Birational Symmetries, Hirota Bilinear Forms and Special Solutions of the Garnier Systems in 2-variables

By Teruhisa Tsuda

Abstract. Hirota bilinear forms of the Garnier system in 2-variables, $G(1,1,1,1,1)$, are given. By using Hirota bilinear forms we construct new birational symmetries of $G(1,1,1,1,1)$. We obtain special solutions of the Garnier system in $n$-variables, which are described in terms of solutions of the Garnier system in $(n-1)$-variables. We investigate also algebraic solutions for $n = 2$.

Introduction

The Painlevé equations $P_J$ ($J = I, \cdots, VI$) are derived from the theory of monodromy preserving deformation of the linear differential equation of second order:

\[
(L_J) \frac{d^2y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x)y = 0.
\]

R. Fuchs ([1]) had obtained the sixth Painlevé equation $P_{VI}$ by considering monodromy preserving deformation of (0.1). In fact, $P_{VI}$ is deduced from complete integrability conditions for an extended system of (0.1). For each of the other Painlevé equations $P_J$ ($J = I, \cdots, V$), such construction from integrability conditions was established firstly by R. Garnier ([2]) without any mention about monodromy property and later more precise consideration has been done by M. Jimbo, T. Miwa, K. Ueno ([4, 16]) and by K. Okamoto ([11]). In this paper we do not enter into details of the theory of monodromy preserving deformation; we give below the list of singularities.
of linear equations $L_J$ ($J = II, \cdots, VI$).

<table>
<thead>
<tr>
<th></th>
<th>Singularities of $L_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{VI}$</td>
<td>(1, 1, 1, 1)</td>
</tr>
<tr>
<td>$P_V$</td>
<td>(1, 1, 2)</td>
</tr>
<tr>
<td>$P_{IV}$</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>$P_{III}$</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>$P_{II}$</td>
<td>(4)</td>
</tr>
</tbody>
</table>

In this table, $(r_1, r_2, \cdots, r_m)$ means that $L_J$ has $m$ singular points with the Poincaré ranks, $r_1 - 1, r_2 - 1, \cdots, r_m - 1$, respectively. We can thus regard each Painlevé equation $P_J$ ($J = II, \cdots, VI$) as corresponding to a partition of 4 through the monodromy preserving deformation. Note that the first Painlevé equation $P_I$ contains no constant parameter and there is no correspondence to any partition of 4.

A generalization of the sixth Painlevé equation $P_{VI}$ was also obtained by R. Garnier ([2]) from the viewpoint of the theory of monodromy preserving deformation. He considered the monodromy preserving deformation of the linear differential equation of second order of the form (0.1), with $n + 3$ regular singularities and $n$ apparent singularities $x = \lambda_j$, whose Riemannian scheme is given by

\[
\begin{pmatrix}
  x = 0 & x = 1 & x = \infty & x = t_i & x = \lambda_j \\
  0 & 0 & \alpha & 0 & 0 \\
  \kappa_0 & \kappa_1 & \alpha + \kappa_\infty & \theta_i & 2
\end{pmatrix}, \quad i, j = 1, \cdots, n,
\]

(0.3)

where

\[
\alpha = -\frac{1}{2} \left( \kappa_0 + \kappa_1 + \kappa_\infty + \sum_i \theta_i - 1 \right).
\]

Then he obtained the system of nonlinear partial differential equations for $\lambda_j = \lambda_j(t)$, called the Garnier system in $n$-variables.

It is known ([3, 7, 12]) that through a certain change of variables the Garnier system in $n$-variables is equivalent to the following Hamiltonian system:

\[
\frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i, j = 1, \cdots, n),
\]

(0.4)
Bilinear Forms of the Garnier Systems

with Hamiltonians:

\[ H_i = \frac{1}{s_i(s_i - 1)} \left( \sum_{j,k} E_{jk}^i(s,q)p_j p_k - \sum_j F_j^i(s,q)p_j + \kappa q_i \right). \]

Here \( E_{jk}^i = E_{kj}^i \), \( F_j^i \in \mathbb{C}(s)[q] \) are given by

\[ E_{jk}^i = \begin{cases} 
q_i q_j q_k, & \text{if } i \neq j \neq k \neq i, \\
q_i q_j (q_j - R_{ji}), & \text{if } i \neq j = k, \\
q_i q_k (q_i - R_{ik}), & \text{if } i = j \neq k, \\
q_i (q_i - 1)(q_i - s_i) - \sum_{l(\neq i)} S_{il} q_i q_l, & \text{if } i = j = k,
\end{cases} \]

\[ F_j^i = \begin{cases} 
A q_i q_j - \theta_i R_{ij} q_j - \theta_j R_{ji} q_i, & \text{if } i \neq j, \\
(q_0 - 1)q_i(q_i - 1) + \kappa_1 q_i(q_i - s_i) \ \\
+ \theta_i(q_i - 1)(q_i - s_i) \ \\
+ \sum_{k(\neq i)} (\theta_k q_i(q_i - R_{ik}) - \theta_i S_{ik} q_k), & \text{if } i = j,
\end{cases} \]

with

\[ R_{ij} = \frac{s_i(s_j - 1)}{s_j - s_i}, \quad S_{ij} = \frac{s_i(s_i - 1)}{s_i - s_j}, \]

\[ A = \kappa_0 + \kappa_1 + \sum_l \theta_l - 1, \quad \kappa = \frac{1}{4}(A^2 - \kappa_\infty^2). \]

