Irrationality of Fast Converging Series of Rational Numbers

By Daniel Duverney

Abstract. We say that the series of general term $u_n \neq 0$ is fast converging if $\log|u_n| \leq c2^n$ for some $c < 0$. We prove irrationality results and compute irrationality measures for some fast converging series of rational numbers, by using Mahler’s transcendence method in the form introduced by Loxton and Van der Poorten. With very weak assumptions on sequence $u_n$, this method allows to obtain only irrationality results.

1. Introduction

Let $(u_n)$ be a sequence of complex numbers. We say that $\sum_{n=0}^{+\infty} u_n$ is a fast converging series if

$$|u_n| \leq C h^{2^n}$$

for some constants $C \in ]0, +\infty[ \ and \ h \in ]0, 1[.$

In this paper, we will be interested in the irrationality properties of fast converging series of rational numbers of the form

$$S = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n},$$

with $a_n \in \mathbb{Z} \setminus \{0\}, \ b_n \in \mathbb{Z} \setminus \{0\}, \ u_n \in \mathbb{N} \setminus \{0\}$ and satisfying

$$\begin{cases}
\lim_{n \to +\infty} u_n = +\infty \\
 cu_n^2 \leq u_{n+1} \leq c' u_n^2 \\
a_n = O(u_n^\alpha) \\
b_n = O(u_n^\varepsilon)
\end{cases} \quad \text{for some positive constants } c \text{ and } c' \text{ for some constant } \alpha \in ]0, 1[ \text{ for every } \varepsilon > 0.$$
It is well-known since Liouville [15] that one can prove irrationality and transcendence results by approximating real numbers by sequences of rationals with good convergence properties. More precisely, the basic (and elementary) result in diophantine approximation theory, with respect to irrationality problems, is the following

**Theorem 1.1.** Let \( \alpha \in \mathbb{R} \). Suppose that there exists a sequence \( P_n/Q_n \) of rational numbers satisfying

\[
0 < \left| \alpha - \frac{P_n}{Q_n} \right| \leq \frac{\varepsilon(n)}{Q_n}
\]

with \( \lim_{n \to +\infty} \varepsilon(n) = 0 \). Then \( \alpha \) is irrational.

For a proof, see for example [5, Theorem 1.5].

In some elementary cases, Theorem 1.1 allows to prove the irrationality of \( \alpha \) when \( \alpha \) is a series of rational numbers and \( P_n/Q_n \) is the partial sum of order \( n \). The most classical proofs of irrationality using this method date back to Fourier [10], who gave the now standard elementary proof of the irrationality of \( e \), and to Liouville himself [15], who proved the irrationality of

\[
\theta = \sum_{n=0}^{+\infty} \frac{1}{m^{n^2}}, \quad m \in \mathbb{N} \setminus \{0, 1\}.
\]

However, these two proofs rely heavily on two facts :
- First, the numerators in the series are all equal to 1.
- Second, every denominator in the series divides the following one : \( n! \) divides \( (n + 1)! \), and \( m^{n^2} \) divides \( m^{(n+1)^2} \).

Hence, these two series are Engel series ([5, Chapter 2], [22, Chapter 4], for example), for which theorem 1.1 allows to obtain an irrationality criterion.

However, it seems impossible to obtain a general criterion for series converging so slowly. On the contrary, conditions (1.3), which rest only on the speed of convergence of the series (and not on the arithmetical properties of its terms), allow to obtain such a criterion (Theorem 3.1 below).

The scope of the paper is as follows. In section 2, we will present a brief review of the history of fast converging series and their irrationality
properties. As in this paper we are not interested in their transcendence properties, we will only mention Mahler’s method [20] when it works. In section 3, we will present our new results, including irrationality statements and computation of irrationality measures. Section 4 will be devoted to the proof of our main irrationality statement (Theorem 3.1.). In section 5, we will prove corollaries to Theorem 3.1. Finally, in section 6, we will prove theorem 3.2, which gives irrationality measures for fast converging series.

2. A Brief History of Fast Converging Series

2.1. Sylvester and Lucas

The oldest result on the irrationality of fast converging series seems to be due to Sylvester [24], who proved in 1880

**Theorem 2.1.** Let $\alpha \in ]0, 1]$. Then $\alpha$ can be expanded, in a unique way, in a series of the form

\begin{equation}
\alpha = \sum_{n=0}^{+\infty} \frac{1}{u_n},
\end{equation}

with $u_n \in \mathbb{N} \setminus \{0, 1\}$ and $u_{n+1} \geq u_n^2 - u_n + 1$ for every $n \in \mathbb{N}$. Moreover, $\alpha$ is rational if and only if,

\begin{equation}
u_{n+1} = u_n^2 - u_n + 1
\end{equation}

for every $n \geq N_0$.

For a proof of Theorem 2.1, see [24], [22, Chapter 4], [11, Chapter 1], [5, Exercise 2.9]. As a matter of fact, it is very easy to verify that $\alpha$ in (2.1) is rational when (2.2) is satisfied; indeed

\[ u_{n+1} = u_n^2 + u_n + 1 \iff \frac{1}{u_{n+1} - 1} = \frac{1}{u_n - 1} - \frac{1}{u_n}, \]

so that, in this case,

\[ \sum_{n=N_0}^{+\infty} \frac{1}{u_n} = \sum_{n=N_0}^{+\infty} \left( \frac{1}{u_n - 1} - \frac{1}{u_{n+1} - 1} \right) = \frac{1}{u_{N_0} - 1} \in \mathbb{Q}. \]
Sylvester’s theorem 2.1 shows that fast converging series of rational numbers can achieve rational values. Another example can be obtained from a formula given by Lucas in 1878 ([17], [18, p. 184]):

\[
\sum_{n=0}^{+\infty} \frac{x^{2n}}{1-x^{2n+1}} = \frac{x}{1-x} \quad (|x| < 1).
\]

In the formula (2.3) as in Sylvester’s theorem, the terms of the series cancel each other, because

\[
\frac{x^{2n}}{1-x^{2n+1}} = \frac{x^{2n}}{1-x^{2n}} - \frac{x^{2n+1}}{1-x^{2n+1}},
\]

which proves (2.3).

For \( x = a/b \), with \( a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\} \) and \( |a| < b \), we obtain a fast converging series whose sum is rational :

\[
\sum_{n=0}^{+\infty} \frac{a^{2n}b^{2n}}{b^{2n+1} - a^{2n+1}} = \frac{a}{b-a}.
\]

It is interesting to observe, following Lucas, that (2.3) also gives a fast converging series whose sum is an irrational quadratic number; indeed, if we take \( x = 1/\Phi \), where

\[
\Phi = \frac{1 + \sqrt{5}}{2}
\]

is the golden number, we obtain

\[
\sum_{n=0}^{+\infty} \frac{1}{F_{2n}} = \frac{7 - \sqrt{5}}{2},
\]

where \( F_n \) is Fibonacci sequence, defined by

\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1).
\]

Naturally, in (2.6) sequence \( F_n \) can be replaced by any Lucas sequence [23, p. 41].
2.2. Golomb, Erdős and Strauss

Now we have to jump over more than 80 years, exactly until 1963. At this time, Golomb [12] proved the irrationality of the sum of the reciprocals of the Fermat numbers

\[ S_1 = \sum_{n=0}^{+\infty} \frac{1}{2^n + 1}. \]

The proof is very complicated for such a result (in fact, \( S_1 \) is transcendental by Mahler’s theorem ([19], [20, p.5], [5, Exercise 12.13]), but contains some interesting remarks. It rests on the formula

\[ \sum_{n=0}^{+\infty} \frac{x^{2^n}}{1 + x^{2^n}} + \sum_{n=0}^{+\infty} \frac{x^{2^n}}{1 - x^{2^n}} = \frac{2x}{1-x}, \quad (|x| < 1), \]

which is a direct consequence of (2.3). Taking \( x = \frac{1}{2} \) in (2.9), Golomb obtains

\[ S_1 = 2 - S_2, \quad S_2 = \sum_{n=0}^{+\infty} \frac{1}{2^{2n} - 1}. \]

He then proves that the expansion of \( S_2 \) in base 2 is not periodic, which proves that \( S_2 \), and therefore \( S_1 \), is irrational. However, this part of the proof is too complicated, because the partial sums of the series giving \( S_2 \) are sufficient to prove its irrationality as indicated in section 1.

Golomb’s paper motivated an important work of Erdős and Strauss [7]; these authors studied irrationality of fast converging series of the form

\[ S_3 = \sum_{n=0}^{+\infty} \frac{1}{u_n}, \]

where \( u_n \in \mathbb{N} \setminus \{0\} \). They called such series Ahmes series (Ahmes was the Egyptian mathematician who wrote the Rhindt papyrus more than 3000 years ago). They didn’t completely succeeded in giving a criterion of irrationality without arithmetical conditions on the \( u_n \)’s. For example, they proved
**Theorem 2.2.** Let \( u_n \) be an increasing sequence of positive integers satisfying

\[
\limsup \frac{u_{n+1}}{u_n^2} \geq 1
\]

\[
LCM(u_0, u_1, \ldots, u_n)/u_{n+1} \text{ is bounded.}
\]

Then the series \( S_3 \) is rational if and only if \( u_{n+1} = u_n^2 - u_n + 1 \) for all \( n \geq n_0 \).

As an application, Erdös and Strauss obtained the following remarkable generalization of Golomb’s result.

**Theorem 2.3.** Let \( a \in \mathbb{N} \setminus \{0, 1\} \), and let \( b_n \in \mathbb{Z} \), such that the series \( \sum |b_n a^{-2^n}| \) is convergent and \( a^{2^n} + b_n \neq 0 \) for every \( n \geq 0 \). Then

\[
S_4 = \sum_{n=0}^{+\infty} \frac{1}{a^{2^n} + b_n}
\]

is irrational.

Erdös’ works on this subject led him to set the following question ([8, p. 64], [9, p. 105]) :

\[
(2.15) \text{Is it true that if } \frac{u_{n+1}}{u_n^2} \to 1 \text{ then } \sum_{n=0}^{+\infty} \frac{1}{u_n} \text{ is irrational unless } u_{n+1} = u_n^2 - u_n + 1 \text{ for } n \geq n_0 ?
\]

We will give a partial answer to this question in Corollary 3.2.

### 2.3. Recent results

In 1993 Badea [2] generalized Sylvester’s results and proved the following

**Theorem 2.4.** Let \( a_n, u_n \) be sequences of positive integers such that the series

\[
S_5 = \sum_{n=0}^{+\infty} \frac{a_n}{u_n}
\]

is convergent. Suppose that

\[
(2.16) \quad u_{n+1} \geq \frac{a_{n+1}}{a_n} (u_n^2 - u_n) + 1.
\]
Then $S_5$ is a rational number if and only if

$$u_{n+1} = \frac{a_{n+1}}{a_n}(u_n^2 - u_n) + 1$$

for every $n \geq N$.

