Fourier Expansion of Holomorphic Siegel Modular Forms of Genus $n$ along the Minimal Parabolic Subgroup

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Abstract. The aim of this paper is to establish a theory of Fourier expansion of holomorphic Siegel modular forms of genus $n$ along the minimal parabolic subgroup. There are two known Fourier expansions of holomorphic Siegel modular forms, i.e. classical Fourier expansion and Fourier-Jacobi expansion (cf. §6). We also give a comparison of our expansion with them.

0. Introduction

In this paper, we study a Fourier expansion of scalar-valued holomorphic Siegel modular forms of arbitrary genus with respect to the minimal parabolic subgroup. In the theory of automorphic forms, the investigation of their Fourier expansions along various parabolic subgroups is fundamental and gives us significant information on such theory. For example, it gives a starting point for the construction of automorphic $L$-functions (cf. A.N.Andrianov [1], W.Kohnen and N.P.Skoruppa [8]).

The Fourier expansions of holomorphic Siegel modular forms along the maximal parabolic subgroups have already been studied. Up to conjugation, there are $n$ maximal parabolic subgroups of the real symplectic group $Sp(n; \mathbb{R})$ of degree $n$. We have the unique maximal parabolic subgroup whose unipotent radical is abelian. This should be called the Siegel parabolic subgroup. The Fourier expansion along this parabolic subgroup is the most classical one. Its detailed investigation was initiated by C.L.Siegel (cf. [14]) in his theory of quadratic forms. The Fourier expansions along the other maximal parabolic subgroups are called Fourier-Jacobi expansions (cf. I.I.Piatetskii-Shapiro [12]). There are some detailed results [4], [17] etc. in the literature. In this paper, we are interested in the Fourier expansion along the minimal parabolic subgroup.

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Let us formulate our problem for a real semi-simple Lie group $G$ of Hermitian type, constructed from $\mathbb{R}$-rational points of a simple $\mathbb{Q}$-algebraic group. The group $Sp(n; \mathbb{R})$ gives an example of such a group. Additionally, let $K$ denote a maximal compact subgroup of $G$. The group $G$ has unitary representations called holomorphic discrete series and we can define holomorphic automorphic forms on $G$. Given some irreducible finite dimensional representation $(\tau, V_\tau)$ of the complexification $K_C$ of $K$, let $f$ be a $V_\tau$-valued holomorphic automorphic form on $G$ with respect to an arithmetic subgroup $\Gamma$. Let $N$ be the unipotent radical of the minimal parabolic subgroup of $G$ and $N_\Gamma := N \cap \Gamma$. We regard $f(xg)$ as a function in $x \in N$ with a fixed $g \in G$. From the $\Gamma$-invariance of $f$, we deduce $f(xg) \in L^2(N_\Gamma \backslash N) \otimes V_\tau$, where $L^2(N_\Gamma \backslash N)$ denotes the space of square-integrable functions on the quotient $N_\Gamma \backslash N$. Since $N_\Gamma \backslash N$ is compact, the space $L^2(N_\Gamma \backslash N)$ decomposes discretely into

$$L^2(N_\Gamma \backslash N) \cong \bigoplus_{(\eta,H_\eta) \in \hat{N}} \text{Hom}_N(\eta, L^2(N_\Gamma \backslash N)) \otimes H_\eta,$$

where $\bigoplus$ denotes the Hilbert space direct sum, $\hat{N}$ the unitary dual of $N$ and note $\dim_{\mathbb{C}} \text{Hom}_N(\eta, L^2(N_\Gamma \backslash N)) < \infty$. According to this decomposition, we have

$$f(*g) = \sum_Q \sum_{\eta \in \hat{N}} \sum_{1 \leq m \leq \dim(\eta)} (\Theta^m_\eta \otimes W^m_{\eta,Q}(g)) \otimes v_Q,$$

where $Q$ runs through an index set for a basis $\{v_Q\}$ of $V_\tau$, $\{\Theta^m_\eta\}_{1 \leq m \leq \dim(\eta)}$ is a basis of $\text{Hom}_N(\eta, L^2(N_\Gamma \backslash N))$ and $W^m_{\eta,Q}(g) \in H_\eta$ the $(\eta, m, Q)$-component of the decomposition. Via the evaluation map at $x \in N$, $\Theta^m_\eta \otimes W^m_{\eta,Q}(g)$ is identified with an element $\Theta^m_\eta(W^m_{\eta,Q}(g))(x)$ of $L^2(N_\Gamma \backslash N)$. Hence the decomposition above can be rewritten as

$$f(xg) = \sum_Q \sum_{\eta \in \hat{N}} \sum_{1 \leq m \leq \dim(\eta)} \Theta^m_\eta(W^m_{\eta,Q}(g))(x) \cdot v_Q.$$

We call this decomposition the Fourier expansion of the form $f$ along the minimal parabolic subgroup. In this paper, we consider the case where $G = Sp(n; \mathbb{R})$ and $\dim_{\mathbb{C}} V_\tau = 1$.

To investigate such an expansion, the following questions are fundamental:

1. Determine $W^m_{\eta,Q}$ explicitly;
(2) Describe the multiplicity \( m(\eta) \) concretely and find a basis of \( \text{Hom}_N(\eta, L^2(N_\Gamma \backslash N)) \) for each \( \eta \in \hat{N} \).

The function \( W^m_\eta = \sum Q W^m_\eta,Q \cdot v_Q \) is found to be a generalized Whittaker function for holomorphic discrete series with \( K \)-type \( \tau \) (for a definition, see Definition 4.1, which treats the case of the one-dimensional \( K \)-type). An explicit formula of \( W^m_\eta \) is obtained by solving the differential equations arising from the “Cauchy-Riemann condition” (for a definition, see the end of §4.1), which characterizes the minimal \( K \)-type of holomorphic discrete series. The results on the generalized Whittaker functions are given as Theorem 4.5, Theorem 4.7, Theorem 4.12 and Theorem 4.13. These are solutions for the problem (1). Here we state our explicit formula of generalized Whittaker functions.

**Theorem 0.1 (Theorem 4.7).** Let \( \pi_\kappa \) be a holomorphic discrete series with the one-dimensional minimal \( K \)-type \( \tau_\kappa \simeq \text{det}^\kappa \) (cf. §2). If there exists a non-zero generalized Whittaker functions for \( \pi_\kappa \) attached to \( \eta_l \in \hat{N} \) parametrized by \( l \in n^* \) (for a detail on \( \eta_l \), see §3), its explicit formula is given as

\[
W_{\kappa,l}(x_L a) = C(a_1 a_2 \cdots a_n)^\kappa \exp(-2\pi \text{Tr}(t(X_L A_n) Y_n(l)(X_L A_n))),
\]

where \( n^* \) denotes the dual space of \( n := \text{Lie}(N) \), \( (x_L, a) = \left( \left( X_L \right)^t X_L^{-1}, \left( A_n \right)^{-1} \right) \) \( \in N_L \times A \) with \( N_L \) (resp. \( A \)) denoting the subgroup of \( N \), canonically isomorphic to the standard maximal unipotent subgroup \( U_n \) of \( GL_n(\mathbb{R}) \) (resp. split component of an Iwasawa decomposition of \( G \)), \( Y_n(l) \) is a certain symmetric matrix of degree \( n \) attached to \( l \) (cf. §3) and \( C \) denotes an arbitrary constant.

By a result of L.Corwin and F.P.Greenleaf [2], we can describe the multiplicity \( m(\eta) \) in terms of the integral coadjoint orbit of a character inducing \( \eta \in \hat{N} \), and find a basis of the space of intertwining operators in the problem (2) by using a notion of theta series on \( N_\Gamma \backslash N \).

These solutions for the two problems (1) and (2) give our Fourier expansion, stated as Theorem 5.8 or equivalently as Theorem 5.10.

**Theorem 0.2 (Theorem 5.8).** Let \( (x, a) \in N \times A \) and let \( f \) be a holomorphic Siegel modular form on \( G \) of weight \( \kappa \) with respect to \( \text{Sp}(n; \mathbb{Z}) \).
Then a Fourier expansion of $f$ along the minimal parabolic subgroup is written as
\[
f(xa) = \sum_{l \in \tilde{L}} \sum_{l' \in \mathcal{M}(l)} C^l_{l'} \Theta_{l'}(W_{\kappa, l'}(*)a)(x),
\]
where $\Theta_{l'}(W_{\kappa, l'}(*)a)(x) :=
\sum_{l'' \in \text{Ad}_S^*N_L(Z)l'} \chi_{l''}(x_S)(a_1 a_2 \cdots a_n)^\kappa \exp(-2\pi \text{Tr}(l'(X_L A_n) Y_n(l'')(X_L A_n))).$

**Theorem 0.3 (Theorem 5.10).** Let $F_f$ be the holomorphic Siegel modular form on the Siegel upper half space $\mathcal{H}_n$ of degree $n$, constructed from the form $f$ on $G$ (cf. §5). The Fourier expansion in Theorem 5.8 is rewritten as
\[
F_f(Z) = \sum_{S \in \Omega_{n,Z}} \sum_{T \in \mathcal{M}(S)} C^S_T \Theta_T(Z),
\]
where $\Theta_T(Z) = \sum_{R \in \Omega_{n}(T)} \exp 2\pi \sqrt{-1} \text{Tr}(RZ)$.

Here we explain the notations in the two theorems above. The set $\Omega_{n,Z}$ denotes the $U_n(Q)$-equivalence classes of the set $\Omega_{n,Z}$ of symmetric positive semi-definite semi-integral matrices of degree $n$, with $U_n(Q) := U_n \cap GL_n(Q)$. The sets $\tilde{L}$ is the quotient of $L$ by co-adjoint $\text{Ad}_L(Q)$-action (denoted by $\text{Ad}_S^*$), where $L$ is a lattice in a certain subspace $n_S^*$ (cf. §5) of $n^*$, canonically bijective with $\Omega_{n,Z}$ and where $\text{N}_L(Q) := \text{N}_L \cap Sp(n; Q)$. These two sets $\Omega_{n,Z}$ and $\tilde{L}$ parametrize irreducible unitary representations of $N$ contributing to our Fourier expansion. For an $S \in \Omega_{n,Z}$ (resp. $l \in L$), $\mathcal{M}_n(S)$ (resp. $\mathcal{M}(l)$) denotes the quotient of the $U_n(Q)$ (resp. $\text{N}_L(Q)$)-equivalence class of $S$ (resp. $l$) in $\Omega_{n,Z}$ (resp. $\tilde{L}$) by $U_n(Z)$ (resp. $\text{N}_L(Z)$)-equivalence, where $U_n(Z) := U_n \cap GL_n(Z)$ and $\text{N}_L(Z) := \text{N}_L \cap Sp(n; Z)$. The cardinalities of two sets $\mathcal{M}(l)$ and $\mathcal{M}_n(S)$ are equal to the multiplicity $m(\eta_l)$ when $S$ corresponds to $l$ via $\Omega_{n,Z} \simeq L$. For a $T \in \Omega_{n,Z}$, $\Omega_n(T) := \{^t uTu \mid u \in U_n(Z)\}$. The theta series $\Theta_{l'}(W_{\kappa, l'}(*)a)(x)$ and $\Theta_T(Z)$ correspond to theta series $\Theta^m_{\eta_n}(W_{\eta_n,Q}^m(g))(x)$ in the formulation of our Fourier expansion. The constants $C^l_{l'}$ and $C^S_T$ denote the Fourier coefficients.

Here we give some remarks on our results. Generalized Whittaker functions for admissible representations are of great interest in terms of representation theory (cf. B.Kostant [9], H.Yamashita [16]). It is known that
holomorphic discrete series on semi-simple Lie groups of Hermitian type
do not admit any Whittaker models attached to non-singular characters
(for a definition of “non-singular characters”, see [9],§2.3). For holomor-
phic discrete series of $Sp(n;\mathbb{R})$ with one-dimensional $K$-type, our results,
Theorem 4.5, Theorem 4.7, Theorem 4.12 and Theorem 4.13 completely
describe Whittaker functions attached to all irreducible unitary representa-
tions of $N$. With regard to our theory of Fourier expansion, we have
already obtained such a Fourier expansion for vector valued holomorphic
Siegel modular forms of genus 2 and those of genus 3 (cf. [10],[11]). The
results in these previous papers are prototypes of our present study here.

Now we explain the contents of this paper. In §1, we introduce some
basic notations for the real symplectic group, its standard subgroups, the
associated Lie algebras, and the root systems for them. In §2, we give a
parametrization of holomorphic discrete series using Harish-Chandra’s the-
ory on discrete series of semi-simple Lie groups. In §3, we recall a classifica-
tion of irreducible unitary representations of $N$, using the “orbit method”
for nilpotent Lie groups, established by A.A.Kirillov. We also give a formula
for the infinitesimal actions of the representations of $N$. In §4, we obtain an
explicit formula for the generalized Whittaker function. To be more precise,
we first define the generalized Whittaker function in §4.1. In §4.2, we de-
duce the differential equations characterizing it from the Cauchy-Riemann
condition. In §4.3, we get an explicit formula for the generalized Whittaker
function by solving the differential equations. In §5, we express our Fourier
expansion using the generalized Whittaker functions obtained above and
the results of Corwin and Greenleaf [2]. In fact, this is accomplished by
constructing theta series from the generalized Whittaker functions. In §6,
we compare our expansion with the other two known Fourier expansions,
i.e. classical Fourier expansion and Fourier-Jacobi expansion. In §6.1, we
obtain a relation between Fourier coefficients of the classical expansion and
those of our expansion. The result is

**Theorem 0.4 (Theorem 6.1).** Let $T \in \Omega_{n,\mathbb{Z}}$ belong to $\mathfrak{m}_n(S)$ with
some $S \in \Omega_{n,\mathbb{Z}}$ and $C^S_T$ (resp. $C_T$) denote the Fourier coefficient of our
Fourier expansion in Theorem 5.10 (resp. classical Fourier expansion).
Then we have

$$C^S_T = C_T$$
and, for every \( u \in U_n(\mathbb{Z}) \),
\[
C_{tu}Tu = C^S_T.
\]

In §6.2, we study a relation between Fourier-Jacobi coefficients and theta series appearing in our Fourier expansion. Our result is stated as

**Theorem 0.5 (Theorem 6.4, Corollary 6.5).** (1) Let \( \phi_{R_1} \) be the Fourier-Jacobi coefficients of Fourier-Jacobi expansion of \( f \) indexed by \( R_1 \in \Omega_{j,\mathbb{Z}} \) with \( 1 \leq j \leq n - 1 \). Then one has
\[
\sum_{T_1 \in \mathcal{M}_j(S_1)} \sum_{R_1 \in \Omega_j(T_1)} \phi_{R_1}(Z_2, Z_3) \exp(2\pi \sqrt{-1} \text{Tr } R_1 Z_1) = \sum_{S \in \tilde{\Omega}_{S_1}} \sum_{T \in \mathcal{M}_n(S)} C^S_T \Theta_T(Z),
\]
where \( \Omega_{S_1} := \left\{ S = \begin{pmatrix} S_1 & S_2 \\ S_2^t & S_3 \end{pmatrix} \in \Omega_{n,\mathbb{Z}} \mid S_2 \in M_{j,n-j}(\mathbb{Q}), \ S_3 \in M_{j-n}(\mathbb{Q}) \right\} \)
and \( \tilde{\Omega}_{S_1} \) denotes the \( U_n(\mathbb{Q}) \)-equivalence classes of \( \Omega_{S_1} \).

(2) When \( j = 1 \), this formula becomes
\[
\phi_{S_1}(Z_2, Z_3) \exp 2\pi \sqrt{-1} \text{Tr } S_1 Z_1 = \sum_{T \in \Omega_{S_1}} \sum_{S \in \mathcal{M}_n(T)} C^S_T \Theta_S(Z).
\]

Theorem 0.4 and Theorem 0.5 tell us how the known two Fourier expansions and our Fourier expansion are related to each other. We hope that these two theorems provide us some new information on the two known expansions in terms of our theory of Fourier expansion.