In the same way as the Painlevé equations, we can regard the Garnier system in \( n \)-variables as corresponding to the partition \((1, 1, \cdots, 1)\) of \( n + 3 \).

It is well known that the confluence of singularities of \( L_J \) causes the step-by-step degeneration of the Painlevé equations \( P_J \) ([12]). In a way similar to the Painlevé equations, the degeneration of the Garnier system can be considered. Many degenerate Garnier systems are studied by several authors ([5, 6, 8, 9, 15]). Each of degenerate Garnier systems corresponds to a certain partition of natural number through the theory of monodromy preserving deformation. In this paper we denote by \( G(#) \) the Painlevé equation or the (degenerate) Garnier system corresponding to the partition (#). For example, we refer by \( G(1,1,1,1) \) the Garnier system in 2-variables, by \( G(1,3) \) the fourth Painlevé equation \( P_{IV} \) and so on; see Figure 1.
Painlevé equations

\[
(1, 1, 1, 1) \rightarrow (1, 1, 2) \rightarrow (1, 3) \rightarrow (4)
\]

Garnier systems in 2-variables

\[
(1, 1, 1, 1) \rightarrow (1, 1, 2) \rightarrow (1, 1, 3) \rightarrow (1, 4) \rightarrow (5)
\]

Fig. 1. degeneration scheme

In this paper we study special solutions, Hirota bilinear forms, and birational symmetries of the Garnier system in 2-variables \(G(1, 1, 1, 1, 1)\). Particular solutions of the system which are described in terms of hypergeometric functions in 2-variables, are known ([14]), and we discuss in the present article other types of particular solutions; we consider special solutions of \(G(1, 1, 1, 1, 1)\), given in terms of solutions of the sixth Painlevé equation \(P_{VI}\); see Theorem 2.1. Moreover, we will see that for \((#) = (1, 1, \cdots, 1)\), the Garnier system \(G(1, #)\) has particular solutions given in terms of solutions of \(G(#)\). It is natural to make the following conjecture.

**Conjecture.** For any partition \((#)\) of an integer \(n (\geq 4)\), \(G(1, #)\) has a particular solution written in terms of solutions of \(G(#)\).

If the statement of the conjecture is true, we denote this fact simply by

\[
G(1, #) \supset G(#).
\]

For example, we obtain (0.10) for \((#) = (1, 1, \cdots, 1)\); see Theorem 6.1. Moreover we can verify (0.10) for \((1, 1, 2), (1, 3), (4), (5)\); details will be discussed in forthcoming papers.

The second subject of the investigation concerns Hirota bilinear forms of the Garnier system \(G(1, 1, 1, 1, 1)\), which plays very important roles in this paper. In fact, we study special solutions and birational symmetries by means of them.
Let us consider, for example, \( P_{II} \), which is equivalent to the following Hamiltonian system \( \mathcal{H}(\alpha) \):

\[
\frac{dq}{ds} = \frac{\partial H}{\partial p}, \quad \frac{dp}{ds} = -\frac{\partial H}{\partial q},
\]

(0.11)

with the Hamiltonian \( H = H(\alpha) \):

\[
H = \frac{1}{2}p^2 - \left(q^2 + \frac{s}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q.
\]

(0.12)

By defining \( d\log \tau(\alpha) = H(\alpha)ds \), Hirota bilinear forms of \( P_{II} \) are described as

\[
\left(D^2 + \frac{s}{2}\right)g \cdot f = 0,
\]

(0.13)

\[
\left(D^3 + \frac{s}{2}D - \alpha\right)g \cdot f = 0,
\]

(0.14)

where \( f = \tau(\alpha), g = \tau(\alpha - 1) \) and \( D \) is the Hirota derivative with respect to \( d/ds \). If we put \( f = 1 \) and \( \alpha = -1/2 \), above bilinear forms reduce to the linear differential equation for \( g \):

\[
\left(\frac{d^2}{ds^2} + \frac{s}{2}\right)g = 0,
\]

which is the Airy differential equation. This gives a classical solution of \( P_{II} \); see [15].

