Hodge Number of Cohomology of Local Systems on the Complement of Hyperplanes in $\mathbb{P}^3$

By Yukihito Kawahara

Abstract. The cohomology of the local system on the complement of hyperplanes has a Hodge structure as the $\alpha$-invariant cohomology of a Kummer covering ramified along their hyperplanes for a generic character $\alpha$. In this paper we study the case of arrangements of hyperplanes in the three dimensional complex projective space. We construct a resolution for an arrangement of hyperplanes and compute its Chow group. By computing the first Chern class of logarithmic 1-forms, we can obtain the Euler characteristic and the Hodge numbers of cohomology of local systems using the intersection set of the arrangement of hyperplanes and binomial coefficients.

1. Introduction

A finite set of hyperplanes is called an arrangement of hyperplanes. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{P}^N = \mathbb{P}^N(\mathbb{C})$ and $U = \mathbb{P}^N - \bigcup_{i=1}^n H_i$ be its complement. Let $V_\alpha$ be the rank one local system on $U$, whose monodromy around the hyperplane $H_k$ is $\exp(2\pi i \alpha_k)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a collection of them. The cohomology groups $H^N(U, V_\alpha)$ of are studied well as generalized hypergeometric functions, we refer to [1], [5] and [22]. In the case of rational exponents it is realized geometrically as the cohomology of a Kummer covering of $\mathbb{P}^N$ ramified along $\mathcal{A}$. When $N = 2$ a covering for a certain arrangement is well-known as a Hirzebruch’s example: the surfaces obtained by a Kummer covering of $\mathbb{P}^2$ is of general type with $c_1^2 = 3c_2$ (see [12]). In general, the cohomology group $H^N(U, V_\alpha)$ has a Hodge structure as follows (see [5]).

We fix a positive integer $m$. Let $\pi_m : Y_m \to \mathbb{P}^N$ be the abelian covering of $\mathbb{P}^N$ ramified only along every $H_i$ with the ramification index $m$ and the

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Key words: Hodge structure, cohomology of local system, arrangement of hyperplanes, Kummer covering, Euler characteristic, blowing up, logarithmic form, Chow group, Chern class.
Galois group $G \simeq (\mathbb{Z}/m\mathbb{Z})^{n-1}$. Then the function field $K$ of $Y_m$ is given by the abelian extension

$$K = \mathbb{C}(z_1/z_0, z_2/z_0, \ldots, z_N/z_0)((h_2/h_1)^{1/m}, \ldots, (h_n/h_1)^{1/m})$$

of the function field $\mathbb{C}(z_1/z_0, z_2/z_0, \ldots, z_N/z_0)$ of $\mathbb{P}^N$ where $h_i$ is a linear form defining $H_i = \ker h_i$.

Let $\tilde{Y} \to Y_m$ be a resolution of $Y_m$. Then the cohomology $H^i(\tilde{Y}, \mathbb{C})$ of $\tilde{Y}$ has the action of $G$ and a pure Hodge structure $H^i(\tilde{Y}, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}$. So for a character $\alpha$ of $G$ we put

$$H^i(\tilde{Y}, \mathbb{C})_\alpha = \{ \omega \in H^i(X, \mathbb{C}) \mid g^*(\omega) = \alpha(g)\omega, \text{ for all } g \in G \},$$

$$H^{p,q}(\alpha) = H^{p+q}(\tilde{Y}, \mathbb{C})_\alpha \cap H^{p,q}$$

and then have the eigenspace decomposition

$$H^i(\tilde{Y}, \mathbb{C}) = \bigoplus_{\alpha \in G^*} H^i(\tilde{Y}, \mathbb{C})_\alpha.$$

They induce the Hodge decomposition

$$H^i(\tilde{Y}, \mathbb{C})_\alpha = \bigoplus_{p+q=i} H^{p,q}(\alpha).$$

On the other hand the cohomology $H^N(U, V_\alpha)$ is isomorphic to $H^N(\tilde{Y}, \mathbb{C})_\alpha$ for generic $\alpha$. Therefore $H^N(U, V_\alpha)$ has the Hodge decomposition.

In this paper our purpose is to compute these Hodge numbers $\dim H^{p,q}(\alpha)$ when $N = 3$. It is clear that the dimension and Hodge numbers of $H^N(U, V_\alpha)$ are combinatorial. For example the dimension of $H^N(U, V_\alpha)$ for arrangement in general position is $\binom{n-2}{N}$. So we give their descriptions with the intersection set $L(A)$ of a arrangement $A$ and binomial coefficients.