Finally, the author would like to express his profound gratitude to Professor Takayuki Oda for his suggestion of this problem and constant encouragement, and also to Professor Werner Hoffmann for various advice, comments and reference to the paper [2].

1. **Basic Notations**

Let \( G = \text{Sp}(n; \mathbb{R}) \) be the real symplectic group of degree \( n \), defined by
\[
\{ g \in GL_{2n}(\mathbb{R}) \mid {}^t g J_n g = J_n \}.
\]
with \( J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \). We often use the block notation \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with \( A, B, C \) and \( D \in M_n(\mathbb{R}) \). Let \( \theta \) denote the Cartan involution defined by \( G \ni g \mapsto g^{-1} \) and \( K := \{ g \in G \mid \theta(g) = g \} \), which coincides with

\[
\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \right\}.
\]

Then the group \( K \) is a maximal compact subgroup of \( G \), which is isomorphic to the unitary group \( U(n) \) of degree \( n \) under the map

\[
K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B.
\]

Let \( \mathfrak{g} = \mathfrak{sp}(n; \mathbb{R}) \) be the Lie algebra of \( G \), which is given as

\[
\{ X \in M_{2n}(\mathbb{R}) \mid tX J_n + J_n X = 0_{2n} \}.
\]

We denote also by \( \theta \) the Cartan involution on \( \mathfrak{g} \) given by

\[
X \mapsto -tX.
\]

Let \( \mathfrak{k} = \{ X \in \mathfrak{g} \mid \theta(X) = X \} \) and \( \mathfrak{p} = \{ X \in \mathfrak{g} \mid \theta(X) = -X \} \). Then \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{g} \) admits a Cartan decomposition

\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.
\]

Throughout the subsequent argument, \( E_{ij} \) denotes the matrix unit \((\delta_{ij} \delta_{pq})_{1 \leq p, q \leq N}\) in the matrix algebra \( M_N \) (either \( N = n \) or \( N = 2n \)). In order to formulate the Iwasawa decomposition of \( \mathfrak{g} \), we prepare the restricted root system of it. Let \( \mathfrak{a} = \sum_{k=1}^{n} \mathbb{R} H_k \) with \( H_k = E_{kk} - E_{k+n,k+n} \), which is a maximal abelian subalgebra of \( \mathfrak{p} \). We write \( A \) for

\[
\exp(\mathfrak{a}) = \left\{ a = \begin{pmatrix} A_n \\ A_n^{-1} \end{pmatrix} \mid A_n = \text{diag}(a_1, a_2, \ldots, a_n), \ a_i \in \mathbb{R}_+ \right\}.
\]

The root system \( \Delta(\mathfrak{a}, \mathfrak{g}) \) of \((\mathfrak{g}, \mathfrak{a})\) is of type \( C_n \) and given by

\[
\{ \pm e_i \pm e_j, \ \pm 2e_k \mid 1 \leq i < j \leq n, \ 1 \leq k \leq n \},
\]

where \( e_i \) denotes the linear functional on \( \mathfrak{a} \) defined by \( e_i(H_j) = \delta_{ij} \). We denote by \( E_\alpha \) the root vector corresponding to a root \( \alpha \in \Delta(\mathfrak{a}, \mathfrak{g}) \), explicitly given as

\[
E_{e_i+e_j} = E_{i,j+n} + E_{j,i+n}, \ E_{e_i-e_j} = E_{ij} - E_{j+n,i+n},
\]

\[
E_{2e_k} = E_{k,k+n}, \ E_{-\alpha} = tE_\alpha.
\]
Let $\Delta^+(a, g) = \{e_i \pm e_j, \ 2e_k \mid 1 \leq i < j \leq n, \ 1 \leq k \leq n\}$ be the standard set of positive roots. Furthermore set $n = \sum_{\alpha \in \Delta^+(a, g)} \mathbb{R}E_\alpha$, which is the nilradical of the minimal parabolic subalgebra. Then we have an Iwasawa decomposition of $g$:

$$g = n \oplus a \oplus k.$$  

With $N := \exp(n)$, the group $G$ also has such decomposition

$$G = NAK.$$  

Additionally, we introduce a compact Cartan subalgebra $t$ given by

$$t = \oplus_{1 \leq k \leq n} \mathbb{R}T_k,$$

with $T_k = E_{k,k+n} - E_{k+n,k}$. Consider the root system $\Delta(t_C, g_C)$ of $(g_C, t_C)$, where $g_C$ and $t_C$ denote the complexifications of $g$ and $t$ respectively. This root system is also of type $C_n$ and given by

$$\{\pm f_i \pm f_j, \pm 2f_k \mid 1 \leq i < j \leq n, \ 1 \leq k \leq n\},$$

where $f_i$ denotes the linear functional on $t_C$ defined by $f_i(T_j) = \sqrt{-1}\delta_{ij}$. We denote by $F_\beta$ the root vector for $\beta \in \Delta(t_C, g_C)$, explicitly given as

$$F_{f_i+f_j} = E_{ij} + E_{ji} - E_{i+n,j+n} - E_{j+n,i+n}$$

$$+ \sqrt{-1}(E_{i,j+n} + E_{j,i+n} + E_{i+n,j} + E_{j+n,i}),$$

$$F_{2f_k} = E_{kk} - E_{k+n,k+n} + \sqrt{-1}(E_{k,k+n} + E_{k+n,k}),$$

$$F_{f_i-f_j} = E_{ij} - E_{ji} + E_{i+n,j+n} - E_{j+n,i+n}$$

$$- \sqrt{-1}(E_{i+n,j} + E_{j+n,i} - E_{i,j+n} - E_{j,i+n}),$$

$$F_{-\beta} = \bar{F}\beta.$$

The set $\Delta^+ = \{f_i \pm f_j, \ 2f_k \mid 1 \leq i < j \leq n, \ 1 \leq k \leq n\}$ forms the standard positive root system and $\Delta^+_n = \{f_i + f_j, \ 2f_k \mid 1 \leq i < j \leq n, \ 1 \leq k \leq n\}$ the set of non-compact positive roots. Put

$$p^+ = \oplus_{\beta \in \Delta^+_n} \mathbb{C}F_\beta, \ p^- = \oplus_{\beta \in \Delta^-_n} \mathbb{C}F_{-\beta} = \overline{p^+}.$$  

Then, in the complexification $p_C$ of $p$, these two spaces $p^+$ and $p^-$ form its holomorphic part and anti-holomorphic part respectively, and we have a decomposition of $g_C$:

$$g_C = t_C \oplus p^+ \oplus p^-.$$
In §4, we will consider the infinitesimal actions of the generators of $\mathfrak{p}^-$. For that purpose we introduce the Iwasawa decomposition of $F_{-\beta}$ for $\beta \in \Delta_n^+$, which is settled by direct computation.

**Lemma 1.1.** Let $\text{Ad} a$ denote the adjoint action of $a \in A$ on $\mathfrak{n}$. Then we have the following decompositions:

$$F_{-f_i-f_j} = 2a_i a_j^{-1} \text{Ad}(a^{-1}) E_{e_i-e_j} - 2a_i a_j \sqrt{-1} \text{Ad}(a^{-1}) E_{e_i+e_j} - F_{-f_i+f_j},$$
$$F_{-2f_k} = -2a_k^2 \sqrt{-1} \text{Ad}(a^{-1}) E_{2e_k} + H_k + \sqrt{-1} T_k.$$ 

2. **Holomorphic Discrete Series of $Sp(n; \mathbb{R})$**

We recall a notion of holomorphic discrete series representations of $Sp(n; \mathbb{R})$ in terms of Harish-Chandra’s parametrization of the discrete series representations of a semisimple Lie group (cf. [6], Chap.IX, §7, Theorem 9.20, Chap.XII, §5, Theorem 12.21). Consider an arbitrary continuous character on the compact Cartan subgroup $T := \exp(t)$, which is of the form:

$$T \ni \exp(\sum_{1 \leq i \leq n} \theta_i T_i) \mapsto \exp(\sqrt{-1}(\sum_{1 \leq i \leq n} \Lambda_i \theta_i)) \in U(1) \ (\theta_i \in \mathbb{R}),$$

where $(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \in \mathbb{Z}^{\oplus n}$ and $U(1)$ is the set of complex numbers of absolute value 1. The vector $(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ is identified with a differential of the above character and with a linear functional $\Lambda = \Lambda_1 f_1 + \Lambda_2 f_2 + \cdots + \Lambda_n f_n$ on $\mathfrak{t}_\mathbb{C}$. Such $\Lambda$ is called an analytically integral weight (cf. [6], Chap.IV, §5, Proposition 4.13). The subset $\{f_i - f_j \mid 1 \leq i < j \leq n\}$ of $\Delta(\mathfrak{t}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ forms a set of compact positive roots. We denote by $\rho$ and $\rho_c$ the half-sum of positive roots and that of compact positive roots, respectively. Due to the Harish-Chandra’s parametrization of discrete series, holomorphic discrete series representations of $Sp(n; \mathbb{R})$ can be parametrized by the following set of analytically integral weights:

$$\{\Lambda \mid \rho + \Lambda \text{ is analytically integral and} \Lambda \text{ is regular dominant with respect to } \Delta^+ \} \simeq \{\Lambda = \Lambda_1 f_1 + \Lambda_2 f_2 + \cdots + \Lambda_n f_n \mid (\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \in \mathbb{Z}^{\oplus n}, \ \Lambda_1 > \Lambda_2 > \cdots > \Lambda_n > 0\}$$
Such $\Lambda$'s are called Harish-Chandra parameters for holomorphic discrete series. We denote by $\pi_\Lambda$ the holomorphic discrete series with Harish-Chandra parameter $\Lambda$. The highest weight of the minimal $K$-type of $\pi_\Lambda$ is given by the special weight $\lambda = \Lambda + \rho - 2\rho_c = (\Lambda_1 + 1)f_1 + (\Lambda_2 + 2)f_2 + \cdots + (\Lambda_n + n)f_n$, which is called the Blattner parameter.

Let the minimal $K$-type $\tau_\lambda$ of $\pi_\Lambda$ be one-dimensional. Then the Harish-Chandra parameter $\Lambda$ (resp. Blattner parameter $\lambda$) is given as $(\kappa - 1)f_1 + (\kappa - 2)f_2 + \cdots + (\kappa - n)f_n$ (resp. $\kappa f_1 + \kappa f_2 + \cdots + \kappa f_n$) with $\kappa > n$. The minimal $K$-type $\tau_\lambda$ can be expressed as $\det^\kappa$ on $U_n$ via the isomorphism $K \simeq U_n$ in §1. We denote this $\tau_\lambda$ by $\tau_\kappa$ and $\pi_\kappa$ by the holomorphic discrete series with the minimal $K$-type $\tau_\kappa$.

3. Classification of Unitary Representation of $N$

The group $N = \exp(n)$ is the standard maximal unipotent subgroup of $G$. We want to describe the unitary dual $\hat{N}$ of $N$ using Kirillov’s construction of irreducible unitary representations of nilpotent Lie groups.

First we introduce some notations. The group $N$ can be written as $N_S \ltimes N_L$ with

$$N_S := \left\{ x_S = \begin{pmatrix} 1_n & X_S \\ 0_n & 1_n \end{pmatrix} \bigg| X_S \in M_n(\mathbb{R}), \, ^tX_S = X_S \right\},$$

$$N_L := \left\{ x_L = \begin{pmatrix} X_L & 0_n \\ 0_n & tX_L^{-1} \end{pmatrix} \bigg| X_L \in U_n \right\},$$

where $U_n$ denotes the standard maximal unipotent subgroup of $GL_n(\mathbb{R})$. We denote the $(i, j)$-component of $X_S$ (resp. $X_L$) by $x_{ij}$ (resp. $x'_{ij}$). Let $n_S$ (resp. $n_L$) be the Lie algebra of $N_S$ (resp. $N_L$). Then these two Lie algebras are given as

$$n_S = \oplus_{1 \leq i < j \leq n} \mathbb{R}E_{e_i + e_j}, \quad n_L = \oplus_{1 \leq i < j \leq n} \mathbb{R}E_{e_i - e_j}$$

and we have

$$n = n_S \oplus n_L.$$
For $l \in \mathfrak{n}^*$, let $\mathfrak{m}$ denote a polarization subalgebra with respect to an inner product $l([*,*])$ on $\mathfrak{n}$ (for a definition, see [3], p27-p28), where $[*,*]$ denotes the bracket product on $\mathfrak{n}$. Furthermore, set $M := \exp(\mathfrak{m})$ and let $\chi_l : M \to U(1)$ be a character defined as $\chi_l(m) := \exp(2\pi \sqrt{-1}l(\log(m)))$ for $m \in M$. Using these notations, we state the following theorem established by A.A.Kirillov (cf. [3], Theorems 2.2.1-2.2.4).

**Proposition 3.1.** (1) Every $\eta \in \hat{N}$ is unitarily equivalent to a representation of the form

$$\eta_l := L^2 - \text{Ind}_{\mathfrak{m}}^N \chi_l$$

with some $l \in \mathfrak{n}^*$. Up to unitary equivalence, $\eta_l$ does not depend on the choice of $M$. Additionally, we remark that, if $\eta_l$ is not a character, a representation space $H_{\eta_l}$ of $\eta_l$ is given as

$$\begin{cases} h : \text{measurable function on } N & h(mx) = \chi_l(m)h(x) \\ & \text{for } (m,x) \in M \times N \\ & \|h\|^2_l := \int_{M \setminus N} h(x)\overline{h}(x)d\hat{x} < \infty \end{cases},$$

where $d\hat{x}$ denotes an invariant measure on the quotient $M \setminus N$. 

(2) Two representations $\eta_l$ and $\eta_{l'}$ with $l$ and $l' \in \mathfrak{n}^*$ are unitarily equivalent if and only if $l = \text{Ad}^* x \cdot l'$ with some $x \in N$. That is, we have a bijection $\hat{N} \simeq \mathfrak{n}^*/\text{Ad}^* N$.

In order to find convenient choices of polarization subalgebras for our argument, we state

**Lemma 3.2.** (1) This time, let $\mathfrak{n}$ be a general $m$-dimensional nilpotent Lie algebra with a chain of ideals

$$\{0\} \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \ldots \subset \mathfrak{n}_m = \mathfrak{n},$$

where $\dim \mathfrak{n}_i = i$. For $l \in \mathfrak{n}^*$, we set $l_i := l|_{\mathfrak{n}_i}$ and

$$r_{\mathfrak{n}_i}(l_i) := \{X \in \mathfrak{n}_i \mid l_i([X,Y]) = 0 \ \forall Y \in \mathfrak{n}_i\}.$$
Then $\sum_{i=1}^{m} r_n (l_i)$ forms a polarization subalgebra for $l$.