Return to bilinear forms (0.13)-(0.14), it is easy to see that these are invariant under the action \( w : (f, g; \alpha) \mapsto (g, f; -\alpha) \). This trivial symmetry can be lifted to a birational canonical transformation of \( \mathcal{H}(\alpha) \). And the fixed solution with respect to \( w \), \( (f, g; \alpha) = (\exp(-s^3/24), \exp(-s^3/24); 0) \), gives a rational solution of \( P_{II} \), \( (q, p; \alpha) = (0, s/2; 0) \).

In the present article we study mainly the Garnier system in 2-variables \( G(1, 1, 1, 1, 1) \). We will give particular solutions which are described in terms of solutions of \( P_{VI} \) (Theorem 2.1), Hirota bilinear forms (Theorem 3.2), birational symmetries (Theorem 4.1, 4.3) and consider algebraic solutions. Finally we consider the Garnier system in \( n \)-variables, where we denote it by \( G_n \). We obtain particular solutions of \( G_n \) given in terms of solutions of \( G_{n-1} \) (Theorem 6.1).
1. Hamiltonian System of $G(1,1,1,1,1)$

The Garnier system $G(1,1,1,1,1)$ is equivalent to the Hamiltonian system

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i, j = 1, 2),$$

with Hamiltonians:

$$(1.1) \quad s_1(s_1 - 1)H_1 = \left( q_1(q_1 - 1)(q_1 - s_1) - \frac{s_1(s_1 - 1)}{s_1 - s_2}q_1q_2 \right)p_1^2$$

$$+ 2q_1q_2 \left( q_1 - \frac{s_1(s_2 - 1)}{s_2 - s_1} \right)p_1p_2$$

$$+ q_1q_2 \left( q_2 - \frac{s_2(s_1 - 1)}{s_1 - s_2} \right)p_2^2$$

$$- \left\{ (\kappa_0 - 1)q_1(q_1 - 1) + \kappa_1q_1(q_1 - s_1) \right.\right.$$

$$\left. + \theta_1(q_1 - 1)(q_1 - s_1) \right.\right.$$

$$\left. + \theta_2q_1 \left( q_1 - \frac{s_1(s_2 - 1)}{s_2 - s_1} \right) - \theta_1 \frac{s_1(s_1 - 1)}{s_1 - s_2}q_2 \right\} p_1$$

$$- \left\{ \theta q_1q_2 - \theta_2q_1 \frac{s_2(s_1 - 1)}{s_1 - s_2} - \theta_1q_2 \frac{s_1(s_2 - 1)}{s_2 - s_1} \right\} p_2$$

$$+ \kappa q_1,$$

and $H_2$ is of the form obtained by the replacement

$$\{ q_1 \leftrightarrow q_2, p_1 \leftrightarrow p_2, s_1 \leftrightarrow s_2, \theta_1 \leftrightarrow \theta_2 \},$$

in $H_1$. Here we consider $\vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \theta_2) \in \mathbb{C}^5$ as parameters and put $\kappa = (\theta - \kappa_\infty)(\theta + \kappa_\infty)/4$, $\theta = \kappa_0 + \kappa_1 + \theta_1 + \theta_2 - 1$.

2. Particular Solutions

It is known ([14]) that $G(1,1,1,1,1)$ with certain special values of parameters admits a particular solution expressed in terms of Appell’s hypergeometric function $F_1(\alpha, \beta, \beta'; \gamma; x, y)$. In this section we show that $G(1,1,1,1,1)$ admits a particular solution expressed in terms of the sixth Painlevé transcendent; in fact we have the
Theorem 2.1. If $\theta_2 = 0$, then $G(1, 1, 1, 1, 1)$ has a particular solution of the form:

$$q_2 = 0, \quad \frac{\partial q_1}{\partial s_2} = \frac{\partial p_1}{\partial s_2} = 0.$$  

Moreover $(q_1, p_1)$ satisfy

$$s_1(s_1 - 1) \frac{\partial q_1}{\partial s_1} = 2q_1(q_1 - 1)(q_1 - s_1)p_1$$

$$- \left\{ (\kappa_0 - 1)q_1(q_1 - 1) + \kappa_1 q_1(q_1 - s_1) + \theta_1(q_1 - 1)(q_1 - s_1) \right\},$$

$$s_1(s_1 - 1) \frac{\partial p_1}{\partial s_1} = - \left\{ 3q_1^2 - 2(s_1 + 1)q_1 + s_1 \right\} p_1^2$$

$$+ \left\{ (\kappa_0 - 1)(2q_1 - 1) + \kappa_1 (2q_1 - s_1) + \theta_1(2q_1 - s_1 - 1) \right\} p_1 - \kappa,$$

which is equivalent to the sixth Painlevé equation $P_{VI}$. And $p_2$ satisfies Riccati type equations whose coefficients are polynomials in $(q_1, p_1)$.