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2. The Hodge Structure of Cohomology of Local Systems on the Complement of Hyperplanes

Let $A = \{ H_1, H_2, \ldots, H_n \}$ be an arrangement of hyperplanes in $\mathbb{P}^N$ and $U = M(A)$ be its complement. The set $L = L(A)$ of nonempty intersections
of elements of $\mathcal{A}$ is called the intersection set of $\mathcal{A}$ and we denote by $L_p = L_p(\mathcal{A})$ the set of elements of $L$ whose codimension in $\mathbb{P}^N$ is $p$. Obviously we see that $L = \bigcup_{p \geq 1} L_p$ and $L_1 = \mathcal{A}$.

2.1. Hodge decomposition of cohomology of local systems

Definition 2.1 (Blow ups for an arrangement). Let $\mathcal{L}$ be a subset of the intersection set $L(\mathcal{A})$ of an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ and set $L_p = \mathcal{L} \cap L_p(\mathcal{A})$.

$\tau : X \to \mathbb{P}^N$ is called a blowing up of $\mathbb{P}^N$ along $\mathcal{L}$ if $\tau$ is the composition of the sequence

$$X = X_{N-1} \xrightarrow{\tau_{N-1}} X_{N-2} \xrightarrow{\tau_{N-2}} \cdots \xrightarrow{\tau_2} X_1 \xrightarrow{\tau_1} X_0 = \mathbb{P}^N$$

where $X_s \xrightarrow{\tau_s} X_{s-1}$ is the blowing up along the proper transform of $\bigcup_{H \in L_{N-s} + 1} H$ under $\tau_1 \circ \cdots \circ \tau_{s-1}$. Furthermore when the total transform $D$ of $\bigcup_{H \in \mathcal{A}} H$ is a normal crossing divisor, we call $\mathcal{L}$ a singular set of $\mathcal{A}$. The intersection of all singular sets of $\mathcal{A}$ is called the minimal singular set of $\mathcal{A}$.

Remark. $\mathcal{L} = L(\mathcal{A})$ is a singular set of $\mathcal{A}$, obviously. Due to [6] and [18] $\mathcal{L}$ consist of all dense edges of $\mathcal{A}$ is also a singular set of $\mathcal{A}$.

Let $\tau : X \to \mathbb{P}^N$ be a blowing up of $\mathbb{P}^N$ along a singular set $\mathcal{L}$ with a normal crossing divisor $D = \tau^{-1} \bigcup_{H \in \mathcal{A}} H$. Let $\pi_m : Y_m \to \mathbb{P}^N$ be the abelian covering of $\mathbb{P}^N$ ramified only along every $H \in \mathcal{A}$ with the ramification index $m$ and the Galois group $G$. This induce the covering $Y \xrightarrow{\pi} X$ ramified only along $D$ with the Galois group $G$. Since it is abelian, we have the eigenspace decomposition

$$\pi_* \mathcal{O}_Y = \bigoplus_{\alpha \in G^*} \mathcal{V}_\alpha.$$ 

In general $Y$ has rational singularities and then let $\sigma : \tilde{Y} \to Y$ be a desingularization of $X$ such that $\tilde{D} = (\pi \circ \sigma)^{-1} D$ is a normal crossing divisor too. Each $\mathcal{V}_\alpha$ is an invertible sheaf on $X$ endowed with a logarithmic connection

$$\nabla_\alpha : \mathcal{V}_\alpha \to \Omega^1_X(\log D) \otimes \mathcal{V}_\alpha.$$
along $D$ induced by the Kähler differential $d: \mathcal{O}_{\tilde{Y}} \to \Omega_{\tilde{Y}}^1(\log \tilde{D})$. Then we have
\[
R(\pi \circ \sigma)_* \Omega_{\tilde{Y}}^\bullet(\log \tilde{D}) = \Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_{\tilde{Y}}.
\]
Since the Hodge to de Rham spectral sequence for hypercohomology on $\tilde{Y}$ degenerates at $E_1$, the $E_1$-spectral sequence
\[
H^q(X, \Omega_X^p(\log D) \otimes V_\alpha) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D) \otimes V_\alpha)
\]
degenerates at $E_1$ (see [7] and [8]).

On the other hand, denote $U = X \setminus D = \mathbb{P}^N \setminus \cup_{H \in \mathcal{A}} H$ and let $j: U \to X$ be the inclusion. For $V_\alpha$ we have a local system $V_\alpha = \text{Ker}(\nabla_\alpha|_U)$ on $U$.

**Definition 2.2.** If none of monodromies of $V_\alpha$ around components of $D$ has one as eigenvalue, $\alpha$ is called to be *generic* for $\mathcal{L}$ (or *non-resonant* in [9]).