(2) Let $\mathfrak{n}$ be our Lie algebra. It has a filtration of ideals satisfying the condition in (1), given by

\[
\{0\} \subset \mathfrak{n}_{11} \subset \mathfrak{n}_{12} \subset \ldots \subset \mathfrak{n}_{1n} \subset \mathfrak{n}_{22} \subset \ldots \subset \mathfrak{n}_{2n} \subset \ldots \\
\subset \mathfrak{n}_{ii} \subset \ldots \subset \mathfrak{n}_{in} \subset \ldots \subset \mathfrak{n}_{nn} = \mathfrak{n}_S \\
\subset \mathfrak{n}^\prime_{1n} \subset \mathfrak{n}^\prime_{1,n-1} \subset \ldots \subset \mathfrak{n}^\prime_{12} \subset \mathfrak{n}^\prime_{2n} \subset \ldots \subset \mathfrak{n}^\prime_{23} \subset \ldots \\
\subset \mathfrak{n}^\prime_{in} \subset \ldots \subset \mathfrak{n}^\prime_{i,i+1} \subset \ldots \subset \mathfrak{n}^\prime_{n-1,n} = \mathfrak{n}.
\]

Here

\[
\mathfrak{n}_{ij} := \bigoplus_{u<v} \mathbb{R} \mathfrak{e}_{eu+ev} \bigoplus_{i \leq v \leq j} \mathbb{R} \mathfrak{e}_{ei+ev} \subset \mathfrak{n}_S \quad \text{for } 1 \leq i \leq j \leq n,
\]

and

\[
\mathfrak{n}^\prime_{ij} := \mathfrak{n}_S \bigoplus \bigoplus_{u<v} \mathbb{R} \mathfrak{e}_{eu-ve} \bigoplus_{v \geq j} \mathbb{R} \mathfrak{e}_{ei-ev} \quad \text{for } 1 \leq i < j \leq n.
\]

**Proof.** For a proof of (1), see [3], Theorem 1.3.5. Regarding (2), we can check that each subspace in the filtration above forms an ideal of $\mathfrak{n}$ by direct computation. \qed

For each $l$, we can take a polarization subalgebra $\mathfrak{m}$ so that it contains $\mathfrak{n}_S$. In fact, since $\mathfrak{r}_{mn} (l|\mathfrak{n}_n) = \mathfrak{n}_S$, Lemma 3.2 implies that there exists such a polarization subalgebra. In terms of Proposition 3.1 (1), there is no loss of generality if we impose the following condition on $\mathfrak{m}$:

**Assumption 1.** From now on, we assume that the polarization subalgebra $\mathfrak{m}$ for an $l$ contains $\mathfrak{n}_S$.

For $l \in \mathfrak{n}^*$, we set

\[
Y_n (l) := \begin{pmatrix}
\xi_{11} & \xi_{12}/2 & \cdots & \xi_{1n}/2 \\
\xi_{12}/2 & \xi_{22} & \cdots & \xi_{2n}/2 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{1n}/2 & \xi_{2n}/2 & \cdots & \xi_{nn}
\end{pmatrix}.
\]
Consider the total set of indices $[1,n] = \{1,2,\ldots,n\}$. For $l \in \mathfrak{n}^*$, we define a subset $I(l)$ of indices by

$$I(l) := \begin{cases} 
\{i \in [1,n] \mid \text{there exists } j \in [1,n] \\
\text{such that } \xi_{ij} \neq 0 \text{ or } \xi_{ji} \neq 0\} & \text{if } Y_n(l) \neq 0_n; \\
\{n\} & \text{if } Y_n(l) = 0_n.
\end{cases}$$

We set $r = \# I(l)$ and write $I(l)$ as $\{n_1,n_2,\ldots,n_r\}$ with $n_1 < n_2 < \ldots < n_r$.

**Proposition 3.3.** Let $l \in \mathfrak{n}^*$ satisfy $I(l) \neq \{n\}$.

1. A polarization subalgebra $\mathfrak{m}$ for $l$ satisfies $$E_{e_{n_p} - e_j} \not\in \mathfrak{m} \text{ for any } (n_p,j) \text{ with } n_p \in I(l) \text{ and } j > n_p.$$  

2. We define an $r \times r$ symmetric matrix $Y(l)$ by

\[
\begin{pmatrix}
\xi_{n_1 n_1} & \xi_{n_1 n_2} / 2 & \cdots & \xi_{n_1 n_r} / 2 \\
\xi_{n_1 n_2} / 2 & \xi_{n_2 n_2} & \cdots & \xi_{n_2 n_r} / 2 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n_1 n_r} / 2 & \xi_{n_2 n_r} / 2 & \cdots & \xi_{n_r n_r}
\end{pmatrix},
\]

and its minor matrices $Y(l)_s$ by

\[
\begin{pmatrix}
\xi_{n_1 n_1} & \xi_{n_1 n_2} / 2 & \cdots & \xi_{n_1 n_s} / 2 \\
\xi_{n_1 n_2} / 2 & \xi_{n_2 n_2} & \cdots & \xi_{n_2 n_s} / 2 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n_1 n_s} / 2 & \xi_{n_2 n_s} / 2 & \cdots & \xi_{n_s n_s}
\end{pmatrix}
\]

for $1 \leq s \leq r$. Moreover, we set

$$n_l := \begin{cases} \bigoplus_{i,j \text{ s.t. } i \not\in I(l), j > i} \mathbb{R}E_{e_i - e_j} & (I(l) \neq [1,n]); \\
\{0\} & (I(l) = [1,n]).
\end{cases}$$

Assume that $\det Y(l)_s \neq 0$ for $1 \leq s \leq r$ (resp. $1 \leq s \leq r-1$) if $n_r \neq n$ (resp. $n_r = n$) in $I(l)$. Then a polarization subalgebra $\mathfrak{m}$ satisfies

$$\mathfrak{m} \subset \mathfrak{n}_s \oplus n_l.$$
In particular, if \( l \) additionally satisfies \( l(n_I) = \{0\} \), then \( m = n_S \oplus n_I \).

**Proof.** (1) Let \( m \) contain an \( E_{e_{n_p},e_j} \) for some \( (n_p,j) \) with \( n_p \in I(l) \) and \( j > n_p \). Then \( l([E_{e_{n_p},e_j},n_S]) = \{0\} \) has to hold since we assume that \( m \supset n_S \). But there is a non-zero \( \xi_{n_p,n_q} \) or \( \xi_{n,q} \) with some \( n_q \in I(l) \), so that \( l([E_{e_{n_p},e_j},E_{e_{n_q},e_j}]) \neq 0 \) or \( l([E_{e_{n_p},e_j},E_{e_{j},e_{n_q}}]) \neq 0 \). This is a contradiction. Hence \( E_{e_{n_p},e_j} \notin m \).

(2) Let \( X := \sum_{n_p \in I(l), \ j > n_p} a_{n_p,j} E_{e_{n_p},e_j} \) with \( a_{n_p,j} \in \mathbb{R} \) belong to a subspace \( \bigoplus_{n_p \in I(l), \ j > n_p} \mathbb{R}E_{e_{n_p},e_j} \), complementary to \( n_S \oplus n_I \) in \( n \). In order to prove (2), it suffices to show the following claim:

\[
\quad
\begin{align*}
\quad
l([X,n_S]) = \{0\} \text{ means } X = 0.
\end{align*}
\]

If this is settled, we obtain \( m \subset n_S \oplus n_I \). In fact, decompose \( Y \in n \) into \( Y = Y_{S,I} + Y' \) with \( Y_{S,I} \in n_S \oplus n_I \) and \( Y' \) in the complementary subspace. In order that \( Y \) belongs to \( m \), \( l([Y,n_S]) = \{0\} \) has to hold. Since \( l([n_S \oplus n_I,n_S]) = \{0\} \), \( l([Y',n_S]) = \{0\} \) is satisfied. Therefore we get \( Y' = 0 \) if the claim is proved. Hence \( m \subset n_S \oplus n_I \). Moreover if \( l(n_I) = \{0\} \), \( m = n_S \oplus n_I \) holds since \( l([n_S \oplus n_I,n_S \oplus n_I]) = \{0\} \).

We start proving the claim. Let \( l([X,n_S]) = \{0\} \). By direct computation, we see that

\[
\begin{align*}
\quad
l([X,E_{e_{n_p},e_j}]) =
\begin{cases}
\quad
\sum_{1 \leq s \leq p-1} \xi_{n_s,n_p} a_{n_s,j} + 2 \xi_{n_p,n_p} a_{n_p,j} \\
\quad
+ \sum_{p+1 \leq s \leq k(j)} \xi_{n_p,n_s} a_{n_s,j} = 0 & (j \notin I(l), j > n_p);
\end{cases}
\end{align*}
\]

where we set \( k(j) := \max\{s \mid n_s < j, n_s \in I(l)\} \). From these formulas, we
have

\[ Y(l)_{k(j)} \begin{pmatrix} a_{nj} \\ a_{nj} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{dcases} 0_k(j) & \text{for } j \notin I(l) \text{ with } j > n_1; \\
-\frac{\xi_n}{2}a_{nj}n_2 & \text{for } j \in I(l) \text{ with } j > n_1, \\
-\sum_{1 \leq t \leq k(j)-1} \frac{\xi_n}{2}a_{nt}n_{k(j)} & \end{dcases} \]

where \( 0_k(j) \) denotes the zero vector in \( \mathbb{R}^{k(j)} \). From the assumption on the minor matrices \( Y(l)_{k(j)} \), we see that \( a_{nj} = 0 \) for any \( (n_p, j) \) with \( n_p \in I(l) \) and \( j > n_p \). Therefore we obtain \( X = 0 \) and complete the proof. \( \square \)

Let \( H^\infty_{\eta} \) denote the space of \( C^\infty \)-vectors in \( H_{\eta} \). We calculate the infinitesimal actions of generators of \( \mathfrak{n} \) on \( H^\infty_{\eta} \) via the differential \( d\eta \) of \( \eta \). For that purpose, we denote by \( \xi_{ij}(tX_LY_n(l)X_L) \) the coefficient of \( \frac{1}{2}(E_{ij} + E_{ji}) \) in \( tX_LY_n(l)X_L \) for \( 1 \leq i \leq j \leq n \).

**Proposition 3.4.** (1) \( d\eta(E_{e_i+e_j}) = 2\pi \sqrt{-1} \xi_{ij}(tX_LY_n(l)X_L) \) for \( 1 \leq i \leq j \leq n \).
(2) \( d\eta(E_{e_i-e_j}) = \frac{d}{dx_{ij}} + \sum_{1 \leq u < i} x'_{ui} \frac{d}{dx_{u_j}} \) for \( 1 \leq i < j \leq n \).

**Proof.** Let \( h \in H^\infty_{\eta} \). Then \( y = yS_{yL} \in N \) with \( yS \in N_S \) and \( yL \in N_L \) acts on \( H^\infty_{\eta} \) by right translation \( R_N \):

\[ R_N(y)h(x) = h(x_S x_L y_S y_L) = h(x_S(yS x_{yS} x_L^{-1}) x_L y_L) = \chi_l(x_S(yS x_{yS} x_L^{-1})) h(x_L y_L). \]

We compute \( d\eta(X) = dR_N(X) \) for each generator \( X \in \mathfrak{n} \). To show the formula (1), the following obvious equality is convenient.

**Lemma 3.5.** The matrix \( Y_n(l) \) is characterized by the following property:

\[ l(\log(x_S)) = \text{Tr}(Y_n(l)X_S) \text{ with } x_S = \begin{pmatrix} 1_n \\ X_S \end{pmatrix} \in N_S. \]
Noting this and a well-known formula $\text{Tr}(XY) = \text{Tr}(YX)$ for two matrices $X$ and $Y$ with the same degree, we have

$$d\eta_l(E_{e_i+e_j}) = d\chi_l(x_L E_{e_i+e_j} x_L^{-1}) = 2\pi \sqrt{-1} l(x_L E_{e_i+e_j} x_L^{-1})$$

$$= 2\pi \sqrt{-1} \text{Tr}(Y_n(l) X_L(E_{ij} + E_{ji})^t X_L)$$

$$= 2\pi \sqrt{-1} \text{Tr}((t^t X_L Y_n(l) X_L)(E_{ij} + E_{ji}))$$

$$= 2\pi \sqrt{-1} \xi_{ij}(t^t X_L Y_n(l) X_L)$$

for $(i, j)$ with $i < j$. We can compute $d\eta_l(E_{2e_k})$ similarly.

In order to obtain the formula (2), we note the equality

$$x_L \exp(tE_{e_i-e_j}) = 1_{2n} + (x_{ij}' + t) E_{e_i-e_j} + \sum_{1 \leq u < i} (x_{uj}' + x_{ui}' t) E_{e_u-e_j}$$

$$+ \sum_{(u,v) \notin \{(w,j) | 1 \leq w \leq i\}} x_{uv}' E_{e_u-e_v},$$

which can be checked by direct computation. Using this, compute the differential $\frac{d}{dt} \bigg|_{t=0} h(x_L \exp(tE_{e_i-e_j})).$ $\square$

4. Generalized Whittaker Functions on $G$ for Holomorphic Discrete Series

4.1. Definition

Recall that $\pi_\kappa$ denotes the holomorphic discrete series representation of $G$ with the minimal $K$-type $\tau_\kappa$ (cf. §2).

Let $\iota: \tau_\kappa \to \pi_{\kappa K}$ be the inclusion map of $\tau_\kappa$ into the space $\pi_{\kappa K}$ of $K$-finite vectors in $\pi_\kappa$. We simply denote $\pi_{\kappa K}$ by $\pi_\kappa$. Before giving the definition of generalized Whittaker functions, we introduce the following spaces:

$$C^\infty_{\eta_l}(N\backslash G) := \{ F : H^\infty_{\eta_l}\text{-valued } C^\infty\text{-function on } G \mid F(xg) = \eta_l(x) F(g) \};$$

$$C^\infty_{\eta_l}(N\backslash G)_K := \{ F \in C^\infty_{\eta_l}(N\backslash G) \mid F \text{ is } K\text{-finite}\};$$

$$C^\infty_{\eta_l, \tau_\kappa}(N\backslash G/K) := \{ W : H^\infty_{\eta_l}\text{-valued } C^\infty\text{-function on } G \mid W(xgk) = \eta_l(x) \tau_\kappa(k)^{-1} W(g) = \eta_l(x) \tau_\kappa(k) W(g) \},$$
where \((x, g, k) \in N \times G \times K\), \(\tau^*_\kappa\) denotes the contragredient representation of \(\tau_\kappa\). The two spaces \(\pi_\kappa\) and \(C^\infty_{\eta_l} (N \backslash G)_K\) form \((g_C, K)\)-modules respectively (for a definition of a \((g_C, K)\)-module, see [15], Chap.0, §3, Definition 0.3.8).

**Definition 4.1.** Let \(\iota^*\) be a map defined as

\[
\iota^* : \text{Hom}_{(g_C, K)}(\pi_\kappa, C^\infty_{\eta_l} (N \backslash G)_K) \ni F \mapsto F \cdot \iota \in \text{Hom}_K(\tau_\kappa, C^\infty_{\eta_l} (N \backslash G)_K).
\]

An element of \(\text{Im} \iota^*\) is called a generalized Whittaker function on \(G\) for the representation \(\pi_\kappa\) with \(K\)-type \(\tau_\kappa\).

We have a canonical identification

\[
\text{Hom}_K(\tau_\kappa, C^\infty_{\eta_l} (N \backslash G)_K) \simeq C^\infty_{\eta_l, \tau^*_\kappa}(N \backslash G/K).
\]

Now we introduce

\[
S_{\eta_l, \tau^*_\kappa}(N \backslash G/K) := \left\{ W : \mathbb{C}\text{-valued } C^\infty\text{ function on } G \mid \begin{array}{l}
W(gk) = \tau_\kappa(k)W(g) \\
W(xg) \in H^\infty_{\eta_l}
\end{array} \right\},
\]

where \((g, k) \in G \times K\) and we regard \(W(xg)\) as a function in \(x \in N\) with a fixed \(g \in G\). Here note that \(\tau^*_\kappa(k) = \tau_\kappa(k)^{-1}\). Then we have a bijection

\[
C^\infty_{\eta_l, \tau^*_\kappa}(N \backslash G/K) \simeq S_{\eta_l, \tau^*_\kappa}(N \backslash G/K)
\]

via the evaluation map \(W(g)(\ast) \mapsto W(g)(1)\) at \(1 \in N\), since \(W(g)(x_0) = \eta_l(x_0)W(g)(x) = W(x_0g)(x)\) for \(x_0, x \in N\) and \(g \in G\).