In fact, in [2] Theorem 1.5 appeared as a corollary of a more general result whose proof is based on the fact that any non increasing sequence $k_{n+1} \leq k_n$ of positive integers must be constant for $n \geq N$; this proof is similar to the proof of Sylvester’s theorem 1.2, although it is more complicated.

Motivated by Badea’s work, Hančl gave in 1996 another criterion of irrationality for fast converging series of rational numbers [13]. As an application, he obtained

**THEOREM 2.5.** Let $k$ be a positive integer, and let $u_n$ be a sequence of positive integers such that $u_1 > 2$ and

$$k u_{n-1}^2 - (3k - 1)u_{n-1} < u_n < ku_{n-1}^2 - ku_{n-1}$$

for every $n \geq n_0$. Then the number

$$S_6 = \sum_{n=1}^{+\infty} \frac{k^n}{u_n}$$

is irrational.

It should be noted that, for the first time, Badea considered fast converging series of rational numbers with numerators different from 1. However, in Badea’s as well as in Hančl’s results, the numerators must be positive.

Recently, I proved in [6] the following

**THEOREM 2.6.** Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ satisfy

$$\begin{align*}
\lim_{n \to +\infty} u_n &= +\infty \\
\beta &= \frac{u_{n+1}}{u_n^2} + O(u_n^\gamma), \quad \beta \in \mathbb{Q}_+, \quad 0 \leq \gamma < 2 \\
\log |a_n| &= o(2^n), \quad \log |b_n| = o(2^n).
\end{align*}$$
Then $S_7 = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n}$ is rational if and only if

\[(2.20) \quad u_{n+1} = \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}} \]

for $n \geq n_0$.

It is clear that Theorem 2.5 is a direct consequence of Theorem 2.6.

The method used for proving theorem 2.6 is quite different from the one used by Badea and Hančl. It comes directly from Mahler’s transcendence method in the form of Loxton and Van der Poorten [16], and turns out to be rather similar to the method of Erdős and Strauss. Mahler’s transcendence method allows to obtain irrationality results in some cases, as it has been recently observed (see [3], [4], [1]).

The new results presented in section 3 are extensions of Theorem 2.6, basically obtained by following the same ideas.

2.4. More fast converging series with rational sums

Let us give some other amusing examples of fast converging series of rational numbers with rational sums. We will use Theorem 2.6. Suppose that $a_n, b_n$ and $u_n$ in series $S_7$ satisfy (2.20) for $n \geq 1$ and for some $\beta \in \mathbb{Q}_+^*$. Then, by arguing the same way as for Sylvester series, we have

$$u_{n+1} - \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}} = \beta u_n \left( u_n - \frac{a_{n+1} b_n}{\beta a_{n+1} b_{n+1}} \right).$$

Hence

$$\frac{a_{n+1}}{b_{n+1}} u_{n+1} - \frac{1}{\beta a_{n+1} b_{n+2}} = \frac{a_n}{b_n} \left( \frac{1}{u_n - \frac{a_{n+1} b_n}{\beta a_{n+1} b_{n+1}}} - \frac{1}{u_n} \right).$$

Therefore

$$\sum_{n=1}^{+\infty} \frac{a_n}{b_n u_n} = \sum_{n=1}^{+\infty} \left( \frac{a_n}{b_n} u_n - \frac{1}{\beta a_n b_{n+1}} \right) - \frac{a_{n+1}}{b_{n+1}} u_{n+1} - \frac{1}{\beta a_{n+1} b_{n+2}}.$$
and we obtain

\[(2.21) \quad \sum_{n=1}^{+\infty} \frac{a_n}{b_n u_n} = \frac{a_1}{b_1 u_1 - \frac{a_2 b_1}{\beta a_1 b_2}}.\]

**Example 2.1.** Let $\beta = 1$, $a_n = 2^n$, $b_n = 1$, $u_n = \alpha^{2^n} + 1$, where $\alpha \in \mathbb{N} \setminus \{0, 1\}$. It is easy to check that $u_{n+1} = u_n^2 - 2u_n + 2$. Therefore, (2.20) is satisfied and we get, by using (2.21),

\[(2.22) \quad \sigma_1(\alpha) = \sum_{n=1}^{+\infty} \frac{2^n}{\alpha^{2^n} + 1} = \frac{2}{\alpha^2 - 1} \quad (\alpha \in \mathbb{N} \setminus \{0, 1\}).\]

**Example 2.2.** Let $\alpha \in \mathbb{Z} \setminus \{0\}$, and consider polynomial $P(X) = X^2 - \alpha X - 1$. Let $\omega > 1$ be a root of $P$. Then the other root is $-1/\omega$. Define the general Lucas sequence $V_n$ [22, p. 41] by

\[(2.23) \quad V_n = \omega^n + \left(\frac{-1}{\omega}\right)^n.\]

Note that $V_n \in \mathbb{N}$ and that $V_2 = \left(\omega - \frac{1}{\omega}\right)^2 + 2 = \alpha^2 + 2$. In the case where $\alpha = 1$, $\omega$ is the golden number $\Phi$ and $V_n$ is the classical Lucas sequence

\[(2.24) \quad L_n = \Phi^n + \left(\frac{-1}{\Phi}\right)^n.\]

Let $\beta = 1$, $a_n = 4^n$, $b_n = 1$, $u_n = V_{2n} + 2$. One checks easily that, for $n \geq 1$,

\[
u_n^2 - 4u_n + 4 = (\omega^{2^n} + \omega^{-2^n} + 2)^2 - 4(\omega^{2^n} + \omega^{-2^n} + 2) + 4 = \omega^{2^{n+1}} + \omega^{-2^{n+1}} + 2 = u_{n+1}.
\]

Hence (2.21) applies and we get

\[(2.25) \quad \sigma_2(\alpha) = \sum_{n=1}^{+\infty} \frac{4^n}{V_{2n} + 2} = \frac{4}{V_2 - 2} = \frac{4}{\alpha^2}.\]
Example 2.3. With the notations of example 2.2, let $\beta = 1$, $a_n = (-2)^n$, $b_n = 1$, $u_n = V_{2n} - 1$. Then
\[
 u_n^2 + 2u_n - 2 = (\omega^{2n} + \omega^{-2n} - 1)^2 + 2(\omega^{2n} + \omega^{-2n} - 1) - 2 \\
= \omega^{2n+1} + \omega^{-2n+1} - 1 = u_{n+1}.
\]
Therefore, by using (2.21), we have
\[
(2.26) \quad \sigma_3(\alpha) = \sum_{n=1}^{+\infty} \frac{(-2)^n}{V_{2n} - 1} = -\frac{2}{V_2 + 1} = -\frac{2}{\alpha^2 + 3}.
\]

Remark 2.1. Formulas (2.22) and (2.25) are well-known (see for example [21, p.140] and [14, Theorem 3]. Very likely, it is the same for the formula (2.26), but I don’t know any reference. Series $\sigma_1, \sigma_2$ and $\sigma_3$ will appear naturally in Corollary 3.5 below.

It is interesting remarking that formulas (2.22), (2.25), and (2.26) come from sums of rational fractions.

Theorem 2.7. For every $x \in \mathbb{C}$, with $|x| < 1$,
\[
(2.27) \quad \sum_{n=1}^{+\infty} \frac{2^n x^{2n}}{1 + x^{2n}} = \frac{2x^2}{1 - x^2},
\]
\[
(2.28) \quad \sum_{n=1}^{+\infty} \frac{4^n x^{2n}}{(1 + x^{2n})^2} = \frac{4x^2}{(1 - x^2)^2},
\]
\[
(2.29) \quad \sum_{n=1}^{+\infty} \frac{(-2)^n x^{2n}}{x^{2n+1} - x^{2n} + 1} = \frac{-2x^2}{x^4 + x^2 + 1}.
\]

Proof. For proving (2.27), observe that the function
\[
f(x) = \sum_{n=1}^{+\infty} \frac{2^n x^{2n}}{1 + x^{2n}}
\]
is analytic in $D = \{x \in \mathbb{C}/|x| < 1\}$. As (2.27) holds for every $x = 1/\alpha$ with $\alpha \in \mathbb{N} \setminus \{0, 1\}$ by (2.22), (2.27) holds in $D$ by analytic continuation.
The proof of (2.28) and (2.29) is the same; observe that (2.28) and (2.29) hold for every $x = -1/\omega = 1/2(\alpha - \sqrt{\alpha^2 + 4})$ when $\alpha \in \mathbb{N} \setminus \{0\}$ by (2.25) and (2.26). When $\alpha \to +\infty$, $x \to 0$, which proves (2.28) and (2.29) by analytic continuation. □

**Remark 2.2.** One can obtain other formulas from (2.27), (2.28), (2.29) by term-by-term derivation. For example, by deriving (2.27) we obtain (2.28). But (2.29) seems of a different nature. If we differentiate it term-by-term, we get

$$
\sum_{n=1}^{+\infty} \frac{(-1)^n x^{2^n} (1 - x^{2^{n+1}})}{(x^{2^{n+1}} - x^{2^n} + 1)^2} = \frac{2x^2(x^2 - 1)}{(x^4 + x^2 + 1)^2}.
$$

**Example 2.4.** If we replace, in (2.28), (2.29), and (2.30), $x$ by $1/\alpha$ with $\alpha \in \mathbb{N} \setminus \{0,1\}$, we obtain other fast converging series of rational numbers with rational sums. However, these series do not satisfy the assumptions of theorem 3.1 below, contrary to (2.22), (2.25), and (2.26): the numerators $a_n$ are too large.