In this case due to [7] we know that $Rj_* V_\alpha$, $j! V_\alpha$, and $\Omega_X^\bullet(\log D) \otimes V_\alpha$ are quasi-isomorphic. Therefore there is an isomorphism
\[
\mathbb{H}^i(X, \Omega_X^\bullet(\log D) \otimes V_\alpha) = H^i(U, V_\alpha).
\]
Furthermore it is known that
\[
H^i(U, V_\alpha) = 0 \quad \text{for } i \neq N
\]
(see [7], [1], [14]). Then we get the Hodge decomposition
\[
H^N(U, V_\alpha) = \bigoplus_{p+q=N} H^q(X, \Omega_X^p(\log D) \otimes V_\alpha)
\]
of the cohomology of the local system on the complement of hyperplanes. Denote those dimensions by $h^{p,q}(\alpha)$ and called the Hodge numbers. Note that
\[
H^q(X, \Omega_X^p(\log D) \otimes V_\alpha) = \overline{H^p(X, \Omega_X^q(\log D) \otimes V_\alpha)} = H^p(X, \Omega_X^q(\log D) \otimes V_{-\alpha}),
\]
because of $\overline{V_\alpha} = V_{-\pi} = V_{-\alpha}$. 
Remark. The isomorphism
\[ H^n(\tilde{Y}, \mathbb{C}) \supset H^n(Y, \mathbb{C}) = H^n(X, \pi_* \mathbb{C}), \]
is compatible with the action of \( G \), hence induces isomorphisms
\[ H^n(\tilde{Y}, \mathbb{C}) \cong H^n(X, j_! V_\alpha) = H^i(U, V_\alpha) \]
and also
\[ H^{p,q}(\alpha) = H^q(X, \Omega^p_X(\log D) \otimes V_\alpha). \]

2.2. Generic characters

Review our situation

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
Y_m & \xrightarrow{\pi_m} & \mathbb{P}^N
\end{array}
\]

here \( \tau \) is a blowing up along \( \mathcal{L} \), \( \pi_m \) is the abelian covering ramified along \( \bigcup_{H \in \mathcal{A}} H \) with the ramification index \( m \) and the Galois group \( G \), \( \pi \) is the covering induced by \( \pi_m \).

The Galois group
\[
G = Gal(Y/X) = Gal(Y_m/\mathbb{P}^N) = Gal(K/\mathbb{C}(z_1/z_0, \ldots, z_N/z_0)) \simeq (\mathbb{Z}/m \mathbb{Z})^{\otimes (n-1)}
\]
is isomorphic to \( \mu_m^{\otimes n}/\mu_m \) here \( \mu_m \) is the group of \( m \)-th root of unity. Fix a primitive \( m \)-th root of unity \( \zeta \) in \( \mathbb{C} \). The character group \( G^* \) of \( G \) is identified with the subset
\[
B^* = \left\{ (k_H)_{H \in \mathcal{A}} \mid k_H \in \mathbb{Z}, 0 \leq k_H < m, \sum_{H \in \mathcal{A}} k_H \equiv 0 \pmod{m} \right\}
\]
of \( \mathbb{Z}^\mathcal{A} \) in the following manner. An element \( k = (k_H) \) of \( G^* \simeq B^* \) is defined by
\[
k(\sigma) = \zeta^{\sum k_H s_H} \in \mathbb{C} \quad \text{for any } \sigma = (j^{s_H} \mod \mu_m).
\]
In addition we shall allow the identification \( G^* = \frac{1}{m} B^* \), and then \( \alpha \in G^* \) is
\[
\alpha = (\alpha_H), \quad \alpha_H = \frac{k_H}{m} \quad \text{for } (k_H) \in B^*.
\]
For $\alpha = (\alpha_H)$, we define some numerical values as follows. We write

$$\nu(\alpha) = \sum_{H \in A} \alpha_H \quad \text{and} \quad \alpha_X = \sum_{H \supset X} \alpha_H$$

for $X \in L(A)$. Note that $0 \leq \alpha_H < 1$ and $\nu(\alpha)$ is a positive integer. Obviously, $\alpha = (\alpha_H)_{H \in A}$ is generic for $L$, if and only if, $\alpha_H$ is not zero for all $H$ and $\alpha_X$ is not integer for all $X$ in $L$.

Denote the integer and decimal part of $\alpha_X$ by $\beta_X(\alpha)$ and $\varepsilon_X(\alpha)$ respectively, namely

$$\beta_X(\alpha) = \lfloor \alpha_X \rfloor \quad \text{and} \quad \alpha_X = \beta_X(\alpha) + \varepsilon_X(\alpha)$$

here $\beta_X(\alpha) \in \mathbb{Z}$ and $0 \leq \varepsilon_X(\alpha) < 1$.