The holomorphic discrete series \(\pi_\kappa\) forms a highest weight module with highest weight \(\kappa(f_1 + f_2 + \cdots + f_n)\) (cf. [16],Proposition 7.4). Due to this and [16],Proposition 12.2, we obtain a bijection

\[
\text{Im} \iota^* \simeq \{ W \in S_{\eta_l, \tau^*_\kappa}(N \backslash G/K) \mid dR_X W = 0 \ \forall X \in p^- \}
\]

by the method of highest weight module, where \(dR\) denotes the differential of right regular representation \(R\) of \(G\) on the space of \(C^\infty\)-functions on \(G\). The condition

\[
dR_X W = 0 \quad \forall X \in p^-
\]

is called the Cauchy-Riemann condition.
4.2. Explicit formulas for differential equations

From the Cauchy-Riemann condition, we obtain the differential equations characterizing the generalized Whittaker functions. Let \( W_{\kappa,l} \) denote a generalized Whittaker function attached to \( \pi_\kappa \) and \( \eta_l \). It is determined by its restriction to \( NA \) because of its \( K \)-equivariance. Furthermore recall that, for each \( l \in n^* \), a polarization subalgebra \( m \) is assumed to be taken so that \( m \supset n_S \) (cf. §3, Assumption 1). Therefore we see that \( W_{\kappa,l}(x_S x_L a) = \chi_l(x_S) W_{\kappa,l}(x_L a) \) with \( (x_S, x_L, a) \in N_S \times N_L \times A \). Hence it suffices to consider the differential equations for the restriction of \( W_{\kappa,l} \) to \( N_L A \). In order to simplify the equations, we introduce the Euler operators \( \partial_k := a_k \frac{\partial}{\partial a_k} \) for \( 1 \leq k \leq n \). Using the infinitesimal actions \( \partial_k \) and \( d\eta_l \), the Cauchy-Riemann condition can be rewritten as the following differential equations:

**Proposition 4.2.** (1) The conditions \( dR_{F_{-e_i-e_j}} W_{\kappa,l} = 0 \) with \( 1 \leq i < j \leq n \) are equivalent to 
\[
a_i a_j^{-1} d\eta_l(E_{e_i-e_j}) W_{\kappa,l} - \sqrt{-1} a_i a_j d\eta_l(E_{e_i+e_j}) W_{\kappa,l} = 0 \quad (1 \leq i < j \leq n).
\]

(2) The conditions \( dR_{F_{-2e_k}} W_{\kappa,l} = 0 \) with \( 1 \leq k \leq n \) are equivalent to 
\[
\partial_k W_{\kappa,l} - 2\sqrt{-1} a_k^2 d\eta_l(E_{2e_k}) W_{\kappa,l} - \kappa W_{\kappa,l} = 0 \quad (1 \leq k \leq n).
\]

**Proof.** Note that the infinitesimal actions of \( \mathfrak{t} \) via the differential \( d\tau_\kappa \) of \( \tau_\kappa \) are given as follows;
\[
d\tau_\kappa(F_{\pm(f_i-f_j)}) = 0 \text{ for } 1 \leq i < j \leq n \quad d\tau_\kappa(T_k) = \sqrt{-1}\kappa \text{ for } 1 \leq k \leq n.
\]
The formulas in the assertion follow from the Iwasawa decompositions of \( F_{-e_i-e_j} \) and \( F_{-2e_k} \) in Lemma 1.1 and from the above formula of \( d\tau_\kappa \). □

Inserting the formulas in Proposition 3.4 into Proposition 4.2, we get more explicit forms of the differential equations for \( W_{\kappa,l} \).

**Proposition 4.3.** (1) The differential equations in Proposition 4.2 (1) are rewritten as
\[
\left( \frac{d}{dx_{ij}} + \sum_{1 \leq u < i} x_{u}^{t_i} \frac{d}{dx_{ui}} \right) W_{\kappa,l}(x_L a) + 2\pi a_j^2 \xi_{ij} (t^i X_L Y_n(l) X_L) W_{\kappa,l}(x_L a) = 0
\]
for $1 \leq i < j \leq n$.

(2) The differential equations in Proposition 4.2 (2) are rewritten as

$$\partial_k W_{\kappa,l}(x_L a) + 4\pi a^2_\kappa \xi_{kk}(t X_L Y_n(l)(X_L A_n)) W_{\kappa,l}(x_L a) - \kappa W_{\kappa,l}(x_L a) = 0$$

for $1 \leq k \leq n$.

### 4.3. Explicit formula of generalized Whittaker functions

In this subsection, we solve the differential equations in Proposition 4.3 and obtain explicit formulas of generalized Whittaker functions. To simplify the equations, we need

**Lemma 4.4.** (1) For $1 \leq i < j \leq n$, we have

$$\left( \frac{d}{dx'_{ij}} + \sum_{1 \leq u < i} x'_{ui} \frac{d}{dx'_{uj}} \right) \left( \operatorname{Tr}(t X_L A_n) Y_n(l)(X_L A_n) \right) = a^2_\kappa \xi_{ij}(t X_L Y_n(l)(X_L))$$

(2) For $1 \leq k \leq n$, we have

$$\partial_k \operatorname{Tr}(t X_L A_n) Y_n(l)(X_L A_n) = 2a^2_\kappa \xi_{kk}(t X_L Y_n(l)(X_L)).$$

For notations $X_L$ and $A_n$, see the definition of $N_L$ in §3 and the definition of $A$ in §1 respectively.

**Proof.** (1) The proof of Proposition 3.4 (2) implies that a $C^\infty$-function $f$ on $N_L$ satisfies

$$\left( \frac{d}{dx'_{ij}} + \sum_{1 \leq u < i} x'_{ui} \frac{d}{dx'_{uj}} \right) f(x_L) = \left. \frac{d}{dt} \right|_{t=0} f(x_L \exp(tE_{e_i - e_j})).$$

Apply this to $f(x_L) := \operatorname{Tr}(t X_L A_n) Y_n(l)(X_L A_n))$ with a fixed $A_n$. Then we have

$$\left( \frac{d}{dx'_{ij}} + \sum_{1 \leq u < i} x'_{ui} \frac{d}{dx'_{uj}} \right) \operatorname{Tr}(t X_L A_n) Y_n(l)(X_L A_n))$$

$$= \left. \frac{d}{dt} \right|_{t=0} \operatorname{Tr}(t X_L (1_n + tE_{e_i}) A_n) Y_n(l)(X_L (1_n + tE_{e_j}) A_n))$$
\[
\begin{aligned}
&= \lim_{t \to 0} \left( \frac{a_i^2 + a_j^2 t^2}{t} \xi_{ii} (tX_n l X_L) + \frac{a_j^2 t}{t} \xi_{ij} (tX_n l X_L) \right) \\
&= a_j^2 \xi_{ij} (tX_n l X_L).
\end{aligned}
\]

(2) This is settled by a calculation as follows;

\[
\partial_k \Tr( t(X_L A_n) Y_n (l)(X_L A_n)) = a_k \frac{\partial}{\partial a_k} \Tr( tX_n l X_L A_n^2) = 2a_k^2 \xi_{kk} (tX_n l X_L).
\]

We first consider the generalized Whittaker functions attached to a representation \( \eta \) in the following case:

**Case 1.** \( I(l) \neq [1, n] \), and for some \((i, j)\) with \( i \not\in I(l) \) and \( j > i \), \( \xi'_{ij} \neq 0 \) and \( E_{ei-ej} \in m \) holds, i.e. \( d\chi_l(E_{ei-ej}) = 2\pi \sqrt{1-\xi'_{ij}} \neq 0 \).

**Theorem 4.5.** Let \( W_{\kappa,l} \) be a generalized Whittaker functions attached to \( \eta_l \) in Case 1. Then we have \( W_{\kappa,l} \equiv 0 \).

**Proof.** We set \( i(l) := \min\{i \not\in I(l) \mid \xi'_{ij} \neq 0 \text{ and } E_{ei-ej} \in m \text{ for some } j\} \) and \( j(l) := \max\{j \mid \xi'_{ij} \neq 0 \text{ and } E_{ei-l-ej} \in m \} \) for \( l \in n^* \). Furthermore, we give the following order for the set \( I := \{(i, j) \in [1, n] \times [1, n] \mid 1 \leq i < j \leq n \} \):

\[
(i,j) > (i', j') \text{ for any } (j, j') \text{ if } i > i' \\
(i,j) > (i, j') \text{ if } j' > j,
\]

and define a subset \( I_{i(l)j(l)} \) of \( I \) by

\[
I_{i(l)j(l)} := \{(i, j) \in I \mid 1 \leq i < i(l), j > i \} \cup \{(i(l), j) \in I \mid j \geq j(l) \}.
\]

We set

\[
W'_{\kappa,l}(x_l a) := \exp(2\pi \Tr( t(X_L A_n) Y_n (l)(X_L A_n)))W_{\kappa,l}(x_l a).
\]
Here we need

**Lemma 4.6.** (1) For $l \in \mathfrak{n}^*$, we have

$$Y_n(\text{Ad}^* x_L^{-1} \cdot l) = t X_L Y_n(l) X_L$$

for $x_L = \begin{pmatrix} X_L & t X_L^{-1} \end{pmatrix} \in N_L$.

(2) The function $W'_{\kappa,l}(x_L a)$ satisfies

$$W'_{\kappa,l}(m_L x_L a) = \chi_l(m_L) W'_{\kappa,l}(x_L a)$$

for $m_L \in M \cap N_L$.

**Proof.** (1) By direct computation, one obtains

$$\text{Ad}^* x_L^{-1} \cdot l \begin{pmatrix} 0_n & X_S \\ 0_n & 0_n \end{pmatrix} = l \begin{pmatrix} 0_n & X_L X_S^{-1} X_L \\ 0_n & 0_n \end{pmatrix}$$

$$= \text{Tr}(Y_n(l) X_L X_S^{-1} X_L) = \text{Tr}(t X_L Y_n(l) X_L X_S).$$

Lemma 3.5 means

$$Y_n(\text{Ad}^* x_L^{-1} \cdot l) = t X_L Y_n(l) X_L.$$

(2) Since $\chi_l = \chi_{\text{Ad}^* m^{-1} \cdot l}$ and $\chi_l|_{N_S} = \chi_{\text{Ad}^* m^{-1} \cdot l}|_{N_S}$ for $m \in M$, we have $Y_n(l) = Y_n(\text{Ad}^* m^{-1} \cdot l)$. This and the assertion (1) means that $\exp(2\pi \text{Tr}(t X_L A_n) Y_n(l)(X_L A_n)))$ is left $M \cap N_L$-invariant. Since $W'_{\kappa,l}(m_L x_L a) = \chi_l(m_L) W'_{\kappa,l}(x_L a)$ for $m_L \in M \cap N_L$, the assertion (2) holds.

Inserting $W_{\kappa,l}(x_L a) = \exp(-2\pi \text{Tr}(t X_L A_n) Y_n(l)(X_L A_n)))W'_{\kappa,l}(x_L a)$ into the differential equations in Proposition 4.3 (1) and noting Lemma 4.4 (1), we get

$$\left( \frac{d}{dx'_{ij}} + \sum_{1 \leq u < i} x'_{ui} \frac{d}{dx'_{x_{uj}}} \right) W'_{\lambda,l}(x_L a) = 0 \quad (1 \leq i < j \leq n).$$

From these differential equations for $(i, j) \in I_{i(l)j(l)} \setminus \{(i(l), j(l))\}$, we observe that

$$\frac{d}{dx'_{ij}} W'_{\lambda,l} = 0 \quad \text{for any } (i, j) \in I_{i(l)j(l)} \setminus \{(i(l), j(l))\},$$

by induction on the order of $(i, j) \in I$. The validity of this is assured by the condition in Case 1, Lemma 4.6 (2) and Proposition 3.3 (1). In particular,
Proposition 3.3 (1) implies that \( \frac{d}{dx_{ij}} \) for \((i, j)\) with \(i \in I(l)\) and \(j > i\) is the non-trivial derivation in a direction \(E_{e_i-e_j}\), transversal to \(m\). Noting the formulas just above and Lemma 4.6 (2), we observe that the differential equation

\[
\left( \frac{d}{dx_{ij}} + \sum_{1 \leq u < i} x_{ui(l)} \frac{d}{dx_{uj(l)}} \right) W'_{\lambda,l}(x_La) = 0
\]

is equivalent to

\[
\frac{d}{dx_{ij(l)}} W'_{\lambda,l}(x_La) = 2\pi \sqrt{-1} \xi'_{ij(l)} W'_{\lambda,l} = 0.
\]

This implies \(W_{\kappa,l} \equiv 0\). □

Now we consider the generalized Whittaker function attached to a representation \(\eta_l\) not in Case 1. Namely we assume that \(\eta_l\) is in the following case:

**Case 2.** \(\eta_l\) satisfies one of the following conditions:

1. \(I(l) = [1, n]\);
2. \(I(l) \neq [1, n]\), and for any \((i, j)\) with \(i \notin I(l)\) and \(j > i\), \(\xi'_{ij} = 0\) or \(E_{e_i-e_j} \notin m\) holds.

**Theorem 4.7.** If \(\eta_l\) is in Case 2, we obtain the unique solution

\[W_{\kappa,l}(x_La) = C(a_1a_2 \cdots a_n)^\kappa \exp(-2\pi \text{Tr}(X_L A_n Y_n(l)(X_L A_n)))\]

of the differential equations in Proposition 4.3, up to an arbitrary constant \(C\).

**Proof.** Let \(W'_{\kappa,l}(x_La)\) and the set \(I\) be as in the proof of Theorem 4.5. Inserting \(W_{\kappa,l}(x_La) = \exp(-2\pi \text{Tr}(X_L A_n Y_n(l)(X_L A_n)))W'_{\kappa,l}(x_La)\) into the differential equations (1) and (2) in Proposition 4.3 and noting Lemma 4.4, we obtain

\[
(1) \left( \frac{d}{dx_{ij}} + \sum_{1 \leq u < i} x_{ui(l)} \frac{d}{dx_{uj(l)}} \right) W'_{\kappa,l}(x_La) = 0 \quad (1 \leq i < j \leq n),
\]

\[
(2) \partial_k W'_{\kappa,l}(x_La) - \kappa W'_{\kappa,l}(x_La) = 0 \quad (1 \leq k \leq n).
\]
From (1) and the two conditions in Case 2, we see that \( \frac{d}{dx} W_{\kappa,l}(x_L a) = 0 \) for any \((i, j) \in I\) by induction on \((i, j)\) with respect to the order of \(I\) given in the proof of Theorem 4.5. That is, \( W_{\kappa,l}(x_L a) \) does not depend on \( x_L \). From (2), we see that \( W_{\kappa,l}(x_L a) = C(a_1 a_2 \cdots a_n)^{\kappa} \) with an arbitrary constant \( C \). Eventually, we get the solution in the assertion. \( \square \)

We consider a necessary and sufficient condition for the above solution to give the non-zero generalized Whittaker function. For that purpose, we state some lemmas:

**Lemma 4.8.** (1) Let \( W_{\kappa,l}(\ast a) \) with a fixed \( a \in A \) denote a function on \( N \) defined by \( N \ni x \mapsto W_{\kappa,l}(xa) \). Consider the restriction \( W_{\kappa,l}|_{N_L}(\ast a) \) of \( W_{\kappa,l}(\ast a) \) to \( N_L \), explicitly given in Theorem 4.5, 4.7. It satisfies

\[
W_{\kappa,l}|_{N_L}(m_L X a) = W_{\kappa,l}|_{N_L}(x_L a)
\]

for any \( m_L \in M \cap N_L \), i.e. \( W_{\kappa,l}|_{N_L}(\ast a) \) defines a well-defined function on \( M \cap N_L \cap N_L \).

