain complex, giving $HF_\ast(Y_0(K)) = 0$, again a contradiction. □

The method readily generalizes beyond the case of knots in $S^3$. Indeed, for knots in integer homology spheres, the Floer homology of the 0-surgery is defined and we obtain a criterion for the non-triviality of the $A$-polynomial.

**Theorem 3.** Let $K$ be a knot in an integer homology sphere $Y^3$. Suppose that the Floer homology of the 0-surgery of $Y^3$ along $K$ is non-vanishing. Then the $A$-polynomial of $K$ is non-trivial.

In our construction, there is nothing special about the longitudinal Dehn filling other than the fact that we know (for knots in $S^3$) the Floer homology of this 3-manifold to be non-trivial. We can use holonomy perturbations for any other filling, and this gives:

**Theorem 4.** Let $K$ be a knot in a homology sphere $Y^3$ such that for some $r \in \{\infty, 0\} \cup \{1/k \mid k \in \mathbb{Z}^\ast\}$ the Dehn filling $Y_r(K)$ has non-trivial Floer homology. Then there exists an arc $\{A_t\} \subset \mathcal{M}^\ast(Y_K)$ locally parametrized by the holonomy along a peripheral element in $\pi_1(Y_K)$.

Theorem 4 does not quite imply that the $A$-polynomial is non-trivial. We obtain existence of deformations of irreducible $SL(2, \mathbb{C})$–characters into a 1-dimensional family of irreducibles, but the $A$-polynomial could still be trivial. This would happen if all the irreducible characters for the knot complement $Y_K$ send the longitude $\lambda_K$ to $I \in SL(2, \mathbb{C})$, as happens in the example below. From the gauge theory side, this situation illustrates how much information can be lost by restriction to the pillow-case.

**Example 1.** Take any integer homology sphere $Y^3$ with $HF_\ast(Y)$ non-trivial, and consider an unknotted curve $K$ contained in a small 3-ball in $Y^3$. By construction, $\mathcal{M}^\ast(Y)$ is non-empty and the knot complement $Y_K$ has fundamental group $\pi_1(Y_K) = \pi_1(Y) \ast \mathbb{Z}$. It is then clear that $\mathcal{M}^\ast(Y_K)$ contains at least one arc of irreducible flat connection parametrized by the holonomy along the meridian $\mu_K$. The 0-surgery will be $Y \sharp S^1 \times S^2$ and hence $HF_\ast(Y_0(K))$ is trivial. Because the longitude $\lambda_K$ is trivial in $\pi_1(Y_K)$, it follows directly that the $A$-polynomial is trivial.
In the case of a knot $K$ in $S^3$, the non-triviality of $HF_\ast(Y_0(K))$ is equivalent to the non-triviality of the $A$-polynomial of $K$. For knots in arbitrary integer homology spheres, we can use Theorem 4 to see that the non-vanishing of $HF_\ast(Y_0(K))$ is not a necessary condition for the $A$-polynomial of $K$ to be non-trivial. Various examples can be constructed using generalized Mazur manifolds. These are contractible 4-manifolds $W^\pm(l, k)$ for $k, l \in \mathbb{Z}$ whose boundaries are integer homology spheres $\partial W^\pm(l, k)$. We refer the reader to [1] for the construction, as we give below two examples of knots in Mazur homology spheres whose $A$-polynomial is non-trivial but for which $HF_\ast(Y_0(K)) = 0$.

**Example 2.** Consider Mazur’s original manifold $W^+(0, 0)$ whose boundary $\partial W^+(0, 0)$ is the Brieskorn homology sphere $\Sigma(2, 3, 7)$. The Floer homology of Brieskorn manifolds was explicitly computed in [7], but all we need here is that $HF_\ast(\Sigma(2, 3, 7))$ is non-vanishing. Let $K$ be a knot in $\partial W^+(0, 0)$ given as a small linking circle about the 1-handle used in the construction of $W^+(0, 0)$. Performing a 0-surgery along $K$ clearly gives $S^1 \times S^2$, a 3-manifold over which there are no irreducible $SO(3)$–connections, so that $HF_\ast(Y_0(K)) = 0$. Now apply Theorem 4 to the filling $Y_{\infty}(K) = \Sigma(2, 3, 7)$ to construct an arc $\{A_t\} \subset M^\ast(Y_K)$. To conclude that the $A$-polynomial of $K$ is non-trivial, we simply need to show, moreover, that $\{A_t\}$ satisfies $\text{hol}_{A_t}(\lambda_K) \neq I$. But this is immediate as otherwise the arc $\{A_t\}$ would provide irreducible flat $SO(3)$–connections over $Y_K$ which extend to $Y_0(K) = S^1 \times S^2$, a contradiction.

**Example 3.** In $\partial W^+(2, 0)$ let $K$ be the knot given by a linking circle along which the $(+1)$-surgery corresponds to a crossing change between the two links in the Kirby diagram of $W^+(2, 0)$. This $(+1)$-surgery along $K$ is obtained by blowing down $K$ and the 1-handle, therefore $Y_{+1}(K)$ can also be seen as the result of a $(+1)$-surgery along some non-trivial knot in $S^3$. By [11] we know that $HF_\ast(Y_{+1}(K))$ is non-vanishing. Also it is an easy exercise to see that $Y_0(K)$ is again $S^1 \times S^2$. As in Example 2, Theorem 4 therefore enables us to conclude that the $A$-polynomial of $K$ is non-trivial although $HF_\ast(Y_0(K)) = 0$.

The following seems likely to be a difficult question: is it possible to find knots in integer homology spheres whose $A$-polynomial is non-trivial.
but such that the Floer homology groups $HF_*(Y_r(K))$ vanish for all $r \in \{\infty, 0\} \cup \{1/k \mid k \in \mathbb{Z}^*\}$?

References


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