If $(\alpha_H)_{H \in G^*} = \frac{1}{m} B^*$ is generic, $-\alpha$ corresponds to an element $(1 - \alpha_H)_{H} \frac{1}{m} B^*$ and it is denoted by $\alpha^*$. Rational numbers $\nu(-\alpha), \alpha_X(-\alpha), \beta_X(1 - \alpha)$ and $\varepsilon_X(1 - \alpha)$ are denoted by $\nu^*(\alpha), \alpha^*_X(\alpha), \beta^*_X(\alpha)$ and $\varepsilon^*_X(\alpha)$, respectively. The following is clear.

**Lemma 2.1.** If $\alpha = (\alpha_H)_{H \in A}$ is generic for a singular set $L$ of $A$ then

$$\nu + \nu^* = n \quad \text{and} \quad \beta_X + \beta^*_X = p_X - 1$$

for $X \in L$. Here $n$ is the cardinality of $A$ and $p_X$ is the number of hyperplanes in $A$ including $X$ for $X \in L(A)$.

### 3. Facts and Results

Let $A$ be an arrangement of $n$ hyperplanes in $\mathbb{P}^N$ and $U$ be its complement. Then the cohomology group $H^N(U,V_\alpha)$ of $U$ for generic $\alpha$ has the Hodge structure; $H^N(U,V_\alpha) = \oplus_{p+q=N} H^{p,q}(\alpha)$. Denote by $\chi(A)$ the topological Euler characteristic of $U$ and by $h^{p,q}(\alpha)$ the dimension of $H^{p,q}(\alpha)$. If $\alpha$ is generic, the dimension of $H^N(U,V_\alpha)$ is $(-1)^N \chi(A)$. Therefore we note that

$$(-1)^N \chi(A) = \sum_{p+q=N} h^{p,q}(\alpha).$$

We shall arrange notations. Let $L = \cup_{p \geq 1} L_p$ be the intersection set of $A$ and $L = \cup_{p \geq 1} L_p$ be a singular set of $A$. For $X \in L(A)$, $p_X$ is the number of
hyperplanes in $A$ including $X$ and $n^X_p$ (resp. $m^X_p$) is the number of elements of $L_p$ (resp. $L^*_p$) included in $X$.

For generic $\alpha$ we shall use notations defined in the preceding section; $\nu$, $\nu^*$, $\beta_X$, $\beta^*_X$ and so on.

For integers $p$ and $q$, we define the binomial coefficient

$$\binom{p}{q} = \begin{cases} \frac{p!}{q!(p-q)!}, & p \geq q \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that this vanishes when $p < q$ and when $q < 0$ and that $\binom{p}{0} = \binom{p}{p} = 1$ when $p \geq 0$. For positive integers $a$, $b$, and $N$, we can make sure that

$$\binom{a+b}{N} = \sum_{p+q=N} \binom{a}{p} \binom{b}{q}.$$ 

### 3.1. In general position

An arrangement $A$ of hyperplanes in $\mathbb{P}^N$ is said to be in general position if $\text{codim} X = p^X$ for all $X$ of $L(A)$. This means that the union of their hyperplanes is a normal crossing. In this case the topological Euler characteristic is well-known (cf. [19] and [1]).

**Theorem 3.1.**

$$(-1)^N \chi(A) = \binom{n-2}{N}.$$ 

And we have the following fact for Hodge numbers.

**Theorem 3.2** (Terasoma [20] Theorem 5.2.1).

$$h^{p,q}(\alpha) = \binom{\nu^* - 1}{p} \binom{\nu - 1}{q}.$$ 

### 3.2. $N = 2$

Let $A$ be an arrangement of hyperplanes in $\mathbb{P}^2$. We can check easily the combinatorial formula

$$\binom{n}{2} = \sum_{x \in L_2} \binom{p_x}{2}.$$ 

The following results are obtained by T. Oda.
Theorem 3.3 (Oda [15] Theorem 1).

\[ \chi(A) = \binom{n-2}{2} - \sum_{x \in L_2} \binom{p_x - 1}{2}. \]

Theorem 3.4 (Oda [15] Theorem 3, see also [12], [13]).

\[ h^{p,q}(\alpha) = \binom{\nu^* - 1}{p} \binom{\nu - 1}{q} - \sum_{x \in L_2} \binom{\beta^*_x}{p} \binom{\beta_x}{q}. \]

3.3. \( N = 3 \)

Let \( A \) be an arrangement of hyperplanes in \( \mathbb{P}^3 \). We can easily check the following lemma.