(2) If there is a non-zero generalized Whittaker function \( W_{\kappa,l} \) for \( \eta_l \), the character \( \chi_l \) inducing \( \eta_l \) has to satisfy \( \chi_l(M \cap N_L) = \{1\} \).

**Proof.** The first assertion follows from the left \( M \cap N_L \)-invariance of \( \exp(2\pi Tr(t^i(X_L A_n) Y_n(l)(X_L A_n))) \), stated in the proof for Lemma 4.6 (2). The second assertion is an immediate consequence of the first. \( \square \)

**Lemma 4.9.** Every \( l \in \mathfrak{n}^* \) with the non-zero \( Y_n(l) \) is Ad* \( N_L \)-equivalent to a linear form \( l' \) such that \( \det Y(l') \neq 0 \).

**Proof.** Let \( X(i, j; c) := 1_n + c E_{ij} \) for \( 1 \leq i < j \leq n \) and \( c \in \mathbb{R} \). With a suitable choice of a product \( X_0 \) of some \( X(i, j; c) \)'s, we can delete all non-zero linearly dependent column vectors and row vectors in \( Y_n(l) \) by considering \( t X_0 Y_n(l) X_0 \). By Lemma 4.6 (1),

\[
^t X_0 Y_n(l) X_0 = Y_n(\text{Ad}^* x_0^{-1} \cdot l)
\]

with \( x_0 = \begin{pmatrix} X_0 & t X_0^{-1} \\ \end{pmatrix} \in N_L \). Therefore \( \det Y(\text{Ad}^* x_0^{-1} \cdot l) \neq 0 \). We can take \( \text{Ad}^* x_0^{-1} \cdot l \) as \( l' \) in the assertion. \( \square \)
Lemma 4.10. For all the assertions, assume that $l$ has a polarization subalgebra $m$ such that $l(m \cap n_L) = \{0\}$. For the assertions (1) and (2), additionally assume that $l \in n^*$ satisfies $I(l) \neq \{n\}$.

1. Let $l \in n^*$ with $n_r \neq n$ (resp. $n_r = n$) in $I(l)$ and the positive definite $Y(l)$ (resp. $Y(l)_{r-1}$). Then $m$ is equal to $n_S \oplus n_l$.

2. Let $l \in n^*$ be $Ad^* N_L$-equivalent to $l' \in n^*$ with $n_r \neq n$ (resp. $n_r = n$) in $I(l')$ and the positive definite $Y(l')$ (resp. $Y(l')_{r-1}$). Then, for any $x' \in N_L$ such that $Ad^* x_L \cdot l = l'$, we have $m = Ad x_L^{-1} \cdot (n_S \oplus n_{l'})$.

3. The condition $I(l) = \{n\}$ holds for $l$ if and only if $l = \xi_{nn}l_{nn}$. For such an $l$, $m$ is equal to $n$.

**Proof.** (1) We prove $l(n_l) = \{0\}$. Then $m = n_S \oplus n_l$ holds by Proposition 3.3 (2). When $I(l) = [1, n]$, there is nothing to prove since $n_l = \{0\}$. We assume $I(l) \neq [1, n]$. We prove $\xi'_{ij} = 0$ for any $(i, j)$ with $i \notin I(l)$ and $j > i$, which means $l(n_l) = \{0\}$. Let $\xi'_{ij} \neq 0$ for some $(i, j)$ with $i \notin I(l)$ and $j > i$. Set $i(l)' = \min\{i \notin I(l) \mid \xi'_{ij} \neq 0 \}$ for some $j$ and $j(l)' = \max\{j \mid \xi'_{ij} \neq 0 \}$. Since $l(m \cap n_L) = \{0\}$ by the assumption, $E_{e((l)')_j - e((l)')_i} \notin m$. We can check that $l([E_{e((l)')_j - e((l)')_i}, n_S \oplus n]) = \{0\}$ by direct computation, and Proposition 3.3 (2) says that $m \subset n_S \oplus n_l$. Hence $m \oplus \mathcal{R} E_{e((l)')_j - e((l)')_i}$ forms an isotropic subspace with respect to an inner product $l([*, *])$. But this contradicts the maximality of $m$ as an isotropic subspace. Therefore $\xi'_{ij} = 0$ for any $(i, j)$ with $i \notin I(l)$ and $j > i$.

(2) Let $x' \in N_L$ be in the assertion (2) and a pair $(l, m)$ satisfy the condition in the assertion (2). Then $(Ad^* x_L \cdot l, Ad x_L \cdot (m))$ forms a pair with the condition in the assertion (1). Hence we have

$$Ad x_L \cdot (m) = n_S \oplus n_{Ad^* x_L \cdot l}$$

that is,

$$m = Ad x_L^{-1} \cdot (n_S \oplus n_{l'})$$

(3) If $l = \xi_{nn}l_{nn}$, clearly $I(l) = \{n\}$ holds. Conversely, assume that $l$ satisfies $I(l) = \{n\}$. We prove $\xi'_{ij} = 0$ for $1 \leq i < j \leq n$. Then we get $l = \xi_{nn}l_{nn}$. Let some $\xi'_{ij} \neq 0$. Set $i(l)'' = \min\{i \in [1, n] \mid \xi'_{ij} \neq 0 \}$ for some $j > i$ and $j(l)'' = \max\{j \in [1, n] \mid \xi'_{ij} \neq 0 \}$. The assumption $l(m \cap n_L) = \{0\}$ means $E_{e((l)''_j - e((l)''_i)} \notin m$. We check $l([E_{e((l)''_j - e((l)''_i)}, n_L]) = \{0\}$ by direct calculation, and $l([E_{e((l)''_j - e((l)''_i)}, n_S]) = \{0\}$ by the assumption $I(l) = \{n\}$. Therefore $m \oplus \mathcal{R} E_{e((l)''_j - e((l)''_i)}$ forms an isotropic subspace with respect to an inner
In particular, this assertion holds if \( \xi'_i = 0 \). For a linear form \( l = \xi_{nn}l_{nn} \), we check that \( l([n,n]) = \{0\} \) by direct computation. Hence \( m = n. \) □

**Lemma 4.11.** Let \( (l,m) \) be a pair of a linear form \( l \) and a polarization subalgebra \( m \), satisfying \( I(l) \neq \{n\} \) and \( l(m \cap n_L) = \{0\} \). For \( n_1 < i \leq n \), we define \( k(i) := \max \{p \mid n_p \in I(l), n_p < i\} \) and introduce a coordinate

\[
X_i := (x'_{n_1,i}, x'_{n_2,i}, \ldots, x'_{n_{k(i)},i}) \text{ of } \mathbb{R}^{k(i)}.
\]

1. Let \( X(x_i) := x'_{n_1,i}E_{n_1i} + x'_{n_2,i}E_{n_2i} + \cdots + x'_{n_{k(i)},i}E_{n_{k(i)}i} \) with \( x_i \in \mathbb{R}^{k(i)} \). If \( l \) satisfies \( n_r \neq n \) (resp. \( n_r = n \)) in \( I(l) \) and \( \det Y(l)_s \neq 0 \) for \( 1 \leq s \leq r \) (resp. \( 1 \leq s \leq r - 1 \)), the set

\[
\left\{ \left( 1_n + \sum_{1 < i \leq n} X(x_i) \right)^t \left( 1_n + \sum_{1 < i \leq n} X(x_i) \right) \in N_L \mid x_i \in \mathbb{R}^{k(i)} \right\}
\]

is bijective with the quotient \( M \cap N_L \backslash N_L \simeq M \backslash N \). Hence an invariant measure \( d\tilde{x} \) on \( M \backslash N \) can be written as \( \prod_{s,t,j > n_p} dx'_{n_{p}j} \) up to constant multiple.

In particular, this assertion holds if \( l \) satisfies \( n_r \neq n \) (resp. \( n_r = n \)) in \( I(l) \) and the positive-definiteness \( Y(l) \) (resp. \( Y(l)_{r-1} \)).

2. Each diagonal entry of \( ^t (X_L A_n) Y_n(l)(X_L A_n) \) is given by

\[
\begin{cases}
0 & (i < n_1); \\
\alpha^2_{n_1} \xi_{n_1 n_1} & (i = n_1); \\
\alpha^2_{np} (x_{np}, 1) Y(l)_{p-1} ^t (x_{np}, 1) & (i = np \in I(l) \backslash \{n_1\}); \\
\alpha^2_{np} x_i Y(l)(k(i)) ^t x_i & (i \notin I(l), i > n_1),
\end{cases}
\]

where we write \( Y(l)_p = \left( \begin{array}{c} (Y(l)_{p-1} ^t y_{p-1}) \\ y_{p-1} \end{array} \right) \) with \( y_{p-1} = (\xi_{1np}/2, \xi_{2np}/2, \ldots, \xi_{np-1 np}/2) \). The explicit formula of \( W_{\kappa,i}(x_L a) \) can be written as

\[
(a_1 a_2 \cdots a_n)^\kappa \exp(-2\pi a^2_{n_1} \xi_{n_1 n_1}) \prod_{np \in I(l) \backslash \{n_1\}} \exp(-2\pi a^2_{np} (x_{np}, 1) Y(l)_{p-1} ^t (x_{np}, 1)) \prod_{i \notin I(l)} \exp(-2\pi a^2_i x_i Y(l)(k(i)) ^t x_i),
\]

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up to constant multiple.

(3) If \( W_{\kappa,l}|_{N_L}(\ast a) \) is square-integrable on \( M \cap N_L \setminus N_L \), \( Y(l)_{k(i)} \) has to be positive semi-definite for \( n_1 < i \leq n \).

**Proof.** The assertion (1) follows from Proposition 3.3 (2), and the assertion (2) is obtained by direct computation. We prove the assertion (3). The square-integrability of \( W_{\kappa,l}|_{N_L}(\ast a) \) means that, for any \( n_1 < i \leq n \), \( x_i Y(l)_{k(i)} x_i^t \) has to be non-negative, i.e. \( Y(l)_{k(i)} \) is positive semi-definite. In fact, otherwise, there exists a non-zero \( x^0_i \in \mathbb{R}^{k(i)} \) such that \( x^0_i Y(l)_{k(i)} x^0_i \) is negative for some \( n_1 < i \leq n \). Noting the formula of the diagonal entries of \( t(X_L A_n) Y_n(l)(X_L A_n) \) and the formula of \( W_{\kappa,l}(x_L a) \) in the assertion (2), we see that \( W_{\kappa,l}|_{N_L}(\ast a) \) is neither trivial nor square-integrable on

\[
\left\{ \left( 1_n + X(x_i) \right)^t (1_n + X(x_i))^{-1} \left| x_i \in \mathbb{R} \cdot x^0_i \right. \right\} \subset N_L.
\]

Therefore we obtain the assertion (3). \( \square \)

Theorem 4.5 and 4.7 tell us that \( \dim \mathbb{C} \text{Hom}_{(G_C,K)}(\pi_\kappa,C_\eta^\infty(N \setminus G)_K) \leq 1 \). To be more precise, we have

**Theorem 4.12.** \( \dim \mathbb{C} \text{Hom}_{(G_C,K)}(\pi_\kappa,C_\eta^\infty(N \setminus G)_K) = 1 \) holds if and only if \( \eta_l \) satisfies one of the following conditions:

(a) \( \eta_l = \chi_I \) and \( \chi_I(M \cap N_L) = \{1\} \), i.e. \( l = \xi_{nn} l_{nn} \).

(b) \( \eta_l \neq \chi_I \), \( \chi_I(M \cap N_L) = \{1\} \) and \( l \) is \( \text{Ad}^* N_L \)-equivalent to \( l' \in n^* \) such that \( n_r \neq n \) in \( I(l') \) and \( Y(l') \) is positive definite, or to \( l' \in n^* \) such that \( n_r = n \) in \( I(l') \) and \( Y(l')_{r-1} \) is positive definite.

For the condition (a), remark that \( \eta_l = \chi_I \) means \( m = n \) and that \( \eta_l = \chi_I \) if and only if non-zero is at least one of the parameters \( \xi_{nn} \) and \( \xi_{i,i+1} \) of \( l \) with \( 1 \leq i < n \), which correspond to the simple roots of the restricted root system \( \Delta(a,g) \). We see that \( \eta_l = \chi_I \) and \( \chi_I(N_L \cap M) = \{1\} \) if and only if \( l = \xi_{nn} l_{nn} \).

**Proof.** By virtue of Lemma 4.8 (2), it suffices to consider only the case where \( \eta_l \) satisfies \( \chi_I(M \cap N_L) = \{1\} \). Therefore we assume this throughout our proof. Here remark that \( \dim \mathbb{C} \text{Hom}_{(G_C,K)}(\pi_\Lambda,C_\eta^\infty(N \setminus G)_K) = 1 \) if and only if \( W_{\kappa,l}(\ast a) \in H_\eta^\infty \) for any fixed \( a \in A \). Hence, under the condition \( \chi_I(M \cap N_L) = \{1\} \), it suffices to prove
The condition (a) or (b) on $\eta_l$ in the assertion holds if and only if $W_{\kappa,l}(\ast a) \in H^\infty_{\eta_l}$ for any fixed $a \in A$.

Here recall that $W_{\kappa,l}(\ast a)$ denotes a function on $N$ defined by $N \ni x \mapsto W_{\kappa,l}(xa)$ (cf. Lemma 4.8 (1)).

For any $\eta_l = \chi_l$ with $\chi_l(M \cap N_L) = \{1\}$, $W_{\kappa,l}(\ast a)$ with any fixed $a \in A$ belongs to $H^\infty_{\eta_l} \simeq \mathbb{C}$ as its explicit formula

$$C(a_1a_2\ldots a_n)^\kappa \exp(-2\pi a_n^2 \xi_{nn})$$

of $W_{\kappa,l}|_{N_L}(\ast a)$ in Theorem 4.7 indicates. Therefore $\dim \mathbb{C}\text{Hom}_{(\mathbb{G},\mathcal{K})}(\pi_{\Lambda}, C^{\infty}_{\eta_l}(N \setminus G)) = 1$ for such $\eta_l = \chi_l$.

Let $\eta_l \neq \chi_l$. This means $I(l) \neq \{n\}$ under our assumption $\chi_l(M \cap N_L) = \{1\}$. In fact, $\eta_l = \chi_l$ and $\chi_l(N_L \cap M) = \{1\}$ if and only if $l = \xi_{nn}l_{nn}$ as is remarked in the assertion, and we see from this and Lemma 4.10 (3) that $\eta_l = \chi_l$ if and only if $I(l) = \{n\}$ under the assumption $\chi_l(N_L \cap M) = \{1\}$. For any fixed $a \in A$, $W_{\kappa,l}(\ast a) \in H^\infty_{\eta_l}$ holds if and only if $W_{\kappa,l}|_{N_L}(\ast a)$ is a square-integrable function on the quotient $M \cap N_L \setminus N_L$. We prove that such square-integrability condition on $W_{\kappa,l}|_{N_L}(\ast a)$ is equivalent to the condition (b) in the assertion.

First we assume that $\det Y(l) \neq 0$. We prove that the square-integrability condition on $W_{\kappa,l}|_{N_L}(\ast a)$ holds if and only if

$$\begin{aligned}
Y(l) & \text{ is positive definite \quad when } n_r \neq n \text{ in } I(l), \\
Y(l)_{r-1} & \text{ is positive definite \quad when } n_r = n \text{ in } I(l).
\end{aligned}$$

Let $n_r \neq n$ in $I(l)$ and $\det Y(l) \neq 0$. If $Y(l)$ is positive definite, we see from Lemma 4.11 (1), (2) that $W_{\kappa,l}|_{N_L}(\ast a)$ is square-integrable on $M \cap N_L \setminus N_L$. Conversely, if such square-integrability condition on $W_{\kappa,l}|_{N_L}(\ast a)$ holds, Lemmas 4.11 (3) and the assumption $\det Y(l) \neq 0$ means that $Y(l)$ is positive definite.