### 3. Presentation of the results

Our main result will be the following

**Theorem 3.1.** Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ be sequences satisfying conditions (1.3) with $\alpha < 1/7$. Let

$$
S = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n}.
$$

Then, if $S$ is rational, there exist sequences $p_n \in \mathbb{N} \setminus \{0\}$, $q_n \in \mathbb{N} \setminus \{0\}$, depending only on $u_n$ (and not on $a_n$ and $b_n$), such that

$$

u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1} b_n}{u_n} u_n + \frac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}}
$$

Then, if $S$ is rational, there exist sequences $p_n \in \mathbb{N} \setminus \{0\}$, $q_n \in \mathbb{N} \setminus \{0\}$, depending only on $u_n$ (and not on $a_n$ and $b_n$), such that
for every $n \geq N(\alpha)$ and, for every $\mu \in ]3\alpha, 1 - 4\alpha[$,

\begin{equation}
\begin{cases}
p_n = O(u_n^{\mu-2}u_{n+1}), & q_n = O(u_n^{\mu}) \\
\frac{|u_{n+1}^2 - p_n|}{u_n^2 - q_n} \leq \frac{1}{q_n u_n^\alpha}.
\end{cases}
\end{equation}

As for Sylvesters', Erdös and Strauss, and Badea's results, there is reciprocal to this theorem. Indeed, if (3.2) is satisfied, then

\[ u_{n+1} - \frac{a_{n+2}b_{n+1}q_{n+1}}{a_{n+1}b_{n+2}p_{n+1}} = \frac{p_n}{q_n} u_n \left( \frac{u_n - \frac{a_{n+1}b_nq_n}{a_nb_{n+1}p_n}}{a_{n+1}b_{n+2}p_{n+1}} \right). \]

Hence

\[ \frac{a_{n+1}}{b_{n+1}} \cdot \frac{1}{u_{n+1} - \frac{a_{n+2}b_{n+1}q_{n+1}}{a_{n+1}b_{n+2}p_{n+1}}} = \frac{a_n}{b_n} \left( \frac{1}{u_n - \frac{a_{n+1}b_nq_n}{a_nb_{n+1}p_n}} - \frac{1}{u_n} \right). \]

Therefore $\frac{a_n}{b_n u_n}$ can be written as a difference of consecutive terms of the same sequence, and we have, with $N = N(\alpha)$,

\[ \sum_{n=N}^{+\infty} \frac{a_n}{b_n u_n} = \sum_{n=N}^{+\infty} \left( \frac{a_n}{b_n} \cdot \frac{1}{u_n - \frac{a_{n+1}b_nq_n}{a_nb_{n+1}p_n}} - \frac{a_{n+1}}{b_{n+1}} \cdot \frac{1}{u_{n+1} - \frac{a_{n+2}b_{n+1}q_{n+1}}{a_{n+1}b_{n+2}p_{n+1}}} \right) \]

\[ = \frac{a_N}{b_N} \cdot \frac{1}{u_N - \frac{a_{N+1}b_Nq_N}{a_Nb_{N+1}p_N}} \in \mathbb{Q}. \]

Proving theorem 3.1 will be more difficult. This will be done in section 5.

In Theorem 3.1, unfortunately, the sequence $p_n/q_n$ is not explicitly known. However, this restriction can be overcome in some cases. For example, consider the entire function

\begin{equation}
f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{u_n},
\end{equation}
with $u_n \in \mathbb{N} \setminus \{0\}$ and

$$
(3.5) \quad \begin{cases} 
\lim_{n \to +\infty} u_n = +\infty \\
cu_n^2 \leq u_{n+1} \leq c'u_n^2 
\end{cases}
$$

for some constants $c > 0$ and $c' > 0$.

Then almost all values of $f(x)$ at rational points are irrational; more precisely we have

**Corollary 3.1.** Let $(u_n) \in \mathbb{N} \setminus \{0\}$ satisfy (3.5), and let $f(x)$ be defined by (3.4). Then $f(r)$ is irrational for every $r \in \mathbb{Q}^*$, except perhaps for one value of $r$.

As a second example, we can give a partial answer to question (2.15).

**Corollary 3.2.** Let $u_n \in \mathbb{N} \setminus \{0\}$ satisfy $\lim_{n \to +\infty} u_n = +\infty$ and

$$
(3.6) \quad \sum_{n=0}^{+\infty} \left( \frac{u_{n+1}}{u_n^2} - 1 \right) < \infty.
$$

Suppose that $a_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$. Then $\sum_{n=0}^{+\infty} a_n \in \mathbb{Q}$ if and only if

$$
u_{n+1} = u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_n+2}{a_{n+1}}
$$

for every $n \geq N$.

This contains, as a special case, Theorem 2.3 of Erdös and Strauss, as well as its alternate case, namely

**Corollary 3.3.** Let $a \in \mathbb{N} \setminus \{0, 1\}$, and let $b_n \in \mathbb{Z}$ such that the series $\sum |b_n| a^{-2n}$ is convergent and $a^{2n} + b_n \neq 0$ for every $n \geq 0$. Let $\varepsilon = \pm 1$. Then

$$
S_8 = \sum_{n=0}^{+\infty} \frac{\varepsilon^n}{a^{2n} + b_n} \notin \mathbb{Q}.
$$

Theorem 3.1 also allows us to generalize Theorem 2.6 to the case where $\beta \notin \mathbb{Q}$.
Corollary 3.4. Let \( a_n \in \mathbb{Z} \setminus \{0\}, b_n \in \mathbb{Z} \setminus \{0\}, u_n \in \mathbb{N} \setminus \{0\} \) satisfy
\[
\begin{cases}
\lim_{n \to +\infty} u_n = +\infty, \\
u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \quad \beta \in \mathbb{R}^*_+, \quad 0 \leq \gamma < 2,
\end{cases}
\]
(3.8)
\[
\log |a_n| = o(2^n), \quad \log |b_n| = o(2^n).
\]

Then \( S_9 = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n} \) is rational if and only if \( \beta \) is rational and
\[
u_{n+1} = \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}}.
\]
(3.9)

As a special case of corollary 3.4, we obtain irrationality results on series containing linear recurring sequences with subscripts in geometric progression.

Corollary 3.5. Suppose that \( d \in \mathbb{N} \setminus \{0\}, a_n \in \mathbb{Z} \setminus \{0\}, b_n \in \mathbb{Z} \), with
\[
\log |a_n| = o(2^n), \quad \log |b_n| = o(2^n) \text{ if } b_n \neq 0.
\]
(3.10)

Suppose that \( v_n \in \mathbb{N} \) is a linear recurring sequence satisfying
\[
\begin{cases}
v_{n+d} = \sum_{h=1}^{d} \alpha_h v_{n+d-h}, \\
v_n = \sum_{h=1}^{d} A_h \omega_h^n,
\end{cases}
\]
(3.11)

with \( \alpha_h \in \mathbb{Z}, A_h \in \mathbb{R}^*, \omega_h \in \mathbb{R}^* \) for \( h = 1, \ldots, d, \alpha_d \neq 0 \), and
\[
|\omega_1| > |\omega_2| > \cdots > |\omega_d|, \quad |\omega_1| > 1.
\]
(3.12)

Assume that \( v_{2n} + b_n \neq 0 \) for every \( n \in \mathbb{N} \setminus \{0\} \). Then
\[
S_{10} = \sum_{n=1}^{+\infty} \frac{a_n}{v_{2n} + b_n},
\]
is irrational, except if there exist rational numbers \( p \) and \( q \), a rational integer \( \alpha \), and \( i \in \{1, 2, 3\} \), such that \( S_{10} = p + q \sigma_i(\alpha) \); in these cases \( S_{10} \) is rational by (2.22), (2.25) and (2.26).
**Example 3.1.** Let $\alpha \in \mathbb{N} \setminus \{0,1\}$. For $v_n = \alpha^n$, we obtain under hypothesis (3.10)

$$S_{11} = \sum_{n=1}^{+\infty} \frac{a_n}{\alpha^{2n} + b_n} \notin \mathbb{Q},$$

except if $a_n = k2^n$ and $b_n = 1$ for every $n \geq N$, where $k$ is a non zero constant natural number.

This generalizes corollary 3.3, under a slightly stronger hypothesis on $b_n$.

**Example 3.2.** If we consider Fibonacci sequence, which satisfies

$$(3.13) \quad F_n = \frac{1}{\sqrt{5}} \left( \Phi^n - \left( -\frac{1}{\Phi} \right)^n \right),$$

where $\Phi$ is the golden number, we obtain, under hypothesis (3.10):

$$S_{12} = \sum_{n=1}^{+\infty} \frac{a_n}{F_{2n} + b_n} \notin \mathbb{Q}.$$

Note that transcendence and algebraic independence results on series like $S_{12}$ can be obtained by Mahler’s method, but with very strong regularity hypothesis on $a_n$ and $b_n$ ([20, pp. 13 and 99], [21], [25]).

As a last corollary of Theorem 3.1, we can give precise results on the irrationality of $f(r)$ when $u_n$ satisfies (3.8) (compare to Corollary 3.1).

**Corollary 3.6.** Suppose that $u_n \in \mathbb{N} \setminus \{0\}$ satisfies

$$(3.14) \quad \begin{cases} \lim_{n \to +\infty} u_n = +\infty \\ u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \quad \beta \in \mathbb{R}^*_+, \gamma \in ]0,2[. \end{cases}$$

Let $f$ be the entire function defined by

$$(3.15) \quad f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{u_n}.$$
Then \( f(r) \) is irrational for every \( r \in \mathbb{Q}^* \), except when \( r \in \mathbb{Z} \) and when there exist \( \eta \in \mathbb{N} \setminus \{0\}, \delta \in \mathbb{N} \setminus \{0\} \) such that, for every large \( n \),

\[
\begin{align*}
\eta | r \\
u_n &= \delta v_n, \text{ with } v_n \in \mathbb{N} \\
v_{n+1} &= \eta v_n^2 - \frac{r}{\eta}.
\end{align*}
\]

(3.16)

Corollaries 3.1 to 3.6 will be proved in section 5.

Now it it natural to ask if irrationality measures can be given for fast converging series, that is, if one can prove some of them are not Liouville numbers. We will give a positive answer under the hypothesis of corollary 3.4. However, for technical reasons which will appear in the proof, we will have to suppose that \( \beta \) in (3.8) is not a Liouville number.

**Theorem 3.2.** Let \( a_n \in \mathbb{Z} \setminus \{0\}, b_n \in \mathbb{Z} \setminus \{0\}, u_n \in \mathbb{N} \setminus \{0\} \) satisfy

\[
\begin{align*}
\lim_{n \to +\infty} u_n &= +\infty \\
u_{n+1} &= \beta u_n^2 + O(u_n^2), \quad \beta \in \mathbb{R}^+, 0 \leq \gamma < 2 \\
\log |a_n| &= o(2^n), \quad \log |b_n| = o(2^n).
\end{align*}
\]

(3.17)

Assume that \( \beta \) is not a Liouville number, which means that there exist \( K > 0 \) and \( \lambda \geq 2 \) such that, for every rational \( A/B \neq \beta \),

\[
\left| \beta - \frac{A}{B} \right| \geq \frac{K}{|B|^\lambda}.
\]

(3.18)

Assume moreover that, if \( \beta \in \mathbb{Q} \),

\[
u_{n+1} \neq \beta u_n^2 - \frac{a_{n+1}b_n}{a_nb_{n+1}} u_n + \frac{a_{n+2}b_n}{\beta a_{n+1}b_{n+2}}
\]

(3.19)

for every \( n \geq N \). Then, for every \( \varepsilon > 0 \), there exists \( q_0 = q_0(\varepsilon) \in \mathbb{N} \) such that, for every rational \( p/q \) satisfying \( |q| \geq q_0 \),

\[
\sum_{n=0}^{+\infty} \left| \frac{a_n}{b_n u_n} - \frac{p}{q} \right| \geq \frac{1}{|q|^\gamma + \varepsilon},
\]

(3.20)
with $\tau = 4\frac{2\lambda + \omega}{\omega}$, $\omega = \inf(2 - \gamma, 1)$.