Proof. Consider the case $\theta_2 = 0$. Take $q_2 = 0$ then

$$s_1(s_1 - 1)H_1 = q_1(q_1 - 1)(q_1 - s_1)p_1^2$$

$$- \left\{ (\kappa_0 - 1)q_1(q_1 - 1) + \kappa_1 q_1(q_1 - s_1) + \theta_1(q_1 - 1)(q_1 - s_1) \right\} p_1 + \kappa q_1.$$  

This is nothing but the Hamiltonian of $P_{VI}$. And it can be verified by computations,

$$\frac{\partial q_1}{\partial s_2} = \frac{\partial H_2}{\partial p_1} = 0, \quad \frac{\partial p_1}{\partial s_2} = - \frac{\partial H_2}{\partial q_1} = 0.$$  

Thus $q_1(s_1), p_1(s_1)$ are solved by the solutions of $P_{VI}$. Also it can be seen easily that $p_2(s_1, s_2)$ satisfies Riccati type equations:

$$s_1(s_1 - 1) \frac{\partial p_2}{\partial s_1} = \frac{s_2(s_1 - 1)}{s_1 - s_2} q_1 p_2^2$$

$$- \left\{ (2q_1 p_1 - \theta)q_1 + \frac{s_1(s_2 - 1)}{s_2 - s_1} (2q_1 p_1 - \theta_1) \right\} p_2$$
As is shown by Theorem 2.1, we have $G(1,1,1,1,1) \supset G(1,1,1,1,1)$; the sixth Painlevé equation $P_{VI}$ is contained in the two dimensional Garnier system, $G(1,1,1,1,1)$.

3. $\tau$-functions and Hirota Bilinear Forms

We can verify that, for the Hamiltonians of the Garnier system,

$$
\frac{\partial H_i}{\partial s_j} = \sum_{k=1,2} \left( \frac{\partial H_i}{\partial q_k} \frac{\partial q_k}{\partial s_j} + \frac{\partial H_i}{\partial p_k} \frac{\partial p_k}{\partial s_j} \right) + \left( \frac{\partial}{\partial s_j} \right) H_i
$$

$$
= \sum_{k=1,2} \left( \frac{\partial H_i}{\partial q_k} \frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k} \frac{\partial H_j}{\partial q_k} \right) + \left( \frac{\partial}{\partial s_j} \right) H_i
$$

$$
= \left( \frac{\partial}{\partial s_j} \right) H_i,
$$

where $(\partial/\partial s_j)$ denotes differentiation with respect to $s_j$ such that $(q,p)$ are viewed to be independent of $s$. By the use of (1.1), we have

$$
(3.1) \quad \frac{\partial H_1}{\partial s_2} = \frac{\partial H_2}{\partial s_1} = \frac{A(q,p,s)}{(s_1 - s_2)^2},
$$

$$
(3.2) \quad A(q,p,s) = q_1 q_2 p_1^2 - 2q_1 q_2 p_1 p_2 + q_1 q_2 p_2^2
$$

$$
+ (\theta_2 q_1 - \theta_1 q_2) p_1 + (\theta_1 q_2 - \theta_2 q_1) p_2.
$$

It is not difficult to show the

Proposition 3.1 ([7]). The 1-form $\omega \equiv H_1 ds_1 + H_2 ds_2$ is closed.
Then we can define, up to multiplicative constants, the \( \tau \)-function, \( \tau = \tau(\vec{\kappa}) \), related to \( G(1,1,1,1) \) as follows:

\[
(3.3) \quad \omega = d \log \tau.
\]

Now we set a pair of \( \tau \)-functions \((f,g)\) as

\[
\begin{align*}
(3.4) & \quad d \log f = H_1 ds_1 + H_2 ds_2, \\
(3.5) & \quad d \log g = \overline{H}_1 ds_1 + \overline{H}_2 ds_2,
\end{align*}
\]

where

\[
(3.6) \quad s_i \overline{H}_i = s_i H_i + x_i, \quad x_i = -q_i p_i, \quad (i = 1, 2).
\]

**Remark.** (i) Existence of the function \( g = g(s_1, s_2) \) satisfying (3.5) is assured by means of the equation:

\[
\frac{\partial x_1}{s_2^2 \partial s_2} = \frac{\partial x_2}{s_1 \partial s_1}.
\]

(ii) If we write as \( f = \tau(\vec{\kappa}) \), then we will see later that \( g = \tau(\rho(\vec{\kappa})) = \tau(R_\tau(\vec{\kappa})) \), where \( \rho(\vec{\kappa}) = (\kappa_0 + 1, \kappa_1 - 1, \kappa_\infty, \theta_1, \theta_2) \) and \( R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2) \).