Let $n_r = n$ in $I(l)$ and $\det Y(l) \neq 0$. Assume that $W_{\kappa,l}|_{N_L}(\ast a)$ is square-integrable on $M \cap N_L \setminus N_L$. Here recall the formula of the $n$-th diagonal entry of $t(XLA)^n_Y(l)(XLA_n)$ in Lemma 4.11 (2):

$$a_{n_r}^2(x_{n_r}Y(l)_{r-1}^t x_{n_r} + 2y_{r-1}^t x_{n_r} + \xi_{n_r n_r}).$$

By this formula, we check that $Y(l)_{r-1}$ is positive definite. In fact, the square-integrability condition on $W_{\kappa,l}|_{N_L}(\ast a)$ and Lemma 4.11 (3) means
that \( Y(l)_{r-1} \) is positive semi-definite. Let \( \det Y(l)_{r-1} = 0 \). Then there exists a non-zero \( \mathbf{x}^0_{n_r} \in \mathbb{R}^{r-1} \) such that \( \mathbf{x}^0_{n_r} Y(l)_{r-1} = (0, \ldots, 0) \). If \( \mathbf{y}_{r-1}^t \mathbf{x}^0_{n_r} = 0 \), this contradicts the assumption \( \det Y(l) \neq 0 \). If \( \mathbf{y}_{r-1}^t \mathbf{x}^0_{n_r} \neq 0 \), we see that \( W_{\kappa,l}|_{N_L}(\ast a) \) is neither trivial nor square-integrable on

\[
\left\{ \begin{pmatrix} 1_n + X(\mathbf{x}_{n_r}) \\ t(1_n + X(\mathbf{x}_{n_r}))^{-1} \end{pmatrix} \mid \mathbf{x}_{n_r} \in \mathbb{R} \cdot \mathbf{x}^0_{n_r} \right\} \subset N_L,
\]

by noting the above formula of the \( n \)-th diagonal entry and the formula of \( W_{\kappa,l}(x_L a) \) in Lemma 4.11 (2). Therefore \( \det Y(l)_{r-1} \neq 0 \), hence \( Y(l)_{r-1} \) is positive definite.

Conversely, let \( Y(l)_{r-1} \) be positive definite. Then Lemma 4.11 (1) is valid. By the formula of \( W_{\kappa,l}(x_L a) \) in Lemma 4.11 (2), we check that \( W_{\kappa,l}|_{N_L}(\ast a) \) is square-integrable with respect to \( \mathbf{x}_i \) for \( n_1 < i < n \). For \( \mathbf{x}_n = \mathbf{x}_{n_r} \), note that the \( n \)-th diagonal entry is written as

\[
a^2_{n_r}((\mathbf{x}_{n_r} + \mathbf{y}_{r-1}) Y(l)_{r-1}^t (\mathbf{x}_{n_r} + \mathbf{y}'_{r-1}) + (\mathbf{y}_{r-1}, 1) (Y(l)_{r-1}^t + (\mathbf{y}'_{r-1}, 1),
\]

where \( \mathbf{y}'_{r-1} := \mathbf{y}_{r-1} Y(l)_{r-1}^{-1} \). This is checked by direct calculation. From the positive-definiteness of \( Y(l)_{r-1} \), we see that \( W_{\kappa,l}|_{N_L}(\ast a) \) is square-integrable with respect to \( \mathbf{x}_{n_r} \), which means that it is square-integrable with respect to all \( \mathbf{x}_i \). Therefore \( W_{\kappa,l}|_{N_L}(\ast a) \) is square-integrable on \( M \cap N_L \setminus N_L \).

Next assume that \( \eta_l \neq \chi_l \) and \( \det Y(l) = 0 \). Remark that \( Y_n(l) \neq 0_n \) since \( \eta_l \neq \chi_l \) means \( I(l) \neq \{n\} \) as is noted above. By Lemma 4.9, we can take an \( x' \in N_L \) so that \( \det Y(\text{Ad}^* x' \cdot l) \neq 0 \). Set \( l' := \text{Ad}^* x' \cdot l \). Since \( W_{\kappa,l}(x_L a) = W_{\kappa,l'}(x' \cdot x_L a) \), the argument for the case \( \det Y(l) \neq 0 \) means that, when \( n_r \neq n \) (resp. \( n_r = n \)) in \( I(l') \), \( W_{\kappa,l} \) is a generalized Whittaker function if and only if \( Y(l') \) (resp. \( Y(l')_{r-1} \)) is positive definite. This implies the assertion for this case. As a result, we complete the proof. \( \square \)

Recall that \( \| \ast \|_l \) denote the norm on \( H_{\eta_l} \) (cf. Proposition 3.1 (1)), and let \( \| \ast \| \) denote the norm on the matrix algebra \( M_{2n}(\mathbb{R}) \) defined by \( \| Y \| := \text{Tr}^1 Y Y \). Moreover, let \( U(\mathfrak{g}_\mathbb{C}) \) be the universal enveloping algebra of \( \mathfrak{g}_\mathbb{C} \). Now we consider the space \( A_{\eta}(N \setminus G) \) of all \( F \in C^\infty_{\eta} (N \setminus G)_K \) satisfying the moderate growth condition, i.e.

\[
\| X \cdot F(g) \|_l < C \| g \|^m \quad \text{for any } g \in G \text{ and any } X \in U(\mathfrak{g}_\mathbb{C}),
\]

where a constant \( C \) and an integer \( m \) depend only on \( F \) and \( X \). This forms a \((\mathfrak{g}_\mathbb{C}, K)\)-submodule of \( C^\infty(N \setminus G)_K \).
Theorem 4.13. \( \dim \mathbb{C} \text{Hom}_{(\mathfrak{g}_{\mathcal{C}}, K)}(\pi_{\kappa}, A_{\eta}(N \backslash G)) = 1 \) if and only if \( \eta \) satisfies the condition that \( Y_n(l) \) is positive semi-definite and that \( \chi_l(M \cap N_L) = \{1\} \).

Proof. As we remarked in the beginning of the proof of Theorem 4.12, it suffices to consider the case where \( \eta \) satisfies \( \chi_l(M \cap N_L) = \{1\} \). Note that, under this condition, \( \dim \mathbb{C} \text{Hom}_{(\mathfrak{g}_{\mathcal{C}}, K)}(\pi_{\kappa}, A_{\eta}(N \backslash G)) = 1 \) is equivalent to the moderate growth condition of \( W_{\kappa, l} \) in the following sense:

\[
X \cdot W_{\kappa, l}(a) \in H^\infty_{\eta}, \quad \|X \cdot W_{\kappa, l}(a)\| < C\|a\|^m
\]

for any \( a \in A \) and any \( X \in U(\mathfrak{g}_{\mathcal{C}}) \), where a constant \( C \) and an integer \( m \) depend only on \( W_{\kappa, l} \) and \( X \). Assuming the condition \( \chi_l(M \cap N_L) = \{1\} \) on \( \eta \), we prove that this moderate growth condition of \( W_{\kappa, l}(a) \) holds if and only if \( \eta \) satisfies the condition that \( Y_n(l) \) is positive semi-definite.

Let \( \eta = \chi_l \). Then the explicit formula of \( W_{\kappa, l}|_{N_L}(a) \) implies that the moderate growth condition holds if and only if \( \xi_{nn} \geq 0 \), i.e. \( Y_n(l) \) is positive semi-definite.

Let \( \eta \neq \chi_l \). Recall that this implies \( I(l) \neq \{n\} \) as we remarked in the 3rd paragraph of the proof of Theorem 4.12. First assuming \( \det Y(l) \neq 0 \), we prove that the moderate growth condition of \( W_{\kappa, l} \) holds if and only if \( Y(l) \) is positive definite. If \( Y(l) \) is positive definite, we can express \( W_{\kappa, l}(x_La) \) as

\[
(a_1 a_2 \cdots a_n)^{\kappa} \exp(-2\pi a_{n_1} \xi_{n_1 n_1})
\times \prod_{n_p \in I(l) \setminus \{n_1\}} \exp(-2\pi a_{n_p}^2 (-y_{p-1, 1})Y(l)_p \cdot (-y_{p-1, 1}))
\times \prod_{n_p \in I(l) \setminus \{n_1\}} \exp(-2\pi a_{n_p}^2 (x_{n_p} + y_{p-1}')Y(l)_p \cdot (x_{n_p} + y_{p-1}'))
\times \prod_{i \notin I(l)} \exp(-2\pi a_i^2 x_i Y(l)_i k(i) t x_i)
\]

up to constant multiple. Here see Lemma 4.11 for the notation \( k(i) \) and we set \( y_{p-1}' := y_{p-1}Y(l)^{-1}_{p-1} \). In fact, this also holds for \( l \) such that \( n_r = n \) in \( I(l) \) and \( Y(l)^{-1}_{r-1} \) is positive definite. From the positive-definiteness of \( Y(l) \), we see that Lemma 4.11 (1) holds and that the exponential part of \( W_{\kappa, l}(x_La) \) is constant on \( \{(a_1, a_2, \ldots, a_{n_1-1}) \in \mathbb{R}^{n_1-1}_+\} \) but defines a Schwartz function.
on \((M \cap N_L \setminus N_L) \times \{(a_{n_1}, a_{n_1+1}, \ldots, a_n) \in \mathbb{R}_{+}^{n-n_1+1}\}\). We can check that 
\(W_{\kappa,l}\) satisfies the moderate growth condition.

Conversely, if the moderate growth condition is satisfied, \(W_{\kappa,l}(\kappa l) \in H_{\eta_l}^\infty\) and \(\|W_{\kappa,l}(\kappa l)\|_l < C\|a\|^m\) hold with a constant \(C\) and an integer \(m\) depending only on \(W_{\kappa,l}\). Theorem 4.12 and \(W_{\kappa,l}(\kappa l) \in H_{\eta_l}^\infty\) means that \(Y(l)\) (resp. \(Y(l)_{r-1}\)) is positive definite if \(l\) satisfies \(n_r \neq n\) (resp. \(n_r = n\)) in \(I(l)\). Hence it suffices to consider the case where \(l\) satisfies \(n_r = n\) in \(I(l)\) and the positive-definiteness of \(Y(l)_{r-1}\). For such an \(l\), Lemma 4.11 (1) and the expression of \(W_{\kappa,l}(x_L a)\) in the preceding paragraph are valid. Here note the

formula

\[a_{n_r}^2((x_{n_r} + y_{r-1}')(y(l)_{r-1}^t(x_{n_r} + y_{r-1}')) + (-y_{r-1}', 1)Y(l)_{r-1}^t(-y_{r-1}', 1))\]

of the \(n\)-th diagonal entry of \(\{X_L A_n\} Y_n(l)(X_L A_n)\), which is also stated in the 7-th paragraph of the proof of Theorem 4.12. The positive-definiteness of \(Y(l)_{r-1}\) means the non-negativity of \((x_{n_r} + y_{r-1}')(y(l)_{r-1}^t(x_{n_r} + y_{r-1}'))\), and the condition \(\|W_{\kappa,l}(\kappa a)\|_l < C\|a\|^m\) the non-negativity of \((-y_{r-1}', 1)Y(l)_{r-1}^t(-y_{r-1}', 1)\). Hence the \(n\)-th diagonal entry has to be non-negative. It tells us that \(t^t Y(l)x \geq 0\) for a column vector \(x\) with \(r\) entries and with non-zero \(r\)-th component. By this condition and the positive definiteness of \(Y(l)_{r-1}\), we check that \(Y(l)\) is positive definite.

For \(\eta_l \neq \chi_l\) with \(\det Y(l) = 0\), we see that the problem is reduced to the previous case by the same reasoning as in the last paragraph of the proof of Theorem 4.12. Indeed, \(l \in \mathfrak{n}^*\) has the non-zero positive semi-definite \(Y_n(l)\) if and only if \(l\) is \(A_d^* N_l\)-equivalent to \(l' \in \mathfrak{n}^*\) with the positive definite \(Y(l')\). Hence we obtain the result. □

**Remark 4.14.** Theorem 4.13 corresponds to the “Koecher principle” for holomorphic Siegel modular forms (cf. [7], Satz 1, Satz 2).

By reviewing the proof of Theorem 4.13, we have

**Corollary 4.15.** Assume \(\chi_l(M \cap N_L) = \{1\}\). Then \(\eta_l\) satisfies the positive-semi-definiteness of \(Y_n(l)\) if and only if

\[W_{\kappa,l}(\kappa a) \in H_{\eta_l}^\infty, \quad \|W_{\kappa,l}(\kappa a)\|_l < C\|a\|^m\]

with the constant \(C\) and the integer \(m\) depending only on \(W_{\kappa,l}\). 


Construction of the Fourier Expansion

From now on, let $\Gamma := \text{Sp}(n;\mathbb{Z})$, $N_{\mathbb{Z}} := N \cap \Gamma$, $N_S(\mathbb{Z}) := N_S \cap \Gamma$ and $N_L(\mathbb{Z}) := N_L \cap \Gamma$. Furthermore let $N_Q$ and $N_L(Q)$ denote the groups of $\mathbb{Q}$-rational points in $N$ and in $N_L$, respectively. We recall a definition of $\mathbb{C}$-valued holomorphic Siegel modular form with respect to $\Gamma$.

**Definition 5.1.** Let $\kappa > n$ be an integer. A $C^\infty$-function $f : G \to \mathbb{C}$ is called a holomorphic Siegel modular form of weight $\kappa$ with respect to $\Gamma$ if it satisfies

1. $f(\gamma g k) = \det(A + \sqrt{-1}B)^\kappa f(g)$ for any $(\gamma, g, k) \in \Gamma \times G \times K$, where we denote $k$ by

   $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

2. $f$ satisfies the Cauchy-Riemann condition, i.e. $dR_X f = 0$ for any $X \in \mathfrak{p}^-$, where $dR$ denotes the differential of right regular representation $R$ of $G$ on the space of $C^\infty$-functions on $G$.

Here remark that the weight $\kappa$ satisfies $\kappa n \equiv 0 \mod 2$ since $-1_{2n} \in \Gamma \cap K$ and that, if $n = 1$, we have to add the moderate growth condition (for a definition, see the remark just before Lemma 5.7) of $f$ to the definition above.

We formulate the Fourier expansion of a modular form $f$ along the minimal parabolic subgroup. For a fixed $g$, we can regard $f(xg)$ as a function in $x \in N$. Note $f(xg) \in L^2(N_{\mathbb{Z}} \setminus N)$, where $L^2(N_{\mathbb{Z}} \setminus N)$ denotes the space of square-integrable functions on the quotient $N_{\mathbb{Z}} \setminus N$. Since $N_{\mathbb{Z}} \setminus N$ is compact, we have

**Proposition 5.2.** The space $L^2(N_{\mathbb{Z}} \setminus N)$ decomposes discretely into

$$L^2(N_{\mathbb{Z}} \setminus N) \simeq \bigoplus_{(\eta, H_\eta) \in \hat{N}} m(\eta) H_\eta \simeq \bigoplus_{(\eta, H_\eta) \in \hat{N}} \text{Hom}_N(\eta, L^2(N_{\mathbb{Z}} \setminus N)) \otimes H_\eta,$$

where $H_\eta$ denotes a representation space of $\eta$, $\bigoplus$ the Hilbert space direct sum and $m(\eta) = \dim_{\mathbb{C}} \text{Hom}_N(\eta, L^2(N_{\mathbb{Z}} \setminus N)) < \infty$.