Remark 3.1. In the case where $\beta \in \mathbb{Q}$, we can take $K = 1$ and $\lambda = 2$ in (3.18). Hence

$$\tau = 4\frac{4 + \omega}{\omega}.$$ 

So theorem 2 of [6] gives a better irrationality measure (for example 9 instead of 20 when $\omega = 1$). This is due to the fact that the general construction used to prove Theorem 3.2 is not so well suited to the case $\beta \in \mathbb{Q}$ than the one used in [6]. In particular, for rational $\beta$, one could use $\lambda = 1$ in (3.18). As it will appear in the proof, the condition $\lambda \geq 2$ in Theorem 3.2 will only make the proof simpler, and is not necessary.

Example 3.3. Theorem 3.2 applies to the general series $S_{10}$ of Corollary 3.5, because here $\beta = \omega_k$ is an algebraic number and we can take $\lambda = 2 + \varepsilon$ by Roth’s Theorem. For instance, in the special case of series $S_{12}$ in Example 3.2, we have $\beta = \sqrt{5}$ (whence $\lambda = 2$) and $\gamma = 0$ by (3.13). Therefore, $\omega = 1$ and $\tau = 20$.

4. Proof of Theorem 3.1

4.1. Lemmas

In what follows, sequences $a_n, b_n$ and $u_n$, satisfy the hypothesis of Theorem 3.1. However, note that the upper bound $u_{n+1} \leq c' u_n^2$ is not necessary in Lemmas 4.1 to 4.5.

Lemma 4.1. There exists $A > 0$ such that

$$u_0u_1 \ldots u_{n-1} \leq A^n u_n, \quad (n \geq 1).$$

Proof. Put $A = \max(u_0/u_1, 1/c)$. Then Lemma 4.1 follows by induction, because $u_{n+1} \geq c u_n^2$. □

Lemma 4.2. There exists $\theta > 1$ and $B > 0$ such that

$$u_n \geq B\theta^{2^n} \quad (n \geq 0).$$
Proof. Put $v_n = cu_n$. Then $v_{n+1} \geq v_n^2$ for every $n \in \mathbb{N}$. As $\lim_{n \to +\infty} v_n = +\infty$, we can choose $N$ such that $v_N > 1$; by induction, one sees that $v_n \geq (v_N)^{2^{n-N}}$ for $n \geq N$, which is Lemma 4.2 with $\theta = (v_N)^{2^{-N}} > 1$. □

**Lemma 4.3.** $\sum_{k=n+1}^{+\infty} \frac{a_k}{b_k u_k} = O(u_n^{2(\alpha-1)})$.

Proof. Let $M \in \mathbb{N}$ such that $u_n \geq c^{-2}$ for every $n \geq M$. Induction on $j$ shows that $u_{n+j} \geq cu_{n+j+1}$ for every $n \geq M$ and $j \geq 1$ (note that $u^j \geq u_n \geq c^{-2}$). As $a_n = O(u_n^\alpha)$, there exists a constant $D > 0$ such that

$$\left| \sum_{k=n+1}^{+\infty} \frac{a_k}{b_k u_k} \right| \leq \sum_{k=n+1}^{+\infty} \frac{|a_k|}{u_k} \leq D \sum_{k=n+1}^{+\infty} \frac{1}{u_k^\alpha} \leq Dc^{\alpha-1} \sum_{m=2}^{+\infty} \frac{1}{u_n^{\alpha}(1-\alpha)m} = Dc^{\alpha-1} \frac{1}{u_n^{2(1-\alpha)}} \frac{1}{1 - \frac{1}{u_n^{1-\alpha}}}.$$

As $\lim_{n \to +\infty} u_n = +\infty$, Lemma 4.3 is proved.

The following lemma is similar to Dirichlet’s Theorem on diophantine approximation ([5, Theorem 1.6], for example). □

**Lemma 4.4.** Let $\mu \in ]0,1[$. For every $n \geq N_0(\mu)$ there exists $(p_n,q_n) \in \mathbb{N}^2$, $q_n \neq 0$, $p_n \neq 0$, such that

\begin{align*}
&(4.1) \quad \left| \frac{u_{n+1}}{u_n^2} - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n u_n^\mu} \\
&(4.2) \quad q_n = O(u_n^\mu) \\
&(4.3) \quad p_n = O(u_n^{\mu-2} u_{n+1}).
\end{align*}

Proof. Denote, as usual, by $[x]$ the integral part of $x$. Put $Q_n = \lceil u_n^\mu \rceil + 1$, and consider the numbers

$$\alpha_i = i \frac{u_{n+1}}{u_n^2} - \left[ i \frac{u_{n+1}}{u_n^2} \right], \quad i = 0, 1, \ldots, Q_n.$$
Two cases can occur.

**First case:** There exist $i < j$ such that $\alpha_i = \alpha_j$. Then

$$\frac{u_{n+1}}{u_n^2} = \frac{p_n}{q_n},$$

with $q_n = j - i$ and $p_n = \left[j\frac{u_{n+1}}{u_n^2}\right] - \left[i\frac{u_{n+1}}{u_n^2}\right]$. Clearly $q_n \leq Q_n \leq 2u_n^\mu$ for large $n$; thus (4.1) and (4.2) are fulfilled. Moreover

$$p_n = q_n \frac{u_{n+1}}{u_n^2} \leq 2u_n^\mu \frac{u_{n+1}}{u_n^2},$$

which proves (4.3).

**Second case:** The numbers $\alpha_i$ are all distinct for $i = 0, 1, \ldots, Q_n$. Let us divide the interval $[0, 1]$ into $Q_n$ intervals with length $1/Q_n$. By the pigeon-hole principle, at least one of these intervals contains two distinct $\alpha_i$'s. Therefore there exist $i < j$ such that $|\alpha_i - \alpha_j| \leq 1/Q_n$. If we put, as before,

$$q_n = j - i \quad \text{and} \quad p_n = \left[j\frac{u_{n+1}}{u_n^2}\right] - \left[i\frac{u_{n+1}}{u_n^2}\right],$$

we see that

$$\left|q_n \frac{u_{n+1}}{u_n^2} - p_n\right| \leq \frac{1}{Q_n} \leq \frac{1}{u_n^\mu},$$

and Lemma 4.4 is proved. Observe that $p_n \neq 0$ for large $n$ because of (4.1) and the fact that $u_{n+1} \geq cu_n^2$. □

**Lemma 4.5.** Let $\mu \in ]0, 1[$, and let $p_n$ and $q_n$ be defined by Lemma 4.4. Then

$$\frac{p_n u_n}{u_{n+1}} - \frac{q_n}{u_n} = O(u_n^{1-\mu}).$$

**Proof.** By (4.1), we have

$$q_n u_{n+1} = p_n u_n^2 + O(u_n^{2-\mu}).$$
Therefore

\[
\frac{p_n u_n}{u_{n+1}} - \frac{q_n}{u_n} = q_n \left( \frac{p_n u_n}{p_n u_n^2 + O(u_n^{2-\mu})} - \frac{1}{u_n} \right)
\]

\[
= \frac{q_n O(u_n^{2-\mu})}{p_n u_n^2 + O(u_n^{3-\mu})} \sim \frac{q_n}{p_n} O(u_n^{-1-\mu})
\]

As \(u_{n+1} \geq c u_n^2\), we have by (4.1)

\[
\frac{p_n}{q_n} \geq \frac{c}{2} \quad \text{for large } n.
\]

This proves Lemma 4.5. \(\square\)

4.2. Proof of Theorem 3.1

Suppose that \(a_n, b_n, u_n\) satisfy the hypothesis of Theorem 3.1. Let \(S\) be defined by (3.1).

Let \(\mu \in ]0, 1[\), and let \(p_n\) and \(q_n\) be defined by Lemma 4.4 (we will choose the value of \(\mu\) later).

We put

\[
(4.5) \quad A_n = \left( p_n b_n b_{n+1} a_n u_n - a_{n+1} q_n b_n^2 \right) \left( \sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} \right) - p_n b_{n+1} a_n^2.
\]

An easy computation shows that

\[
(4.6) \quad \begin{cases}
A_n = a_n a_{n+1} b_n \left( \frac{p_n u_n}{u_{n+1}} - \frac{q_n}{u_n} \right) + R_n - S_n, \\
R_n = p_n b_n b_{n+1} a_n u_n \sum_{k=n+2}^{+\infty} \frac{a_k}{b_k u_k}, \\
S_n = a_{n+1} q_n b_n^2 \sum_{k=n+1}^{+\infty} \frac{a_k}{b_k u_k}.
\end{cases}
\]

By Lemmas 4.3 and 4.4, we have for every \(\varepsilon \in ]0, 1[\)

\[
S_n = O(u_n^\alpha) O(u_n^{\mu}) O(u_n^{2\varepsilon}) O(u_n^{2(\alpha-1)})
\]
As \( u_n = O(u_{n+1}^{1/2}) \), we get, by keeping one \( u_n^{-1} \) in the last factor,

\[(4.7)\]
\[ S_n = O(u_{n+1}^{2\alpha + (\mu/2) + \epsilon - (1/2)} u_n^{-1}). \]

Similarly :

\[ R_n = O(u_n^{\mu - 2} u_{n+1}) O(u_n^\epsilon) O(u_n^\alpha) u_n O(u_n^{2(\alpha - 1)}) \]

\[(4.8)\]
\[ R_n = O(u_{n+1}^{(5\alpha/2) + (\mu/2) + (3\epsilon/2) - 1} u_n^{-1}). \]

By using (4.7), (4.8), and Lemma 4.5, we obtain

\[(4.9)\]
\[ A_n = O(u_{n+1}^{(3\alpha/2) - (\mu/2) + (\epsilon/2)} u_n^{-1}) + O(u_{n+1}^{(5\alpha/2) + (\mu/2) + (3\epsilon/2) - 1} u_n^{-1}) + O(u_{n+1}^{2(\alpha + (\mu/2) + \epsilon - (1/2)} u_n^{-1}). \]

We will choose \( \mu \) in such a way that each of the three numbers \( \frac{3\alpha}{2} - \frac{\mu}{2} \), \( \frac{5\alpha}{2} + \frac{\mu}{2} - 1 \), and \( 2\alpha + \frac{\mu}{2} - \frac{1}{2} \) is negative. Put \( \mu = t\alpha \) ; then we must have

\[ t > 3, \quad (t + 5)\alpha < 2, \quad (t + 4)\alpha < 1. \]

The third condition implies the second one. Hence we have to find \( t \) such that

\[ 3 < t < \frac{1}{\alpha} - 4, \]

which is possible only if \( \alpha < 1/7 \), and is equivalent to find \( \mu \) satisfying

\[(4.10)\]
\[ \mu \in [3\alpha, 1 - 4\alpha]. \]