Now we recall the definition of Hirota derivatives (in 2-variables):

\[
(3.7) \quad P(D_1, D_2)g \cdot f = P(D_1, D_2)(g(s + t)g(s - t))|_{t_1 = t_2 = 0},
\]

where \( P(D_1, D_2) \) is a polynomial in \((D_1, D_2)\) and \( D_i \) is a derivation. In this paper we deal with

\[
(3.8) \quad D_i = s_i \frac{\partial}{\partial s_i}.
\]

By definition we have

\[
\begin{align*}
(3.9) & \quad D_ig \cdot f = (D_ig)f - g(D_if), \\
(3.10) & \quad D_iD_jg \cdot f = (D_iD_jg)f - (D_ig)(D_jf) - (D_jg)(D_if) + g(D_iD_jf),
\end{align*}
\]
for $i, j = 1, 2$. It is easy to verify the following identities:

\begin{align}
\tag{3.11} D_i \log \frac{g}{f} &= \frac{D_i g \cdot f}{g \cdot f}, \\
\tag{3.12} D_i D_j \log gf &= \frac{D_i D_j g \cdot f}{g \cdot f} - \frac{D_i g \cdot f D_j g \cdot f}{g \cdot f}, \\
\tag{3.13} D_i^2 D_j \log \frac{g}{f} &= \frac{D_i^2 D_j g \cdot f}{g \cdot f} - \frac{D_i^2 g \cdot f D_j g \cdot f}{g \cdot f} \\
&\quad - 2 \frac{D_i D_j g \cdot f D_i g \cdot f}{g \cdot f} + 2 \left( \frac{D_i g \cdot f}{g \cdot f} \right)^2 \frac{D_j g \cdot f}{g \cdot f},
\end{align}

for $i, j = 1, 2$.

For the pair of $\tau$-functions $(f, g)$, we have the

**Theorem 3.2.** The pair of $\tau$-functions $(f, g)$ satisfies bilinear equations of the forms:

\begin{align}
\tag{3.14} B_1(g, f; \kappa) + s_1(s_2 - 1)B_2(g, f; \kappa) &= 0, \\
\tag{3.15} \frac{s_1 - 1}{s_1} B_3(g, f; \kappa) + \frac{(s_1 - 1)^2}{s_1} B_4(g, f; \kappa) + \frac{s_1^2 - s_2}{s_1 s_2} B_5(g, f; \kappa) \\
&\quad + 2(\kappa_0 - \kappa_1)(s_1 - s_2)B_2(g, f; \kappa) + B_6(g, f; \kappa) = 0,
\end{align}

and satisfies also the equations obtained by the replacement \{ $s_1 \leftrightarrow s_2, \theta_1 \leftrightarrow \theta_2$ \} in (3.14), (3.15). Here $D_i$ is the Hirota derivative and $B_i(g, f; \kappa)$ are given by:

\begin{align}
\tag{3.16} B_1(g, f; \kappa) &= (s_1 - 1)D_1^2 g \cdot f \\
&\quad + \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}D_1 g \cdot f \\
&\quad + (s_1 + 1)g \cdot D_1 f, \\
\tag{3.17} B_2(g, f; \kappa) &= \frac{1}{s_1 + s_2} \left( 2D_1 D_2 g \cdot f + \theta_2 D_1 g \cdot f + \theta_1 D_2 g \cdot f \right), \\
\tag{3.18} B_3(g, f; \kappa) &= (s_1 - 1)D_1^3 g \cdot f \\
&\quad + \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}D_1^2 g \cdot f \\
&\quad + (s_1 + 1)D_1 g \cdot D_1 f \\
&\quad + \frac{s_1}{s_1 - 1} \left( \{(\kappa_0 - \kappa_1 - \kappa_\infty)(\kappa_0 - \kappa_1 + \kappa_\infty) + \theta_1(2\kappa_0 + 2\kappa_1 + \theta_1 - 2)\}D_1 g \cdot f \right)
\end{align}
\[
+(\kappa_1 - \kappa_0)(g \cdot D_1 f + D_1 g \cdot f) + 2\theta_1 \kappa g \cdot f,
\]

(3.19) \( B_4(g, f; \vec{\kappa}) = 2D_1^2D_2 g \cdot f + \theta_2 D_1^2 g \cdot f + \theta_1 D_1D_2 g \cdot f, \)

(3.20) \( B_5(g, f; \vec{\kappa}) = (s_2 - 1)D_1D_2^2 g \cdot f
\]
\[
+ \{(\kappa_1 + \theta_2)s_2 - (\kappa_0 + \theta_2 - 1)\}D_1D_2 g \cdot f
\]
\[
+(s_2 + 1)D_1 g \cdot D_2 f,
\]

(3.21) \( B_6(g, f; \vec{\kappa}) = \theta_2(2\theta_1 + \theta_2)D_1 g \cdot f + 2\theta_1(\kappa_0 + \kappa_1 - 1)D_2 g \cdot f. \)

**Remark.** (i) If we put \( \theta_2 = 0, \frac{\partial g}{\partial s_2} = \frac{\partial f}{\partial s_2} = 0, \) then above bilinear forms (3.14)-(3.15) reduce to the following:

(3.22) \( (s_1 - 1)D_1^2 g \cdot f + \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}D_1 g \cdot f
\]
\[
+(s_1 + 1)D_1 g \cdot D_1 f = 0,
\]

(3.23) \( (s_1 - 1)D_1^2 g \cdot f + \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}D_1^2 g \cdot f
\]
\[
+(s_1 + 1)D_1 g \cdot D_1 f
\]
\[
+ \frac{s_1}{s_1 - 1}\left(\{(\kappa_0 - \kappa_1 - \kappa_\infty)(\kappa_0 - \kappa_1 + \kappa_\infty)
\]
\[
+ \theta_1(2\kappa_0 + 2\kappa_1 + \theta_1 - 2)\}D_1 g \cdot f
\]
\[
+(\kappa_1 - \kappa_0)(g \cdot D_1 f + D_1 g \cdot f) + 2\theta_1 \kappa g \cdot f\right) = 0.
\]

These are equivalent to the bilinear forms of \( P_{VI} \) ([15]).

(ii) If we put \( \kappa = 0, f = 1, \) then the bilinear forms of \( G(1, 1, 1, 1, 1) \) reduce to the system of linear partial differential equations for \( g, \) which is equivalent to Appell’s hypergeometric differential equation.

**Proof of Theorem 3.2.** Recall the definitions of \( \tau \)-functions \( (f, g): \)

(3.24) \( s_i H_i = D_i \log f, \quad s_i \overline{H}_i = D_i \log g, \)

where

(3.25) \( s_i \overline{H}_i = s_i H_i + x_i, \quad x_i = -q_i p_i, \)

for \( i = 1, 2. \) Using the formulae (3.11)-(3.13), we have expressions of Hirota derivatives of \( (f, g), \) in terms of \( x_i \) and \( H_i, \) as follows:

(3.26) \( \frac{D_i g \cdot f}{g \cdot f} = x_i, \)
366 Teruhisa Tsuda

\[(3.27) \quad \frac{\partial_i \partial_j g \cdot f}{g \cdot f} = 2D_j(s_i H_i) + D_j x_i + x_i x_j,\]

\[(3.28) \quad \frac{\partial_i^2 \partial_j g \cdot f}{g \cdot f} = D_i^2 x_j - (2D_i(s_i H_i) + D_i x_i + x_i^2) x_j
- 2(2D_j(s_i H_i) + D_j x_i + x_i x_j) x_i + 2x_i^2 x_j.\]

Put these into \(B_i(g, f; \vec{\kappa})\), we can verify the bilinear relations (3.14) and (3.15) by computations. □

4. Birational Symmetries

In this section we consider birational symmetries of \(G(1, 1, 1, 1, 1)\). The Hamiltonians \(H_i (i = 1, 2)\) are invariant under the action: \(\kappa_\infty \mapsto -\kappa_\infty\). This trivial symmetry can be lifted to a birational canonical transformation of \(G(1, 1, 1, 1, 1)\).

On the other hand, from the viewpoint of monodromy preserving deformations, H. Kimura constructed birational symmetries of \(G(1, 1, 1, 1, 1)\) which act on the parameters as permutations; see [3].

Then combining the above results, we obtain the following theorem.

**Theorem 4.1.** There exist birational canonical transformations

\[\mathcal{H}(\vec{\kappa}) \rightarrow \mathcal{H}(R_\Delta(\vec{\kappa}))\]

of \(G(1, 1, 1, 1, 1)\), where \(\mathcal{H}(\vec{\kappa}) = (q(\vec{\kappa}), p(\vec{\kappa}), H(\vec{\kappa}), s)\). Here the transformations \(R_\Delta: (q, p) \mapsto (Q, P)\) are given as follows:

<table>
<thead>
<tr>
<th>(R_\Delta)</th>
<th>action on (\vec{\kappa})</th>
<th>(Q_i) ((i = 1, 2))</th>
<th>(P_i) ((i = 1, 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_{\kappa_\infty})</td>
<td>(\kappa_\infty \mapsto -\kappa_\infty)</td>
<td>(Q_i = q_i)</td>
<td>(P_i = p_i)</td>
</tr>
<tr>
<td>(R_{\kappa_1})</td>
<td>(\kappa_1 \mapsto -\kappa_1)</td>
<td>(Q_i = q_i)</td>
<td>(P_i = p_i - \frac{\kappa_1}{q_1 + q_2 - 1})</td>
</tr>
<tr>
<td>(R_{\kappa_0})</td>
<td>(\kappa_0 \mapsto -\kappa_0)</td>
<td>(Q_i = q_i)</td>
<td>(P_i = p_i - \frac{\kappa_0}{s_i(q_1/s_1 + q_2/s_2 - 1)})</td>
</tr>
<tr>
<td>(R_{\theta_1})</td>
<td>(\theta_1 \mapsto -\theta_1)</td>
<td>(Q_i = q_i)</td>
<td>(P_1 = p_1 - \theta_1/q_1, \quad P_2 = p_2)</td>
</tr>
<tr>
<td>(R_{\theta_2})</td>
<td>(\theta_2 \mapsto -\theta_2)</td>
<td>(Q_i = q_i)</td>
<td>(P_1 = p_1, \quad P_2 = p_2 - \theta_2/q_2)</td>
</tr>
</tbody>
</table>

\[(4.1)\]
Furthermore, another birational symmetry can be derived from the Hirota bilinear forms.