For a proof, see [5], Chap I, §2.3.
Let \( \{ \Theta_{\eta}^{\eta} \}_{1 \leq m \leq m(\eta)} \) be a basis of \( \text{Hom}_N(\eta, L^2(N\mathbb{Z}\setminus N)) \). According to the decomposition above, \( f(xg) \) decomposes into

\[
f(xg) = \sum_{\eta \in N} \sum_{1 \leq m \leq m(\eta)} \Theta_{\eta}^{\eta}(W_{\eta,m}^f(g))(x),
\]

where \( W_{\eta,m}^f(g) \in H_\eta \) denotes the \((\eta,m)\)-component of \( f(xg) \) and we regard \( \Theta_{\eta}^{\eta}(W_{\eta,m}^f(g)) \) as a function on \( N \). For two \( x_1, x_2 \in N \), we deduce

\[
\Theta_{\eta}^{\eta}(W_{\eta,m}^f(g))(x_1 x_2) = \Theta_{\eta}^{\eta}(\eta(x_2)W_{\eta,m}^f(g))(x_1)
\]

from the \( N \)-equivariance of \( \Theta_{\eta}^{\eta} \), and

\[
\Theta_{\eta}^{\eta}(W_{\eta,m}^f(g))(x_1 x_2) = \Theta_{\eta}^{\eta}(W_{\eta,m}^f(x_2 g))(x_1)
\]

from the trivial formula \( f(x_1 x_2 \cdot g) = f(x_1 \cdot x_2 g) \). Therefore we obtain

\[
W_{\eta,m}^f(xg) = \eta(x)W_{\eta,m}^f(g)
\]

for any \( x \in N \). From the right-\( K \)-equivariance of \( f \) and the holomorphy of \( f \), we find \( W_{\eta,m}^f(g) \) a generalized Whittaker function for holomorphic discrete series \( \pi_\kappa \) with \( K \)-type \( \tau_\kappa \) (for the notations \( \pi_\kappa \) and \( \tau_\kappa \), see \( \S 2 \)). We have already obtained the explicit formula of \( W_{\eta,m}^f \).

Our remaining work is to determine the dimension \( m(\eta) \) and a basis of the space \( \text{Hom}_N(\eta, L^2(N\mathbb{Z}\setminus N)) \). For such determination, we recall some results established by L.Corwin and F.P.Greenleaf [2]. The paper treats a spectral decomposition of \( L^2(\text{Ind}_N^N \rho) \), where this time \( N \) is a general simply connected nilpotent Lie group with some \( \mathbb{Q} \)-rational structure and \( \rho \) denotes a character on a uniform discrete subgroup \( N_\Gamma \) in \( N \). There is another result by L.Richardson [13], which treats only the special case where \( \rho \) is trivial. We first state their result on \( m(\eta) \).

**Proposition 5.3.** In this assertion, we do not assume the assumption 1 (cf. \( \S 3 \)) for the polarization subalgebras \( \mathfrak{m} \) and \( \mathfrak{m}_0 \).

1. If \( \eta_l \) occurs in \( L^2(N\mathbb{Z}\setminus N) \), a coadjoint orbit \( \text{Ad}^* N \cdot l \) contains a \( \mathbb{Q} \)-rational \( l' \in \mathfrak{n}^* \), i.e. \( l'(\log(N\mathbb{Z})) \subset \mathbb{Q} \).
2. We define, by \( \text{Ad}^* x \cdot (\chi_0, M_0) := (\chi_0(x^{-1} * x), xM_0x^{-1}) \) with \( x \in N \), the action \( \text{Ad}^* \) of \( N \) on the set of pairs \( (\chi_0, M_0) \), where \( M_0 := \exp(\mathfrak{m}_0) \).
with a polarization subalgebra $m_0$ for some linear form in $n^*$, and $\chi_0$ is a character on $M_0$.

Let $l \in n^*$ be $Q$-rational and $M := \exp(m)$ with a $Q$-rational polarization subalgebra $m$ for $l$, i.e. $m \cap n_Q$ forms a $Q$-structure of $m$ with $n_Q := Q$-span of $\{\log(N_{\mathbb{Z}})\}$ (the existence of such an $m$ is proved in [3], Proposition 5.2.6).

Let $\eta_l$ be induced from $(\chi_l, M)$ with $l$ and $M$ above and $O(\eta_l)_\mathbb{Z} := \{(\chi_{l'}, M') \in \text{Ad}^* N \cap (\chi_l, M) \mid \chi_{l'}(N_{\mathbb{Z}} \cap M') = \{1\}\}$. Then the representation $\eta_l$ occurs in $L^2(N_{\mathbb{Z}} \setminus N)$ if and only if $O(\eta_l)_\mathbb{Z}$ is non-empty. The multiplicity $m(\eta_l)$ of the representation $\eta_l$ in $L^2(N_{\mathbb{Z}} \setminus N)$ is equal to the cardinality of $\mathfrak{M}(\eta_l) := O(\eta_l)_\mathbb{Z}/\text{Ad}^* N_{\mathbb{Z}}$.

For a proof, see Theorem 5.1 in [2].

Let $n^*_S := \{l \in n^* \mid l(n_L) = \{0\}\}$. Let $\text{Ad}_{S}^*$ denote the coadjoint action of $N$ on $n^*_S$ and also denote the action of $N$ on the set of pairs $(l, M)$ with $l \in n^*_S$ and an associated polarization subgroup $M := \exp(m)$, defined by

$$\text{Ad}_{S}^* x \cdot (l, M) := (\text{Ad}_{S}^* x \cdot l, xMx^{-1})$$

with $x \in N$. Remark that both actions satisfy the triviality of $\text{Ad}_{S}^* |_{N_S}$.

Theorem 4.12 and Proposition 5.3 implies that a representation $\eta_l$ occurring in the Fourier expansion is attached to $(l, m)$ in (1), (2) or (3) of Lemma 4.10 with a $Q$-rational $l$. There is no loss of generality if we assume $l \in n^*_S$. In fact, we check that, under the assumption 1, the polarization subalgebra $m$ for $l$ in Lemma 4.10 (1), (2) or (3) such that $l(m \cap n_L) = \{0\}$ coincides with the polarization subalgebra for $l' \in n^*_S$ with $Y_n(l') = Y_n(l)$, which implies $\eta_l = \eta_{l'}$. From now on, we assume

**Assumption 2.** $l$ is in $n^*_S$, $Q$-rational and satisfies the condition in Lemma 4.10 (1), (2) or (3).

For a $Q$-rational $l \in n^*_S$ with the condition in Lemma 4.10 (1) (resp. Lemma 4.10 (3)), the polarization subalgebra $m$ is $Q$-rational as its explicit form $m = n_S \oplus n_l$ (resp. $m = n$) indicates. For a $Q$-rational $l \in n^*_S$ with the condition in Lemma 4.10 (2), the polarization subalgebra $m$ is of the form $\text{Ad}_S^* x'_L^{-1} \cdot (n_S \oplus n_{l'})$, where $x'_L \in N_{L}(Q)$ and $l' \in n^*_S$ with the condition in Lemma 4.10 (1). Hence $m$ is $Q$-rational.
Proposition 5.4. Let $l \in \mathfrak{n}_S^*$ satisfy the assumption 2. Then we have the following identifications:

$$O(\eta)_Z \simeq O(l)_Z := \{ l' \in \text{Ad}^*_S N_L(Q) \cdot l \mid l'(\log(N_S(Z))) \subset Z \};$$

$$\mathcal{M}(\eta_l) \simeq \mathcal{M}(l) := O(l)_Z/\text{Ad}^*_S N_L(Z).$$

Proof. Characters $\chi_l$ for $l \in \mathfrak{n}_S^*$ are determined by its restriction to $N_S$. The set of characters on $N$ is in bijection with $\mathfrak{n}_S^*$. Therefore, in the expression of $O(\eta)_Z$ in Proposition 5.3 (2), we may replace $\chi_{l'}$ and $\chi_l$ by $l'$ and $l$ respectively. Since the action of $N$ on $\chi_l$ via $\text{Ad}^*$ is identical with the action of it on $l \in \mathfrak{n}_S^*$ via $\text{Ad}^*_S$, we can replace $\text{Ad}^*$ by $\text{Ad}^*_S$. Under such replacement, we have a bijection

$$O(\eta)_Z \simeq \{ (l', M') \in \text{Ad}^*_S N_L(Q) \cdot (l, M) \mid l'(\log(M \cap N_Z)) \subset Z \}.$$ 

In order to deduce the bijection $O(\eta)_Z \simeq O(l)_Z$, we insert

Lemma 5.5. For a pair $(l, m)$ with $l \in \mathfrak{n}_S^*$, $x \in N$ satisfies $\text{Ad}^*_S x \cdot l = l$ if and only if $x \in M$. This also holds for any $(l, m)$ with $l \in \mathfrak{n}_S^*$ not satisfying the assumption 2.

Proof. It suffices to prove that, for $x_L \in N_L$, $\text{Ad}^*_S x_L \cdot l = l$ if and only if $x_L \in M$. Let $x_L = \exp(X)$ with $X \in \mathfrak{n}_L$. Since $l$ is trivial on $\mathfrak{n}_L$, we see that $\text{Ad}^*_S x_L \cdot l = l$ if and only if $l([X, Y_S]) = 0$ for any $Y_S \in \mathfrak{n}_S$. But the condition $l([X, \mathfrak{n}_S]) = \{0\}$ is equivalent to $X \in m \cap \mathfrak{n}_L$. In fact, $X \in m \cap \mathfrak{n}_L$ clearly satisfies $l([X, \mathfrak{n}_S]) = \{0\}$. Conversely, assume that $l([X, \mathfrak{n}_S]) = \{0\}$ holds but $X \not\in m \cap \mathfrak{n}_L$. Then $m \oplus \mathbb{R}X$ forms an isotropic subspace for an inner product $l([*, *])$, but this contradicts the maximality of $m$ as an isotropic subspace. Hence $X \in m \cap \mathfrak{n}_L$. As a result, we obtain the assertion. \(\square\)

We return to the proof of the proposition. The condition $l'(\log(M \cap N_Z)) \subset Z$ can be replaced by the condition $l'(\log(N_S(Z))) \subset Z$. Since Lemma 5.5 means that for an $x_L \in N_L$, $\text{Ad}^*_S x_L \cdot (l, M) = (l, M)$ if and only if $\text{Ad}^*_S x_L \cdot l = l$, the bijection $O(\eta)_Z \simeq O(l)_Z$ is obtained by deleting $M$ and $M'$ in the set on the right hand side of the bijection just before Lemma 5.5. The bijection on $\mathcal{M}(\eta_l)$ follows immediately from that on $O(\eta)_Z$. \(\square\)
We recall the construction of a basis of $\text{Hom}_N(\eta_l, L^2(N\mathbb{Z}\backslash N))$ stated in [2]. For each $l' \in \mathcal{M}(l)$, we define $\Theta_{l'} \in \text{Hom}_N(\eta_{l'}, L^2(N\mathbb{Z}\backslash N))$ by

$$\Theta_{l'}(h)(x) := \sum_{\gamma \in N\mathbb{Z} \cap M' \backslash N\mathbb{Z}} h(\gamma x),$$

where $M' = \exp(m')$ with the polarization subalgebra $m'$ for $l' \in \mathcal{M}(l)$. Here remark that $\Theta_{l'}$ may depend on the choice of $m'$ but that thanks to Lemma 4.10 $m'$ is uniquely determined by $l'$. We obtain

**Proposition 5.6.** The space $\bigoplus_{l' \in \mathcal{M}(l)} \Theta_{l'}(H_{\eta_{l'}})$ forms the $\eta_l$-isotypic component of $L^2(N\mathbb{Z}\backslash N)$, where recall that $\bigoplus$ denotes the Hilbert space direct sum.

For a proof, see [2], §6.

By virtue of Proposition 5.6, the $\eta_l$-component of our Fourier expansion is given as

$$\sum_{l' \in \mathcal{M}(l)} C_{l,l'}^{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x) = \sum_{l' \in \mathcal{M}(l)} C_{l,l'}^{l'} \sum_{\gamma \in M' \cap N\mathbb{Z} \backslash N\mathbb{Z}} W_{\kappa,l'}(\gamma x a),$$

where $C_{l,l'}^{l'}$ denotes the constant factor of the Whittaker function $W_{\kappa,l'}$ with the boundary condition $W_{\kappa,l'}(1) = 1$. An element $\gamma \in N\mathbb{Z}$ can be decomposed into $\gamma = \gamma_S \gamma_L$ with $\gamma_S \in N_S(\mathbb{Z})$ and $\gamma_L \in N_L(\mathbb{Z})$ since $N\mathbb{Z} = N_S(\mathbb{Z}) \times N_L(\mathbb{Z})$. Noting this, the Whittaker function $W_{\kappa,l'}(\gamma x a)$ twisted by $\gamma \in N\mathbb{Z}$ can be written as

$$W_{\kappa,l'}(\gamma x a) = W_{\kappa,l'}(\gamma_S \gamma_L x S x L a) = \chi_{\text{Ad}_S^*} \gamma^{-1}_{L \cdot l'}(x S) W_{\kappa,l'}(\gamma_L x L a)$$

$$= \chi_{\text{Ad}_S^*} \gamma^{-1}_{L \cdot l'}(x S) W_{\kappa, \text{Ad}_S^* \gamma^{-1}_{L \cdot l'}}(x L a) = W_{\kappa, \text{Ad}_S^* \gamma^{-1}_{L \cdot l'}}(x a),$$

where we use Lemma 4.6 (1) in order to deduce the third equation. Lemma 5.5 yields a bijection:

$$M' \cap N\mathbb{Z} \backslash N\mathbb{Z} \simeq M' \cap N_L(\mathbb{Z}) \backslash N_L(\mathbb{Z}) \simeq \text{Ad}_S^* N_L(\mathbb{Z}) \cdot l'.$$

Therefore the $\eta_l$-component of the Fourier expansion can be written as

$$\sum_{l' \in \mathcal{M}(l)} C_{l,l'}^{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x) = \sum_{l' \in \mathcal{M}(l)} C_{l,l'}^{l'} \sum_{\gamma \in \text{Ad}_S^* N_L(\mathbb{Z}) \cdot l'} W_{\kappa,l'}(x a),$$
Due to the Koecher principle (cf. [7], Satz 1, Satz 2), a holomorphic Siegel modular form $f$ satisfies the moderate growth condition as follows:

$$|f(g)| < C_f \|g\|^{m_f} \quad \text{for any } g \in G,$$

with some constant $C_f$ and integer $m_f$, where $\|*\|$ denotes the norm on $\mathbb{C}$ defined as $|z| := \overline{z} \cdot z$. In fact, we can check this by observing the relation between modular forms on $G$ and those on the Siegel upper half space, which will be referred to just after Definition 5.9. Here we insert

**Lemma 5.7.** (1) A theta series $\Theta_l(W_{\kappa,l}(*)a)$ contributing to the Fourier expansion of a holomorphic Siegel modular form $f$ satisfies the moderate growth condition in the following sense:

$$\|\Theta_l(W_{\kappa,l}(*)a)(x)\|_{L^2} < C'|a|^{m'},$$

where $\|*\|_{L^2}$ denotes the $L^2$-norm on $L^2(N\mathbb{Z}\backslash N)$, and $C'$ and $m'$ are a constant and an integer not dependent on $a \in A$ respectively.

(2) The moderate growth condition for $\Theta_l(W_{\kappa,l}(*)a)$ stated in (1) holds if and only if $Y_n(l)$ is positive semi-definite.

**Proof.** (1) For a fixed $g \in G$, we regard $f(xg)$ as a function in $x \in N$. It belongs to $L^2(N\mathbb{Z}\backslash N)$ as we remarked in the formulation of the Fourier expansion.

Then, for any $a \in A$, we have

$$\|C_l\Theta_l(W_{\kappa,l}(*)a)\|_{L^2} < \|f(*)a\|_{L^2}$$

for a theta series $\Theta_l(W_{\kappa,l}(*)a)$ contributing to the Fourier expansion of $f$, where $C_l$ denotes the coefficient of $\Theta_l(W_{\kappa,l}(*)a)$ in $f$.