By (4.10), the three numbers \( \frac{3\alpha}{2} - \frac{\mu}{2} \), \( \frac{5\alpha}{2} + \frac{\mu}{2} - 1 \), and \( 2\alpha + \frac{\mu}{2} - \frac{1}{2} \) are negative. Now if we choose \( \epsilon \) small enough, each of the three numbers \( \frac{3\alpha}{2} - \frac{\mu}{2} + \frac{\epsilon}{2} \), \( \frac{5\alpha}{2} + \frac{\mu}{2} + \frac{3\epsilon}{2} - 1 \), and \( 2\alpha + \frac{\mu}{2} + \epsilon - \frac{1}{2} \) is negative. Therefore, by putting

\[(4.11)\]
\[ \delta = \max \left( \frac{3\alpha}{2} - \frac{\mu}{2} + \frac{\epsilon}{2}, \frac{5\alpha}{2} + \frac{\mu}{2} + \frac{3\epsilon}{2} - 1, 2\alpha + \frac{\mu}{2} + \epsilon - \frac{1}{2} \right), \]
we have by (4.9)

$$A_n = O(u_{n+1}^{\delta}u_n^{-1}), \quad \text{with } \delta < 0 \text{ for } \varepsilon < \varepsilon_0(\alpha, \mu).$$

Now we observe that

$$\sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} = S - \sum_{k=0}^{n-1} \frac{a_k}{b_k u_k}. \quad (4.13)$$

Define the following rational integers:

$$\begin{cases} K_n &= b_0 b_1 \ldots b_{n-1} u_0 u_1 \ldots u_{n-1} \\ B_n &= K_n (p_n b_n b_{n+1} a_n u_n - a_{n+1} b_n^2) \\ C_n &= K_n p_n b_{n+1} a_n^2 + B_n \sum_{k=0}^{n-1} \frac{a_k}{b_k u_k}. \end{cases} \quad (4.14)$$

Multiply $A_n$ by $K_n$ in (4.5); by using (4.12), (4.13) and (4.14), we obtain

$$B_n S - C_n = K_n O(u_{n+1}^{\delta}u_n^{-1}). \quad (4.15)$$

But by using Lemma 4.1 we can obtain an upper bound for $K_n$; for $\varepsilon \in ]0, 1[$, there exists $\nu = \nu(\varepsilon) > 0$ such that

$$|K_n| \leq \nu^n u_0^{1+\varepsilon} \ldots u_{n-1}^{1+\varepsilon} \leq \nu^n A^{2n} u_n^{1+\varepsilon} \leq \nu^n A^{2n} u_n^{1+\varepsilon}.$$

Hence

$$K_n = O((\nu A^2)^n u_n^{\varepsilon/2}). \quad (4.16)$$

Therefore (4.15) can be written as

$$B_n S - C_n = O((\nu A^2)^n u_n^{\delta+(\varepsilon/2)}). \quad (4.17)$$

For $\varepsilon < \varepsilon_1(\alpha, \mu)$, we have $\delta + \frac{\varepsilon}{2} < 0$. If we choose such an $\varepsilon$ and fix it, we obtain, by using Lemma 4.2,

$$\lim_{n \to +\infty} (B_n S - C_n) = 0.$$
Assume that $S$ is rational; as $B_n, C_n$ are integers, this implies

\[(4.18) \quad B_n S - C_n = 0 \quad \text{for} \quad n \geq N(\alpha).\]

Now consider the determinant

\[(4.19) \quad \Delta_n = \begin{vmatrix} B_n & -C_n \\ B_{n+1} & -C_{n+1} \end{vmatrix}.\]

By (4.18) we have $\Delta_n = 0$ for $n \geq N(\alpha)$. But if we multiply the first column of $\Delta_n$ by $\sum_{k=0}^{n-1} \frac{a_k}{b_k u_k}$ and add it to the second one, we obtain by using (4.14)

\[
\Delta_n = \begin{vmatrix} B_n & -K_n p_n b_{n+1} a_n^2 \\ B_{n+1} & -K_{n+1} p_{n+1} b_{n+2} a_{n+1}^2 - B_{n+1} b_n u_n \end{vmatrix}.
\]

We replace $B_n$ and $B_{n+1}$ by using the second equality in (4.14), and develop $\Delta_n$. We get

\[
\Delta_n = K_n K_{n+1} \frac{a_{n+1} b_n}{u_n} \left( -a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} p_{n+1} u_n^2 
+ a_{n+1} b_n b_{n+2} p_{n+1} q_n u_n + a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} q_n u_{n+1} 
- a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} q_n q_{n+1} \right).
\]

The term in the brackets is zero for $n \geq N(\alpha)$; if we divide it by $a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} q_n$, we obtain (3.2), and Theorem 3.1 is proved.

**Remark 4.1.** The method used for proving Theorem 3.1 is a weak form of Mahler’s transcendence method in the form introduced by Loxton and Van der Poorten. This will be apparent in the special case where

\[(4.20) \quad u_{n+1} = \beta u_n^2 + \tau_n, \quad \beta \in \mathbb{Q}_+^*, \]

first studied in [6].

If we introduce

\[
\begin{cases}
  f_n(x) = \beta + \tau_n x^2 \\
  \varphi_n(x) = \sum_{k=n}^{+\infty} \frac{a_k x^{2k-n}}{b_k (f_k \circ f_{k-1} \circ \cdots \circ f_{n+1})(x)},
\end{cases}
\]
we easily see that

\[ \sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} = \varphi_n \left( \frac{1}{u_n} \right). \]

Now we can compute explicitly the \([1/1]\) Padé-approximants to \(\varphi_n\), namely find \(\mu_n, \nu_n, \rho_n\) satisfying

\[ (\mu_n x + \nu_n) \varphi_n - \rho_n x = O(x^3). \]

By (4.21) we see that

\[ \varphi_n(x) = \frac{a_n}{b_n} x + \frac{a_{n+1}}{b_{n+1} \beta} x^2 + O(x^4), \]

so that (4.23) is equivalent to

\[
\begin{align*}
\mu_n \frac{a_n}{b_n} + \nu_n \frac{a_{n+1}}{b_{n+1} \beta} &= 0 \\
\frac{a_n}{b_n} - \rho_n &= 0.
\end{align*}
\]

If we put \(\beta = \eta/\delta\) and look for integers \(\mu_n, \nu_n, \rho_n\) we can take

\[
\begin{align*}
\mu_n &= -a_{n+1} \delta b_n^2 \\
\nu_n &= \eta b_n b_{n+1} a_n \\
\rho_n &= \eta b_{n+1} a_n^2.
\end{align*}
\]

This explains the number \(A_n\) defined in [6], formula (10). In the present paper, we have only modified the definition of \(A_n\) by replacing \(\eta\) by \(p_n\) and \(\delta\) by \(q_n\) in order to obtain more general results (see formula (4.5)).

5. Proof of Corollaries 3.1 to 3.6

5.1. Proof of Corollary 3.1

Suppose that \(f(r) \in \mathbb{Q}\) and \(f(r') \in \mathbb{Q}\), with \(r = a/b, \ r' = a'/b'\), \(a, a', b, b' \in \mathbb{Z} \setminus \{0\}, r \neq r'\). By Theorem 3.1 we have for \(n \geq N\)

\[
u_{n+1} = \frac{p_n}{q_n} u_{n+1} - \frac{a}{b} u_n + \frac{a}{b} q_{n+1}
\]

\[
u_{n+1} = \frac{p_n}{q_n} u_{n+1} - \frac{a}{b} u_n + \frac{a}{b} q_{n+1}
\]
Subtracting (5.2) to (5.1), we obtain

\[ u_n = \frac{q_{n+1}}{p_{n+1}}. \]

Therefore, if we replace in (5.1) \( q_{n+1}/p_{n+1} \) by \( u_n \) and \( q_n/p_n \) by \( u_{n-1} \), we get for \( n \geq N + 1 \)

\[ u_{n+1} = \frac{u_n^2}{u_{n-1}}. \]

Hence \( \lim\limits_{n \to +\infty} \frac{u_{n+1}}{u_n^2} = 0 \), contrary to the assumption.

### 5.2. Proof of Corollary 3.2

Suppose that \( \sum_{n=0}^{+\infty} \frac{a_n}{u_n} \in \mathbb{Q} \), \( a_n \in \{-1, 1\} \). By Theorem 3.1 we can write for large \( n \)

\[ u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_{n+2} q_{n+1}}{a_{n+1} p_{n+1}}. \]

For every \( n \), put \( \frac{p_n}{q_n} = \frac{p'_n}{q'_n} \) where \( p'_n \) and \( q'_n \) are prime to each other. Then we have

\[ u_{n+1} = \frac{p'_n}{q'_n} u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_{n+2} q'_{n+1}}{a_{n+1} p'_{n+1}}. \]

As \( \frac{a_{n+1}}{a_n} \in \{-1, 1\} \) and \( u_n \in \mathbb{N} \), \( p'_{n+1} \) must divide \( q'_n \). This implies \( p'_{n+1} \leq q'_n \) for \( n \geq N \), that is

\[ p'_{n+1} \leq \frac{q'_n}{p_n} p'_n \] for \( n \geq N \).

Therefore, by induction we have

\[ p'_n \leq p'_N \prod_{k=N}^{n-1} \frac{q'_k}{p'k} \] for \( n \geq N \).
But we have by (3.3), for every $\mu \in ]0,1[,$
\[
\frac{u_{n+1}}{u_n^2} - \frac{p'_n}{q'_n} = O(u_n^{-\mu}).
\]
Hence
\[
1 - \frac{p'_n}{q'_n} = \left(1 - \frac{u_{n+1}}{u_n^2}\right) + O(u_n^{-\mu}).
\]
By (3.6), this implies that the series $\sum \left(1 - \frac{p'_n}{q'_n}\right)$ is convergent: therefore
\[
\lim_{n \to +\infty} \frac{p'_n}{q'_n} = 1
\]
and the infinite product $\prod \frac{p'_n}{q'_n}$ is convergent, so that $p'_n$ is bounded by (5.7). As $\lim_{n \to +\infty} \frac{p'_n}{q'_n} = 1$, $q'_n$ is also bounded, which implies
\[
\frac{p'_n}{q'_n} = 1 \quad \text{for } n \geq N_1,
\]
and completes the proof of Corollary 3.2.