**Proposition 4.2.** Hirota bilinear forms of $G(1, 1, 1, 1, 1)$ are invariant under the action

$$R_\tau : (f, g; \vec{\kappa}) \mapsto (g, f; R_\tau(\vec{\kappa}))$$

where $R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2)$.

**Proof.** If $P(\mathcal{D})$ is a monomial, then we have:

$$P(\mathcal{D})g \cdot f = -P(\mathcal{D})f \cdot g \quad (P : \text{odd}),$$

$$P(\mathcal{D})g \cdot f = P(\mathcal{D})f \cdot g \quad (P : \text{even}),$$

and it is easy to verify that

$$g \cdot D_1 f = D_1 f \cdot g + f \cdot D_1 g,$$

$$D_1 g \cdot D_1 f = -D_1^2 f \cdot g - D_1 f \cdot D_1 g,$$

$$D_1 g \cdot D_2 f = -D_1 D_2 f \cdot g - D_1 f \cdot D_2 g,$$

which we use in the following.

Consider the exchange of $\tau$-functions $f \leftrightarrow g$ in Hirota bilinear forms of $G(1, 1, 1, 1, 1)$, (3.14)-(3.15), then we obtain again Hirota bilinear forms of $G(1, 1, 1, 1, 1)$ with parameters $R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2)$. For example we compute:

\begin{equation}
B_1(g, f; \vec{\kappa}) = (s_1 - 1)D_1^2 g \cdot f \\
+ \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}D_1 g \cdot f \\
+ (s_1 + 1)g \cdot D_1 f \\
= (s_1 - 1)D_1^2 f \cdot g \\
+ \{(-\kappa_1 - \theta_1 + 1)s_1 - (-\kappa_0 - \theta_1)\}D_1 f \cdot g \\
+ (s_1 + 1)f \cdot D_1 g \\
= B_1(f, g; R_\tau(\vec{\kappa})),
\end{equation}

similarly we have $B_2(g, f; \vec{\kappa}) = B_2(f, g; R_\tau(\vec{\kappa}))$ and $B_i(g, f; \vec{\kappa}) = -B_i(f, g; R_\tau(\vec{\kappa}))$ for $i = 3, 4, 5, 6$. We obtain thus Hirota bilinear forms of $G(1, 1, 1, 1, 1)$ with parameters $R_\tau(\vec{\kappa})$. \qed
This symmetry of \( \tau \)-functions can be lifted to a birational canonical transformation of \( G(1, 1, 1, 1, 1) \).

**Theorem 4.3.** There exists a birational canonical transformation

\[
R_\tau : \mathcal{H}(q_i, p_i; \vec{\kappa}) \to \mathcal{H}(Q_i, P_i; R_\tau(\vec{\kappa}))
\]

of \( G(1, 1, 1, 1, 1) \) where \( R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2) \) described as

\[
(4.3) \quad Q_i = \frac{s_ip_i (q_ip_i - \theta_i)}{(\alpha + q_1p_1 + q_2p_2)(\alpha + \kappa_\infty + q_1p_1 + q_2p_2)},
\]

\[
(4.4) \quad Q_iP_i = -q_ip_i,
\]

for \( i = 1, 2 \) with \( \alpha = -(\theta + \kappa_\infty)/2 \).

**Proof.** The transposition of \( \tau \)-functions \( R_\tau : f \leftrightarrow g \) yields the transposition of Hamiltonians \( H_i \leftrightarrow \overline{H}_i \); we have (4.4) from (3.6). On the other hand, since \( g = \tau(R_\tau(\vec{\kappa})) \), the following relation holds:

\[
(4.5) \quad H_i(Q, P, s, R_\tau(\vec{\kappa})) = \overline{H}_i = H_i(q, p, s, \vec{\kappa}) - \frac{q_ip_i}{s_i}.
\]

Using this and (4.4), we obtain (4.3). \( \square \)