There exists an $x_{max} \in N$ such that $|f(x_{max}a)|$ is the maximal value of $|f(*)a|$ since $x_{max}$ is determined modulo $N\mathbb{Z}$ and $N\mathbb{Z}\backslash N$ is compact. Then we obtain

$$\|f(*)a\|_{L^2}^2 \leq \text{vol}(N\mathbb{Z}\backslash N)|f(x_{max}a)|^2,$$

where $\text{vol}(N\mathbb{Z}\backslash N)$ denotes the volume of $N\mathbb{Z}\backslash N$. Note that $x_{max}$ may depend on $a \in A$. But the moderate growth condition of $f$ implies

$$|f(x_{max}a)| < C'_f \|a\|^{m'_f}$$
Fourier Expansion

with a constant $C'_f$ and an integer $m'_f$ depending only on $f$. Hence
$\Theta_l(W_{\kappa,l}(*a))$ satisfies the moderate growth condition in the assertion.

(2) Theorem 4.12 and Corollary 4.15 mean that, under the assumption 2,
$Y_n(l)$ is positive semi-definite if and only if
\[ \|W_{\kappa,l}(*a)\|_l < C\|a\|^m \]
for any $a \in A$, with a constant $C$ and an integer $m$ depending only on $W_{\kappa,l}$.
In fact, Theorem 4.12 means that $W_{\kappa,l}(*a) \in H^\infty_{n_l}$ automatically holds under
the assumption 2.

By the definition of $\Theta_l$ given before Proposition 5.6, we have
\[ \|\Theta_l(W_{\kappa,l}(*a))\|_{L^2} = \text{vol}(N_{\mathbb{Z}} \cap M\setminus M)\|W_{\kappa,l}(*a)\|_l, \]
where $M$ denotes the polarization subgroup for $l$ and \text{vol}(N_{\mathbb{Z}} \cap M\setminus M) is the
volume of $N_{\mathbb{Z}} \cap M\setminus M$. Hence $Y_n(l)$ is positive semi-definite if and only if
\[ \|\Theta_l(W_{\kappa,l}(*a))\|_{L^2} < C\text{vol}(N_{\mathbb{Z}} \cap M\setminus M)\|a\|^m \]
with $C$ and $m$ as above. This means the assertion (2). $\square$

In order to express our Fourier expansion, we introduce the following
sets:
\[ \Omega_{n,Z} := \{ T \in M_n(\mathbb{Q}) \mid T \text{ is symmetric positive semi-definite semi-integral}\}, \]
\[ L := \{ l \in \mathfrak{n}^*_\mathcal{S} \mid Y_n(l) \in \Omega_{n,Z} \}, \quad \tilde{L} := L/\text{Ad}^\ast_{\mathcal{S}} N_L(\mathbb{Q}). \]
Here remark that the map $L \ni l \mapsto Y_n(l) \in \Omega_{n,Z}$ gives a bijection $L \simeq \Omega_{n,Z}$.

By virtue of Proposition 5.3 (2), Proposition 5.4 and Lemma 5.7, we see
that the totality of elements in $\tilde{N}$ occurring in our Fourier expansion is in
bijection with the set $\tilde{L}$. Therefore we can write our Fourier expansion of a
modular form $f$ as follows;

**Theorem 5.8.** Let $f$ be a holomorphic Siegel modular form of weight $\kappa$ with respect to $\Gamma$. Its Fourier expansion along the minimal parabolic sub-group can be written as
\[ f(xa) = \sum_{l \in \tilde{L}} \sum_{l' \in \tilde{M}(l)} C^l_{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x), \]
where \( \Theta^l(W_{\kappa,l^*}(a))_l(x) := \)

\[
\sum_{l^* \in \text{Ad}^{\times} \mathbb{N}_L(\mathbb{Z}), l} \chi_{l^*}(x_S)(a_1 a_2 \cdots a_n) \kappa \exp(-2\pi \text{Tr}(t^*(X_L A_n) Y_n(l^*)(X_L A_n)))
\]

**Definition 5.9.** We call the constants \( C^l \) Fourier coefficients of \( f \).

Let \( \mathfrak{H}_n \) be the Siegel upper half space of degree \( n \), defined by

\[
\{ Z = t^*Z \in M_n(\mathbb{C}) \mid \text{Im} Z \text{ is positive definite} \}
\]

where \( \text{Im} Z \) denotes the imaginary part of \( Z \). The group \( G \) acts on this via the linear fractional transformation

\[
\mathfrak{H}_n \ni Z \mapsto g \cdot Z := (AZ + B)(CZ + D)^{-1} \in \mathfrak{H}_n
\]

with \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \). For a \( Z \in \mathfrak{H}_n \), let \( g_Z \) be an element of \( G \) such that

\[
g_Z \cdot \sqrt{-11}_n = Z. \]

Since the stabilizer of \( \sqrt{-11}_n \) in \( G \) is the maximal compact subgroup \( K \), \( g_Z \) is uniquely determined modulo \( K \). For a holomorphic modular form \( f \) on \( G \), we define a function on \( \mathfrak{H}_n \) by

\[
F_f(Z) := \det(C \sqrt{-11}_n + D)^\kappa f(g_Z),
\]

where we write \( g_Z = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \). The map \( f \mapsto F_f \) provides a bijection between the space of holomorphic Siegel modular forms on \( G \) and the space of holomorphic Siegel modular forms on \( \mathfrak{H}_n \). We want to rewrite our Fourier expansion for \( F_f \). For that purpose, we introduce some symbols.

Let \( \Omega_{n,\mathbb{Z}} \) be the quotient of \( \Omega_{n,\mathbb{Z}} \) by an equivalence relation:

\[
S \sim S' \iff \text{there exists a } u \in U_n(\mathbb{Q}) \text{ such that } t^*uSu = S',
\]

where \( U_n(\mathbb{Q}) = U_n \cap GL_n(\mathbb{Q}) \). For a \( S \in \Omega_{n,\mathbb{Z}} \), we define the set \( \mathfrak{M}_n(S) \) as a quotient of the set \( \{ T \in \Omega_{n,\mathbb{Z}} | t^*uTu = S \exists u \in U_n(\mathbb{Q}) \} \) by an equivalence relation

\[
T \sim T' \iff \text{there exists a } u \in U_n(\mathbb{Z}) \text{ such that } t^*uTu = T',
\]
where \( U_n(Z) = U_n \cap GL_n(Z) \). For \( S \in \Omega_n, Z \), let \( l_S \in n^*_S \) such that \( Y_n(l_S) = S \). Lemma 4.6 (1) means that the sets \( \Omega_n, Z \) and \( \mathfrak{m}_n(S) \) are bijective with the sets \( \tilde{L} \) and \( \mathfrak{m}(l_S) \), respectively. Furthermore, for an \( T \in \Omega_n, Z \), set \( \Omega_n(T) := \{ uTu \mid u \in U_n(Z) \} \). This is in bijection with \( \text{Ad}_S^* N_L(Z) \cdot l_T \).

The map \( f \mapsto F_f \) sends the \( \eta \)-component \( \sum_{\nu \in \mathfrak{m}(l)} C^l_{\nu} \Theta\nu(W_{\kappa,\nu}(\ast a))(x) \) of the Fourier expansion of \( f \) to

\[
\sum_{T \in \mathfrak{m}_n(S)} C^S_T \Theta(T)(Z),
\]

where \( l, l' \in L \) correspond to \( S, T \in \Omega_n, Z \) respectively, and we rewrite \( C^l_{\nu} \) as \( C^S_T \) and set \( \Theta(T)(Z) := \sum_{R \in \Omega_n(T)} \exp 2\pi \sqrt{-1} \text{Tr}(RZ) \). As a result, we obtain our Fourier expansion for \( F_f \).

**Theorem 5.10.**

\[
F_f(Z) = \sum_{S \in \Omega_n, Z} \sum_{T \in \mathfrak{m}_n(S)} C^S_T \Theta(T)(Z).
\]

**Remark 5.11.** Let \( \Omega := \{ Y \in M_n(\mathbb{R}) \mid Y:\text{symmetric positive-definite} \} \). The theta series \( \Theta(T)(Z) \) defines a holomorphic function on \( \mathfrak{h}_n \).

In fact, since \( \Omega \simeq N_L A \) via the linear functional transformation, the uniform convergence of the absolute value of \( \Theta(T)(Z) \) on any compact subset of \( \mathfrak{h}_n \) is equivalent to that of \( \Theta(l(W_{\kappa, l}(\ast a))(x) \) on any compact subset of \( N_L A \), with \( l \in L \) such that \( Y_n(l) = T \). The latter condition is justified by Lemma 5.7 (2).

**6. Comparison with the Other Two Fourier Expansions**

In this section, we compare our Fourier expansion with the other two known Fourier expansions, i.e. classical Fourier expansion and Fourier-Jacobi expansion. This section consists of two subsections §6.1 and §6.2. In §6.1 (resp. §6.2), we consider the comparison with the classical expansion (resp. Fourier-Jacobi expansion).
6.1. Comparison with the classical Fourier expansion

Let $F_f$ be as in the previous section. As is well-known, the classical Fourier expansion of $F_f$ can be written as

$$F_f(Z) = \sum_{T \in \Omega_{n,Z}} C_T \exp 2\pi \sqrt{-1} \text{Tr} TZ,$$

where $C_T$ denotes the Fourier coefficient indexed by $T$. Compare this classical expansion with our expansion in Theorem 5.10. Then we obtain a relation between the Fourier coefficients of the classical expansion and those of our expansion.

**Theorem 6.1.** Let $T \in \Omega_{n,Z}$ belong to $\mathfrak{M}_n(S)$ with some $S \in \Omega_{n,Z}$ and $C^S_T$ denote the Fourier coefficient of our Fourier expansion in Theorem 5.10. Then we have

$$C^S_T = C_T$$

and, for every $u \in U_n(Z)$,

$$C_{tu}T_u = C^S_T.$$

**Remark 6.2.** This relation of Fourier coefficients is compatible with a well-known formula

$$C_{\gamma T \gamma} = C_T$$

for any $\gamma \in SL_n(Z)$. Noting this relation, we can deduce our Fourier expansion in Theorem 5.10 from the classical expansion since

$$\sum_{R \in \Omega_n(T)} C_R \exp 2\pi \sqrt{-1} \text{Tr}(RZ) = C_T \Theta_T(Z).$$

6.2. Comparison with the Fourier-Jacobi expansion

For a field $F$, $M_{m,n}(F)$ denotes the set of matrices with their size $m \times n$ and coefficients in $F$. If $m = n$, it is nothing but $M_n(F)$.

Let $Z = \begin{pmatrix} Z_1 & Z_2 \\ tZ_2 & Z_3 \end{pmatrix} \in \mathfrak{H}_n$ with $Z_1 \in M_j(\mathbb{C})$, $Z_2 \in M_{j,n-j}(\mathbb{C})$ and $Z_3 \in M_{n-j}(\mathbb{C})$, where $1 \leq j \leq n - 1$. The Fourier-Jacobi expansion of a holomorphic form $F_f$ on $\mathfrak{H}_n$ is written as

$$F_f(Z) = \sum_{T_1 \in \Omega_{j,Z}} \phi_{T_1}(Z_2, Z_3) \exp 2\pi \sqrt{-1} \text{Tr} T_1 Z_1,$$
where
\[
\phi_{T_1}(Z_2, Z_3) := \sum_{T \in \Omega_{T_1}} C_T \exp 2\pi \sqrt{-1}(\mathrm{Tr}(2^t T_2 Z_2 + T_3 Z_3)),
\]
with \( \Omega_{T_1} := \left\{ T = \begin{pmatrix} T_1 & T_2 \\ \dagger T_2 & T_3 \end{pmatrix} \in \Omega_{n, \mathbb{Z}} \mid T_2 \in M_{j,n-j}(\mathbb{Q}), \ T_3 \in M_{j-n}(\mathbb{Q}) \right\}. \)

As is well-known, \( \phi_{T_1} \) is a Jacobi form of weight \( \kappa \) and index \( T_1 \) (for a definition, see [17], Definition 1.3).

For an \( S_1 \in \Omega_{j, \mathbb{Z}} \), let \( \tilde{\Omega}_{S_1} \) denote the quotient of \( \Omega_{S_1} \) by an equivalence relation
\[
S \sim S' \iff \exists u \in U_n(\mathbb{Q}).
\]

For our purpose, we need

**Lemma 6.3.**
\[
\bigcup_{T_1 \in \mathfrak{M}_j(S_1)} \bigcup_{R_1 \in \Omega_j(T_1)} \Omega_{R_1} = \bigcup_{S \in \tilde{\Omega}_{S_1}} \bigcup_{T \in \mathfrak{M}_n(S)} \Omega_n(T).
\]

**Proof.** The unions appearing in the both sides of the equation above are all disjoint. It suffices to prove that each \( \Omega_{R_1} \) (resp. \( \Omega_n(T) \)) is contained in the right hand side (resp. left hand side). The upper-left \( j \times j \) component of each element in \( \Omega_n(T) \) is in \( \bigcup_{T_1 \in \mathfrak{M}_j(S_1)} \Omega_j(T_1) \). Hence \( \Omega_n(T) \) forms a subset of the set on the left hand side. The set \( \Omega_{R_1} \) can be written as
\[
\Omega_{u_1 S_1 u_1} = \begin{pmatrix} t u_1 & \\ 1_{n-j} & u_1 \\ 1_{n-j} \end{pmatrix} \Omega_{S_1} \begin{pmatrix} u_1 & \\ 1_{n-j} & u_1 \end{pmatrix}
\]

with some \( u_1 \in U_j(\mathbb{Q}) \). So we see that \( \Omega_{R_1} \) is contained in the right hand side. Therefore the assertion is verified. \( \square \)

From this lemma, we deduce a relation between the Fourier-Jacobi coefficients and theta series \( \Theta_T \), stated as

**Theorem 6.4.**
\[
\sum_{T_1 \in \mathfrak{M}_j(S_1)} \sum_{R_1 \in \Omega_j(T_1)} \phi_{R_1}(Z_2, Z_3) \exp(2\pi \sqrt{-1} \mathrm{Tr} R_1 Z_1) = \sum_{S \in \tilde{\Omega}_{S_1}} \sum_{T \in \mathfrak{M}_n(S)} C_T^S \Theta_T(Z).
\]
Proof. Lemma 6.3 means that the equation above formally holds. The convergence of the infinite sums on both sides is justified since the Fourier series of \( f \) in the sense of classical Fourier expansion, which uniformly absolutely converges on \( \Gamma \backslash \mathfrak{g}_n \), is a majorant of them. \( \square \)

Consider the case \( j = 1 \). Then the formula in Lemma 6.3 is rewritten as

\[
\Omega_{S_1} = \bigcup_{S \in \Omega_{S_1}} \bigcup_{T \in \mathcal{M}_n(S)} \Omega_n(T)
\]

since \( U_1 = \{1\} \). This means

**Corollary 6.5.** When \( j = 1 \), one obtains

\[
\phi_{S_1}(Z_2, Z_3) \exp 2\pi \sqrt{-1} \text{Tr} S_1 Z_1 = \sum_{S \in \Omega_{S_1}} \sum_{T \in \mathcal{M}_n(S)} C^S_T \Theta_T(Z).
\]

**Remark 6.6.** Our Fourier expansion along the minimal parabolic subgroup, stated as Theorem 5.8 and Theorem 5.10, is the most coarse one. In this paper, we give a comparison of the Fourier expansions along the extremal parabolic subgroups, i.e. the maximal parabolic subgroups and the minimal parabolic subgroup, in terms of Fourier coefficients and theta series appearing in the expansions. We think that such comparison seems to be also possible for Fourier expansions along arbitrary parabolic subgroups.

**References**


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