5.3. Proof of Corollary 3.3
We apply Corollary 3.2. Here
\[
\begin{align*}
  u_n &= a^{2^n} + b_n \\
  a_n &= \varepsilon^n.
\end{align*}
\]
As the series $\sum b_n a^{-2^n}$ is convergent, we have
\[
u_n \sim a^{2^n}, \quad b_n = o(a^{2^n}).
\]
Moreover
\[
\frac{u_{n+1}}{u_n^2} - 1 = -\frac{2b_n a^{2^n}}{u_n^2} + \frac{b_{n+1}}{u_n^2} - \frac{b_n^2}{u_n^2}.
\]
By (5.10), each of the three series in the right hand side of (5.11) is convergent. Hence $\sum \left(\frac{u_{n+1}}{u_n^2} - 1\right)$ is convergent. By Corollary 3.2, we have for $n \geq N_1$
\[
u_{n+1} = u_n^2 - \varepsilon u_n + \varepsilon.
\]
Multiply (5.11) by \( u_n^2 \), and substract it from (5.12), we obtain
\[
    u_n = a^{2n} + b_n = \varepsilon(2b_na^{2n} - b_{n+1} + b_n^2) + 1.
\]
Hence
\[
    (5.13) \quad \varepsilon b_{n+1} - \varepsilon b_n^2 = a^{2n}(2\varepsilon b_n - 1) + 1 - b_n.
\]

Following [7], observe that (5.13) implies
\[
    b_{n+1} = a^{2n}(b_n(2 + b_n a^{-2n}) - \varepsilon - \varepsilon b_n a^{-2n} + \varepsilon a^{-2n}).
\]

As \( b_n = o(a^{-2n}) \), for every \( n \geq N_2 \geq N_1 \) satisfying \( b_n \neq 0 \), we therefore have
\[
    (5.14) \quad |b_{n+1}| \geq a^{2n} |b_n|.
\]
But \( b_n = 0 \Rightarrow b_{n+1} = \varepsilon(1 - a^{2n}) \neq 0 \), so that (5.14) holds for every \( n \geq N_3 \), with \( N_3 = N_2 \) or \( N_3 = N_2 + 1 \). By induction, we get for every \( k \geq 0 \)
\[
    |b_{N_3+k}| \geq |b_{N_3}| a^{2N_3+k-2N_3}.
\]
Therefore \( \lim_{k \to +\infty} |b_{N_3+k}| a^{-2N_3+k} \neq 0 \), and this contradiction proves Corollary 3.3.

5.4. Proof of Corollary 3.4

By lemma 4.2, we have for every \( \alpha > 0 \)
\[
    (5.15) \quad a_n = O(u_n^\alpha), \quad b_n = O(u_n^\alpha).
\]
Suppose that \( S_9 \in \mathbb{Q} \); then, by Theorem 3.1,
\[
    (5.16) \quad u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}}
\]
for \( n \geq N(\alpha) \). But by hypothesis
\[
    (5.17) \quad u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \quad \text{with } 0 \leq \gamma < 2.
\]
Subtracting (5.17) to (5.16), we obtain

\[(5.18) \quad q_n \beta - p_n = q_n O(u_n^{-2}) - \frac{a_{n+1} b_n}{a_n b_{n+1}} q_n + \frac{a_{n+2} b_{n+1}}{a_n b_{n+2} p_{n+1}} q_n.\]

In (4.10), we choose
\[\mu = (1 - \alpha)3\alpha + \alpha(1 - 4\alpha) = 4\alpha - 7\alpha^2.\]

By (3.3), we have
\[(5.19) \quad q_n = O(u_n^{4\alpha - 7\alpha^2}).\]

Therefore (5.18) implies, by using \(u_{n+1} = O(u_n^2),\)
\[(5.20) \quad q_n \beta - p_n = O(u_n^{-2+4\alpha - 7\alpha^2}) + O(u_n^{7\alpha - 7\alpha^2 - 1}) + O(u_n^{18\alpha - 21\alpha^2 - 2}).\]

Hence, by multiplying (5.20) by \(q_n + 1,\) we obtain
\[(5.21) \quad q_n q_{n+1} \beta - p_n q_{n+1} = O(u_n^{7-2+12\alpha - 21\alpha^2}) + O(u_n^{15\alpha - 21\alpha^2 - 1}) + O(u_n^{26\alpha - 35\alpha^2 - 2}).\]

Similarly, replace \(n\) by \(n + 1\) in (5.20), and multiply by \(q_n;\) we get
\[(5.22) \quad q_n q_{n+1} \beta - p_n q_{n+1} = O(u_n^{2\gamma - 4 + 12\alpha - 21\alpha^2}) + O(u_n^{18\alpha - 21\alpha^2 - 2}) + O(u_n^{40\alpha - 49\alpha^2 - 4}).\]

Now we choose \(\alpha\) so small that each of the numbers \(\gamma - 2 + 12\alpha - 21\alpha^2,\)
\(15\alpha - 21\alpha^2 - 1, 26\alpha - 35\alpha^2 - 2, 2\gamma - 4 + 12\alpha - 21\alpha^2, 18\alpha - 21\alpha^2 - 2,\)
\(40\alpha - 49\alpha^2 - 4\) is negative. With such a choice of \(\alpha,\) (5.21) and (5.22) imply
\[\lim_{n \to +\infty} (q_n q_{n+1} \beta - p_n q_{n+1}) = 0,\]
Therefore
\[(5.23) \quad \left\{ \begin{array}{l} \beta = \lim_{n \to +\infty} p_n / q_n \\ \lim_{n \to +\infty} (p_n q_{n+1} - p_{n+1} q_n) = 0. \end{array} \right.\]

As \(p_n, q_n, p_{n+1}, q_{n+1}\) are integers, this means that \(p_n q_{n+1} - p_{n+1} q_n = 0\)
for \(n\) large enough. Hence, for \(n \geq N_1(\alpha)\)
\[(5.24) \quad \frac{p_n}{q_n} = \frac{p_{n+1}}{q_{n+1}} = \lim_{n \to +\infty} \frac{p_n}{q_n} = \beta.\]

So \(\beta \in \mathbb{Q},\) and Corollary 3.4 is proved by using (5.24) and (5.16).
5.5. Proof of Corollary 3.5

Put $u_n = v_2^n + b_n = A_1 \omega_1^{2n} + \sum_{h=2}^{d} A_h \omega_h^{2n} + b_n$. We then have

\begin{align}
\begin{cases}
  u_n = A_1 \omega_1^{2n} + t_n \\
  t_n = \sum_{h=2}^{d} A_h \omega_h^{2n} + b_n.
\end{cases}
\end{align}

(5.25)

Observe that, by (3.10) and (3.12),

\begin{align}
\begin{cases}
  t_n \sim A_2 \omega_2^{2n} & \text{if } d \geq 2 \text{ and } |\omega_2| > 1 \\
  t_n = O(|\omega_1|^{\varepsilon 2^n}) & \text{(for every } \varepsilon > 0) \text{ otherwise}.
\end{cases}
\end{align}

(5.26)

We see immediately that

\begin{align}
u_{n+1} - \frac{1}{A_1} u_n^2 = t_{n+1} - 2t_n \omega_1^{2n} - \frac{1}{A_1} t_n^2.
\end{align}

(5.27)

As $u_n \sim A_1 \omega_1^{2n}$, we see, by using (5.26), that there exists $\gamma \in ]0, 2[$ such that $u_{n+1} - \frac{1}{A_1} u_n^2 = O(u_n^\gamma)$. Hence Corollary 3.4 applies; so, if $S_{10} \in \mathbb{Q}$, we have $A_1 \in \mathbb{Q}$ and, for every $n \geq N_0$,

\begin{align}
u_{n+1} = \frac{1}{A_1} u_n^2 - \frac{a_{n+1}}{a_n} u_n + A_1 \frac{a_{n+2}}{a_{n+1}}.
\end{align}

(5.28)

Comparing this equality to (5.27) yields

\begin{align}
t_{n+1} - 2t_n \omega_1^{2n} - \frac{1}{A_1} t_n^2 = -\frac{a_{n+1}}{a_n} (A_1 \omega_1^{2n} + t_n) + A_1 \frac{a_{n+2}}{a_{n+1}}.
\end{align}

(5.29)

Now we distinguish four cases.

**First case:** $d \geq 2$ and $|\omega_2| > 1$. By taking the equivalents in (5.28), we obtain in virtue of (5.26)

\begin{align}
-2A_2 \omega_2^{2n} \omega_1^{2n} \sim -\frac{a_{n+1}}{a_n} A_1 \omega_1^{2n}.
\end{align}

Therefore $\omega_2^{2n} \sim \frac{a_{n+1}}{a_n} \frac{A_1}{2A_2}$. 
This is impossible because $|\omega_2| > 1$ and $\log |a_n| = o(2^n)$.

**Second case**: $d = 1$. In this case $t_n = b_n \in \mathbb{Z}$. We write (5.28) in the following form:

$$(5.29) \quad \omega_1^{2^n} \left( A_1 \frac{a_{n+1}}{a_n} - 2t_n \right) = \frac{1}{A_1} t_n^2 - t_{n+1} - \frac{a_{n+1}}{a_n} t_n + A_1 \frac{a_{n+2}}{a_{n+1}}.$$

Suppose that the rational number $\frac{A_1}{a_n} \frac{a_{n+1}}{a_n} - 2t_n$ is different from 0; in this case, denoting by $\delta > 0$ the denominator of $A_1$, we have

$$\left| A_1 \frac{a_{n+1}}{a_n} - 2t_n \right| \geq \frac{1}{\delta |a_n|},$$

which implies by (5.29):

$$\omega_1^{2^n} \leq \delta |a_n| \left| \frac{1}{A_1} t_n^2 - t_{n+1} - \frac{a_{n+1}}{a_n} t_n + A_1 \frac{a_{n+2}}{a_{n+1}} \right|.$$

By using (5.26), we get $\omega_1^{2^n} = O(|\omega_1|^{2^n})$ for every $\varepsilon > 0$, a contradiction. Hence (5.29) implies, for every $n \geq N \geq N_0$,

$$(5.30) \quad \begin{cases} A_1 \frac{a_{n+1}}{a_n} = 2t_n \\ \frac{1}{A_1} t_n^2 - t_{n+1} - \frac{a_{n+1}}{a_n} t_n + A_1 \frac{a_{n+2}}{a_{n+1}} = 0. \end{cases}$$

Replacing $t_n$ and $t_{n+1}$ from the first equality into the second yields

$$\frac{1}{2} \frac{a_{n+2}}{a_{n+1}} = \left( \frac{1}{2} \frac{a_{n+1}}{a_n} \right)^2.$$

Therefore, for every $n \geq N$,

$$(5.31) \quad \frac{1}{2} \frac{a_{n+1}}{a_n} = \left( \frac{1}{2} \frac{a_{N+1}}{a_N} \right)^{2^{n-N}} = \left( \left( \frac{1}{2} \frac{a_{N+1}}{a_N} \right)^{2^{-N}} \right)^{2^n}.$$
So, as \( \log |a_n| = o(2^n) \), \( \left| \frac{a_{N+1}}{2a_N} \right| > 1 \) is impossible. Taking the inverses in (5.31), we see that \( \left| \frac{a_{N+1}}{2a_N} \right| < 1 \) is also impossible. Therefore \( \frac{1}{2} \left| \frac{a_{N+1}}{a_N} \right| = 1 \).