**Remark.** We can construct a birational canonical transformation for \( G(1, 1, 1, 1, 1) \), called a contiguity relation, which realizes the action on the space of parameters as translation. Put

\[
(4.6) \quad \rho = R_{\kappa_1} \circ R_\tau \circ R_{\theta_1} \circ R_{\theta_2} \circ R_{\kappa_\infty} \circ R_{\kappa_0},
\]

then we have a birational canonical transformation

\[
\rho : \mathcal{H}(\vec{\kappa}) \to \mathcal{H}(\rho(\vec{\kappa})),
\]

where \( \rho(\vec{\kappa}) = (\kappa_0 + 1, \kappa_1 - 1, \kappa_\infty, \theta_1, \theta_2) \). The relation between the Hamiltonians is

\[
(4.7) \quad \rho(H_i) = H_i - \frac{q_ip_i}{s_i}, \quad (i = 1, 2),
\]

hence we have

\[
(4.8) \quad \overline{H}_i = \rho(H_i) = R_\tau(H_i), \quad (i = 1, 2),
\]

i.e., \( g = \tau(\rho(\vec{\kappa})) = \tau(R_\tau(\vec{\kappa})) \).
5. Algebraic Solutions

Now consider the birational canonical transformation, \( w = R_{\tau} \circ R_{\theta_1} \circ R_{\kappa_\infty} \):

\[
w : \mathcal{H}(q_i, p_i; \kappa) \rightarrow \mathcal{H}(Q_i, P_i; w(\kappa));
\]
we have \( w(\kappa) = (-\kappa_0 + 1, -\kappa_1 + 1, \kappa_\infty, \theta_1, \theta_2) \) and

\[
Q_i = \frac{s_i p_i (q_i p_i - \theta_i)}{(\alpha + q_1 p_1 + q_2 p_2)(\alpha + \kappa_\infty + q_1 p_1 + q_2 p_2)},
\]
(5.1)

\[
Q_i P_i = -q_i p_i + \theta_i,
\]
(5.2)

for \( i = 1, 2 \).

Put \( \kappa_0 = \kappa_1 = 1/2 \), then there is a fixed point with respect to the action \( w \):

\[
(q_i, p_i) = \pm \left( \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \frac{\kappa_\infty}{2 \sqrt{s_i}} \right)
\]
(5.3)

\( \times \) \( \kappa_\infty \), \( \theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty - \sqrt{s_i} \)

This gives an algebraic solution of \( G(1, 1, 1, 1, 1) \). By using birational symmetries, we can construct many other algebraic solutions. For example, when \( \kappa_0 = 1/2, \kappa_1 = -1/2 \), we have

\[
q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty},
\]
(5.4)

\[
p_i = \frac{\kappa_\infty}{2 \sqrt{s_i}} \cdot \frac{\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty - \sqrt{s_i}}{\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty}.
\]
(5.5)

For \( \kappa_0 = 1/2, \kappa_1 = 3/2 \), we have

\[
q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \frac{(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)^2 - s_i}{\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty} - 1,
\]
(5.6)

\[
p_i = \frac{\kappa_\infty}{2 \sqrt{s_i}} \cdot \frac{(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)^2 - 1}{(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty - \sqrt{s_i})}.
\]
(5.7)

For \( \kappa_0 = \kappa_1 = -1/2 \), we have

\[
q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty},
\]
(5.8)

\[
p_i = \frac{\kappa_\infty}{2 \sqrt{s_i}} \cdot \frac{(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty - \sqrt{s_i}) - 1}{(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)(\theta_1 \sqrt{s_1} + \theta_2 \sqrt{s_2} - \kappa_\infty)}.
\]
(5.9)
6. Particular Solutions of the Garnier System in $n$-variables

**Theorem 6.1.** For special values of parameters, the Garnier system in $n$-variables $\mathcal{G}_n$ admits a particular solution expressed in terms of solutions of $\mathcal{G}_{n-1}$.

**Proof.** If $\theta_n = 0$, $\mathcal{G}_n$ admits a particular solution as $q_n = 0$. Take $q_n = 0$, we obtain

$$\frac{\partial q_i}{\partial s_n} = \frac{\partial H_n}{\partial p_i} = 0,$$

$$\frac{\partial p_i}{\partial s_n} = -\frac{\partial H_n}{\partial q_i} = 0,$$

for $1 \leq i \leq n - 1$, i.e., $(q_i, p_i)$ do not depend on $s_n$. Put $\theta_n = 0$, $q_n = 0$ into the Hamiltonians $H_i$ (0.5) for $1 \leq i \leq n - 1$, then we obtain the Hamiltonians for $\mathcal{G}_{n-1}$. We do not enter into detail of computation. □

**Remark.** In her paper [10], M. Mazzocco obtains the same type of particular solutions, by considering the monodromy preserving deformation of a linear differential equations such that some monodromy matrices are reduced to $\pm I$.

*Acknowledgement.* The author would like to thank Professor K. Okamoto for useful advice.

**References**


Bilinear Forms of the Garnier Systems


(Received May 27, 2002)
(Revised November 29, 2002)

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Tokyo 153-8914, Japan