By (5.31), this implies

\[
\frac{a_{n+1}}{a_n} = 2 \quad \text{for every } n \geq N + 1.
\]

Thus \( t_n = A_1 \) for every \( n \geq N + 1 \) by using (5.30), which means that \( u_n = A_1(\omega_1^{2n} + 1) \). As \( A_1 \in \mathbb{Q} \) and \( u_n \in \mathbb{Z} \), as \( |\omega_1| > 1 \), then \( \omega_1 \in \mathbb{Z} \setminus \{-1, 0, 1\} \), and

\[
S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1\omega_1^{2n}} + b_n + \frac{a_N}{A_1} \sum_{n=1}^{\infty} \frac{2^n}{(\omega_1^{2N})^{2n} + 1}
\]

**Third case:** \( d \geq 2 \) and \( |\omega_2| = 1 \). Let \( P(X) = X^d - \sum_{h=1}^{d} \alpha_h X^{d-h} \) be the characteristic polynomial of \( v_n \), defined in (3.11). As \( \omega_2 = \pm 1 \) is a root of \( P \), we have

\[
P(X) = (X - \omega_2)Q(X), \quad Q \in \mathbb{Z}(X),
\]

and \( \omega_1, \omega_3, \ldots, \omega_d \) are the roots of \( Q \). But \( A_1, A_2, \ldots, A_d \) are the roots of the system

\[
\begin{align*}
 v_0 &= A_1 + A_2 + \cdots + A_d \\
v_1 &= A_1\omega_1 + A_2\omega_2 + \cdots + A_d\omega_d \\
\vdots \\
v_{d-1} &= A_1\omega_1^{d-1} + A_2\omega_2^{d-1} + \cdots + A_d\omega_d^{d-1}.
\end{align*}
\]

Hence, by Cramer’s formula,

\[
A_2 = \frac{\begin{vmatrix}
1 & v_0 & 1 & \cdots & 1 \\
\omega_1 & v_1 & \omega_3 & \cdots & \omega_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_1^{d-1} & v_{d-1} & \omega_3^{d-1} & \cdots & \omega_d^{d-1} \\
\end{vmatrix}}{\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
\omega_1 & \omega_2 & \omega_3 & \cdots & \omega_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_1^{d-1} & \omega_2^{d-1} & \omega_3^{d-1} & \cdots & \omega_d^{d-1}
\end{vmatrix}}.
\]
So $A_2$ is a symmetric rational fraction in $\omega_1, \omega_3, \ldots, \omega_d$ with integer coefficients. Therefore $A_2 \in \mathbb{Q}$. Write $A_2 = r/s$, $r, s \in \mathbb{Z}$, $s \neq 0$. We then have

$$S_{10} = \sum_{n=1}^{+\infty} sa_n + sA_1\omega_1^{2n} + sA_3\omega_3^{2n} + \cdots + sA_d\omega_d^{2n} + r + sb_n.$$ 

Hence, if $d = 2$ we are led back to the second case. If $d \geq 3$, as $|\omega_3| < |\omega_2| = 1$, we are led to the fourth case.

**Fourth case**: $d \geq 2$ and $|\omega_2| < 1$. As $\omega_1\omega_2\ldots\omega_d = -\alpha_d \in \mathbb{Z} \setminus \{0\}$ and $1 > |\omega_2| > |\omega_3| > \cdots > |\omega_d|$, we cannot have $|\omega_1\omega_2| < 1$. Therefore $|\omega_1\omega_2| \geq 1$. We have by (5.29), for every $\varepsilon > 0$,

$$A_1a_{n+1} - 2t_na_n = O(|\omega_1|^{(1+\varepsilon)2^n}).$$

(5.36)

Put $t'_n = t_n - b_n$; then $t'_n \sim A_2\omega_2^{2^n}$ by (5.35), and (5.36) yields

$$A_1a_{n+1} - 2a_nb_n - 2t'_na_n = O(|\omega_1|^{(1+\varepsilon)2^n}).$$

(5.37)

As $A_1 \in \mathbb{Q}$ and $|\omega_2| < 1$, this implies

$$A_1a_{n+1} - 2a_nb_n = 0.$$ (5.38)

for every large $n$, whence

$$-2a_n = O(|\omega_2|^{-2^n})O(|\omega_1|^{(1+\varepsilon)2^n}).$$

(5.39)

Suppose that $|\omega_1\omega_2| > 1$. By choosing $\varepsilon$ small enough, (5.39) implies $a_n = 0$ for every large $n$, a contradiction. Hence $|\omega_1\omega_2| = 1$. This implies that $d = 2$, otherwise $|\omega_1\omega_2\ldots\omega_d|$ would be less than 1. Therefore the characteristic polynomial of sequence $v_n$ is of the form

$$P(X) = X^2 + cX \pm 1, \quad e \in \mathbb{Z}.$$ 

As $|\omega_1| > 1$, $P$ has no rational root. Let $\sigma \neq Id$ be the morphism of conjugaison in quadratic field $K = \mathbb{Q}(\omega_1)$. As $\sigma(\omega_1) = \omega_2$, $v_n \in \mathbb{Z}$, and $A_1 \in \mathbb{Q}$, we have

$$v_n = A_1\omega_1^n + A_2\omega_2^n \Rightarrow v_n = A_1\omega_2^n + \sigma(A_2)\omega_1^n.$$
Therefore $A_2 = A_1$ and

\[(5.40) \quad t_n = A_1 \omega_2^n + b_n.\]

Replace in (5.29) and use (5.38); we get, as $|\omega_1 \omega_2| = 1$,

\[-2A_1 = \frac{1}{A_1} (A_1 \omega_2^n + b_n)^2 - (A_1 \omega_2^{n+1} + b_{n+1}) - \frac{a_{n+1}}{a_n} \left( A_1 \omega_2^n + b_n \right) + A_1 \frac{a_{n+2}}{a_{n+1}}.\]

As $A_1 \in \mathbb{Q}$ and $|\omega_2| < 1$, this implies

\[(5.41) \quad -2A_1 = \frac{1}{A_1} b_n^2 - b_{n+1} - \frac{a_{n+1}}{a_n} b_n + A_1 \frac{a_{n+2}}{a_{n+1}}.\]

By (5.38), we have $\frac{a_{n+1}}{a_n} = \frac{2bn}{A_1}$. Replacing in (5.41), we get

\[(5.42) \quad b_{n+1} = \frac{1}{A_1} b_n^2 - 2A_1.\]

Put $b_n = A_1 c_n$; (5.42) becomes

\[(5.43) \quad c_{n+1} = c_n^2 - 2 = f(c_n); \quad f(x) = x^2 - 2.\]

It is easy to see that $f(x) > x$ for every $x > 2$. Therefore, if $c_0 > 2$, then $c_n$ is increasing; if $c_n$ has a limit $\ell$, then $\ell = \ell^2 - 2$, which is impossible because $c_0 > 2$. Hence $\lim_{n \to +\infty} c_n = +\infty$ and for large $n$ we have, by (5.43), $c_{n+1} \geq \frac{1}{2} c_n^2$. By Lemma 4.2, $b_n = A_1 c_n \geq A_1 C' \theta 2^n$ with $\theta > 1$, a contradiction. Therefore $c_0 \leq 2$. If $c_0 < -2$, then $c_1 > 2$ and we can argue the same way and get a contradiction, whence

\[(5.44) \quad c_0 \in [-2, 2].\]

As $f(x) \in [-2, 2]$ for every $x \in [-2, 2]$, we have

\[(5.45) \quad c_n \in [-2, 2], \quad \forall n \in \mathbb{N}.\]
But $c_n = b_n/A_1$ and $b_n \in \mathbb{Z}$. So (5.45) implies that $c_n$ takes only a finite number of values. Therefore there exist $N \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$ such that $c_{N+k} = c_N$. By using (5.43), we immediately get by induction

\[(5.46) \quad c_{n+k} = c_n = f^k(c_n) \quad \text{for every } n \geq N.\]

But $f^k$ is clearly a monic polynomial with integer coefficients and degree $2^k$. Moreover $f^1(0) = -2$ ; $f^2(0) = f(f(0)) = f(-2) = 2$ ; $f^3(0) = f(f^2(0)) = f(2) = 2 \ldots$. Hence

\[(5.47) \quad f^k(x) = x^{2^k} + \cdots \pm 2; \quad \forall k \in \mathbb{N} \setminus \{0\}.\]

By (5.46), we see that $c_n$ is a rational root of an equation with integer coefficients of the form

\[(5.48) \quad x^{2^k} + \cdots \pm 2 = 0.\]

Therefore we have only four possibilities :

\[(5.49) \quad c_n = -2, c_n = 1, c_n = -1, c_n = 2.\]

But if $c_n = -2$, then $c_{n+1} = 2$, and if $c_n = 1$, then $c_{n+1} = -1$. Therefore $c_n = -1$ or $c_n = 2$ for every $n \geq N + 1$, and

\[(5.50) \quad b_n = -A_1 \text{ or } b_n = 2A_1 \quad \text{for every } n \geq N + 1.\]

In the first case, we have $a_{n+1} = -2a_n$ by (5.38), so that

\[(5.51) \quad S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1(\omega_1^{2^n} + \omega_1^{-2^n}) + b_n} + \frac{a_N}{A_1} \sum_{n=1}^{+\infty} \frac{(-2)^n}{(\omega_1^{2N})^{2n} + (\omega_1^{-2N})^{2n} - 1}.\]

In the second case, we have $a_{n+1} = 4a_n$ by (5.38), and we get

\[(5.52) \quad S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1(\omega_1^{2^n} + \omega_1^{-2^n}) + b_n} + \frac{a_N}{A_1} \sum_{n=1}^{+\infty} \frac{4^n}{(\omega_1^{2N})^{2n} + (\omega_1^{-2N})^{2n} + 2}.\]

The proof of Corollary 3.5 is complete.
5.6. Proof of Corollary 3.6

Let \( a \in \mathbb{Z} \setminus \{0\} \), \( b \in \mathbb{Z} \setminus \{0\} \). By Corollary 3.4, \( f(\frac{a}{b}) \notin \mathbb{Q} \) if \( \beta \in \mathbb{R}_+^* \setminus \mathbb{Q} \).

So, if \( f(\frac{a}{b}) \in \mathbb{Q} \), by (3.9) there exist \((\eta, \delta) \in \mathbb{N}^2\) such that, for \( n \geq N \),

\[
(5.53) \quad u_{n+1} = \frac{\eta}{\delta} u_n^2 - \frac{a}{b} u_n + \frac{\delta a}{\eta b}.
\]

We can suppose

\[
(5.54) \quad GCD(\eta, \delta) = 1 \quad ; \quad GCD(a, b) = 1.
\]

Let \( d = GCD(a, \eta) \), \( \eta = d\eta' \), \( a = da' \). Then (5.53) becomes

\[
(5.55) \quad bu_{n+1} = \frac{\eta b}{\delta} u_n^2 - au_n + \frac{\delta a'}{\eta'}.
\]

The fraction \( \delta a'/\eta' \) is irreducible ; therefore \( \eta'|\delta \) by (5.55). But \( GCD(\eta', \delta) = 1 \); hence \( \eta' = 1 \), \( \eta = d \), \( a = a'\eta \), so that we can write (5.55) in the form

\[
(5.56) \quad bu_{n+1} = \frac{\eta b}{\delta} u_n^2 - a'\eta u_n + \delta a'.
\]

Let \( p \) be any prime divisor of \( \delta \); by (5.56), \( p|u_n \) for every \( n \geq N \). Put \( \delta = p\delta' \), \( u_n = pu_n' \), and replace in (5.56) ; we obtain

\[
(5.57) \quad bu_n' + 1 = \frac{\eta b}{\delta'} u_n'^2 - a'\eta u_n' + \delta'a'.
\]

Hence we see by induction that \( \delta|u_n \). Put

\[
(5.58) \quad u_n = \delta v_n,
\]

and replace in (5.56). We get

\[
(5.59) \quad bv_{n+1} = \eta bv_n^2 - a'\eta v_n + a'.
\]
As $GCD(a', b) = 1$ by (5.54), (5.59) implies that $b|\eta v_n - 1$. Put

\[(5.60) \quad \eta v_n - 1 = bk_n \quad \text{for } n \geq N.\]

Now multiply (5.59) by $\eta$, and replace $\eta v_n$ by using (5.60). We obtain

\[(5.61) \quad bk_{n+1} = (b^2 + 2b - a'\eta)k_n.\]

By (5.60), $b$ and $\eta$ are prime to each other, as well as $b$ and $a'$ by (5.54). Therefore

\[(5.62) \quad GCD(b^2 + 2b - a'\eta, b) = 1.\]

Suppose that $b > 1$; let $p$ be a prime divisor of $b$. By (5.61) and (5.62), $p|k_n$ for every $n \geq N$. More precisely, by putting

\[(5.63) \quad \begin{cases} k_n = k'_n b^{\alpha(n)}, & b = p^\alpha b' \\
p \nmid k'_n, & p \nmid b' \\
\alpha(n) \geq 1, & \alpha \geq 1 \end{cases}\]

we have by (5.61) and (5.62)

\[(5.64) \quad \alpha + \alpha(n + 1) = \alpha(n),\]

which is impossible because $\alpha(n) \geq 1$.

Therefore $b = 1$ and (5.59) becomes

\[(5.65) \quad v_{n+1} = \eta v_n^2 - a'\eta v_n + a',\]

which proves Corollary 3.6.
6. Proof of Theorem 3.2

6.1. A lemma on irrationality measures

We give a complete proof of a classical lemma allowing to compute irrationality measures.

**Lemma 6.1.** Let $\alpha \in \mathbb{R}$. Suppose that there exist constants $a > 0$, $b > 0$, $h \geq 1$, a function $g : \mathbb{N} \rightarrow \mathbb{R}_+^*$, increasing for $n \geq N$ and satisfying $\lim_{n \to +\infty} g(n) = +\infty$, and a sequence $C_n/B_n$ of rational numbers such that

\begin{align*}
(6.1) & \quad |B_n C_{n+1} - B_{n+1} C_n| \neq 0 \quad \text{for every } n \geq N \\
(6.2) & \quad |B_n| = O(g(n)^a) \\
(6.3) & \quad |B_n \alpha - C_n| = O(g(n)^{-1}) \\
(6.4) & \quad g(n+1) \leq b(g(n))^h \quad \text{for every } n \geq N.
\end{align*}

Then, for every $\varepsilon > 0$, there exists $q_0 = q_0(\varepsilon) \in \mathbb{N}$ such that, for every rational $p/q$ satisfying $|q| \geq q_0$

\begin{equation}
(6.5) \quad \left| \alpha - \frac{p}{q} \right| \geq \frac{1}{|q|^{m+\varepsilon}},
\end{equation}

where $m = ah^2 + 1$.

See [5, Theorem 9.7] for an effective version of lemma 6.1 (there is a misprint in this book; the condition ”$b \leq 1$” should be replaced by ”$b > 0$”).

**Proof of Lemma 6.1.** By (6.2) and (6.3), there exist $k > 0$ and $\ell > 0$ such that, for every $n \geq N$

\begin{equation}
(6.6) \quad \begin{cases} 
|B_n| \leq k(g(n))^a \\
|B_n \alpha - C_n| \leq \ell/g(n)
\end{cases}
\end{equation}

Choose $q_1$ such that

\begin{equation}
(6.7) \quad \frac{q_1 \ell}{g(N)} \geq \frac{1}{2}.
\end{equation}
Now let \((p, q) \in \mathbb{Z} \times \mathbb{Z}, \) with \(|q| \geq q_1\). Let \(\nu\) be the least integer satisfying
\[
|q| \ell \frac{g(\nu)}{\ell} < \frac{1}{2}.
\]
It is possible to find such a \(\nu\) because \(\lim_{n \to +\infty} g(n) = +\infty\). By (6.7), we have \(\nu \geq N + 1\). As \(\nu\) is the least integer satisfying (6.8), we have \(g(\nu - 1) \leq 2 |q| \ell\), which implies, by (6.4), \(g(\nu) \leq b(2 |q| \ell)^{h}\). Using (6.4) again, we get
\[
g(\nu) < g(\nu + 1) \leq b^{h+1}(2 |q| \ell)^{h^2}.
\]
Now we consider the determinant
\[
\Delta_\nu = \begin{vmatrix}
B_\nu & C_\nu \\
B_{\nu+1} & C_{\nu+1}
\end{vmatrix},
\]
which is not zero by (6.1); this means that the vectors \((B_\nu, C_\nu)\) and \((B_{\nu+1}, C_{\nu+1})\) form a basis of \(\mathbb{R}^2\). Therefore, one of the two determinants
\[
\begin{vmatrix}
B_\nu & C_\nu \\
q & p
\end{vmatrix} \quad \text{or} \quad \begin{vmatrix}
B_{\nu+1} & C_{\nu+1} \\
q & p
\end{vmatrix}
\]
is not zero. Put \(s = \nu\) or \(\nu + 1\), such that
\[
\delta_s = \begin{vmatrix}
B_s & C_s \\
q & p
\end{vmatrix} \neq 0.
\]
As \(\delta_s \in \mathbb{Z}\), \(|\delta_s| \geq 1\), that is \(|pB_s - qC_s| \geq 1\). Therefore \(1 \leq |q(B_s \alpha - C_s) - B_s(q\alpha - p)|\), which implies
\[
1 \leq |q| |B_s \alpha - C_s| + |B_s| |q\alpha - p|.
\]
By using (6.6), we get
\[
1 \leq \frac{|q| \ell}{g(s)} + kg(s)^a |q\alpha - p|.
\]
Hence, by (6.8), \(\frac{1}{2} < kg(s)^a |q\alpha - p|\).

Using (6.9), we finally get
\[
|q\alpha - p| > \frac{1}{2kb^{a(h+1)}(2 |q| \ell)^{ah^2}},
\]
which proves lemma 6.1 by choosing \(q_0 = q_0(\varepsilon)\) such that \(2^{ah^2}kb^{a(h+1)}\ell^{ah^2} \leq q_0^\varepsilon\). □
6.2. Proof of Theorem 3.2

By (5.15), we have \( a_n = O(u_n^{\alpha}) \) and \( b_n = O(u_n^{\alpha}) \) for every \( \alpha > 0 \). Put

\[
\begin{align*}
\omega_\alpha &= \inf(2 - \gamma, 1 - 3\alpha) \\
\mu &= \frac{\omega_\alpha}{\lambda} - \alpha.
\end{align*}
\]

(6.10)

As \( \lambda \geq 2 \) we see that, for \( \alpha \) small enough

\[
3\alpha < 7\alpha < \mu < \frac{1}{2} < 1 - 4\alpha,
\]

so that (4.10) is fulfilled.

Now we follow the proof of Theorem 3.1 until (4.7). As \( \mu < \frac{1}{2} \), we see that (4.11) becomes, for \( \alpha \) small enough, and \( \varepsilon = \alpha \),

\[
\delta = -\frac{\mu}{2} + 2\alpha,
\]

so that (4.17) can be written as, because \( u_{n+1} = O(u_n^{2}) \) and \( \varepsilon = \alpha \),

\[
B_nS - C_n = O(u_n^{1+6\alpha}).
\]

(6.13)

Similarly we have by (4.16), with \( \varepsilon \) replaced by \( \alpha \),

\[
K_n = O(u_n^{1+2\alpha}).
\]

(6.14)

Now we have to find an upper bound for \( B_n \). By (4.14) and (3.3) we have

\[
|B_n| \leq |K_n| O(u_n^{4\alpha}) O(u_n^{\mu}) u_n.
\]

Hence, by (6.14),

\[
B_n = O(u_n^{2+\mu+6\alpha}).
\]

(6.15)

In Lemma 6.1, we therefore choose

\[
\begin{align*}
g(n) &= u_n^{\mu - 6\alpha} \\
a &= \frac{2 + \mu + 6\alpha}{\mu - 6\alpha} \\
h &= 2 \\
b &= (2\beta)^{\mu - 6\alpha}.
\end{align*}
\]

(6.16)
It remains to prove that (6.1) holds. Assume that there exist infinitely many $n$ such as $B_n C_{n+1} - B_{n+1} C_n = 0$. Then a look at (4.19) shows that (5.16), and hence (5.18), hold for infinitely many $n$. Therefore we have for infinitely many $n$, as $p_{n+1}/q_{n+1} \to \beta$,

$$\beta - \frac{p_n}{q_n} = O(u_n^{-2}) + O(u_n^{-1+3\alpha}) + O(u_n^{-2+6\alpha}).$$

So, for $\alpha$ small enough, we get

(6.17) $$\beta - \frac{p_n}{q_n} = O(u_n^{-\omega_\alpha}).$$

By (3.18) this implies, if $p_n/q_n \neq \beta$, $K/q_n^\lambda = O(u_n^{-\omega_\alpha})$, whence $K = O(u_n^{-\omega_\alpha + \lambda \mu}) = O(u_n^{-\alpha})$. Therefore $K = 0$, which is impossible, and $p_n/q_n = \beta$ for infinitely many $n$; this means that $\beta \in \mathbb{Q}$. So (6.1) holds by (3.19). By lemma 6.1 and (6.16), we get (6.5) with $m = 4(2 + \mu + 6\alpha)/(\mu - 6\alpha)$, which proves Theorem 3.2 by choosing $\alpha$ small enough.

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References


(Received September 13, 2000)
24, Place du Concert
F-59800 LILLE FRANCE
E-mail: dduverney@nordnet.fr