Lemma 3.5. We have the combinatorial Formula

\[ \binom{n}{3} = \sum_{x \in L_3} \binom{p_x}{3} - \sum_{l \in L_2} \left( n_{l3} - 1 \right) \binom{p_l}{3}. \]

Here \( n_{l3} \) is the number of points in \( L_3 \) on \( l \).

Main Theorems in this paper is following.

Theorem 3.6. The topological Euler characteristic is

\[ -\chi(A) = \binom{n-2}{3} - \sum_{x \in L_3} \binom{p_x - 1}{3} \]
\[ + \sum_{l \in L_2} \left\{ (n_{l3} - 1) \binom{p_l - 1}{3} + \binom{p_l - 1}{2} \right\}. \]

Theorem 3.7. The Hodge number is

\[ h^{p,q}(\alpha) = \binom{\nu^* - 1}{p} \binom{\nu - 1}{q} - \sum_{x \in L_3} \binom{\beta^*_x}{p} \binom{\beta_x}{q} \]
\[ + \sum_{l \in L_2} \left( n_{l3} - 1 \right) \binom{\beta^*_l}{p} \binom{\beta_l}{q} - \mathcal{E}^{p,q}(\alpha) \]
Here we put
\[ g_l = g_l(\alpha) = \nu - \beta_l - \sum_{x \in L_3, x \subset l} (\beta_x - \beta_l) \]
for \( l \in L_2 \), and \( \mathcal{E}^{p,q}(\alpha) \) is given by
\[ \mathcal{E}^{p,q}(\alpha) = \sum_{l \in L_2} \left\{ (g_l - 1) \left( \frac{\beta_l^p}{p} \right) \left( \frac{\beta_l}{q-1} \right) + (g_l^* - 1) \left( \frac{\beta_l^*}{p-1} \right) \left( \frac{\beta_l}{q} \right) \right\}. \]

**Problem 3.8.** In higher dimensional case, express Euler characteristic of the complement of hyperplanes and Hodge numbers of cohomology of local systems by the binomial coefficient like theorems above.

## 4. Proofs of Main Theorems

### 4.1. Resolution and Chow ring

In this section we shall construct the blowing up of \( \mathbb{P}^3(\mathbb{C}) \) and compute the structure of its Chow ring due to [11, pp. 621–624].

Let \( \mathcal{A} \) be an arrangement of hyperplanes in \( \mathbb{P}^3 \), \( L(\mathcal{A}) \) its intersection set and \( \mathcal{L} \) a subset of \( \mathcal{A} \). We construct the blowing up \( \tau \) along \( \mathcal{L} \) which is the composition
\[ X := X_2 \xrightarrow{\tau_2} X_1 \xrightarrow{\tau_1} X_0 = \mathbb{P}^3 \]
of \( \tau_1 \) and \( \tau_2 \) as follows.

\( \tau_1 : X_1 \to \mathbb{P}^3 \) is the blowing up at points in \( \mathcal{L}_3 \). We denote by \( E_x \) the exceptional divisor over \( x \in \mathcal{L}_3 \), by \( L_x \) a generic line in \( E_x \cong \mathbb{P}^2 \) and by \( H \) the pullback of a hyperplane in \( \mathbb{P}^3 \). \( \tau_2 : X \to X_1 \) is the blowing up along the proper transforms \( \hat{l} \) of \( l \in \mathcal{L}_2 \). We denote by \( F_l \) the exceptional divisor over \( \hat{l} \), and by \( M_l \) a fiber of the \( \mathbb{P}^1 \)-bundle \( \tau_2 : F_l \to \hat{l} \). The proper transform of \( L_x \) and \( E_x \) in \( X \) is also denoted by \( L_x \) and \( E_x \). Then we have
\[
\begin{align*}
H^2(X_1) &= \mathbb{C}\{H, E_x\}_{x \in \mathcal{L}_3} \\
H^4(X_1) &= \mathbb{C}\{H^2, L_x\}_{x \in \mathcal{L}_3} \\
H^2(X) &= \mathbb{C}\{H, E_x, F_l\}_{x \in \mathcal{L}_3, l \in \mathcal{L}_2} \\
H^4(X) &= \mathbb{C}\{H^2, L_x, M_l\}_{x \in \mathcal{L}_3, l \in \mathcal{L}_2}
\end{align*}
\]
and the intersection pairing of Chow ring is given by Table 1 and 2.
Table 1. $H^2 \times H^2 \to H^4$.

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$E_x$</th>
<th>$F_l$</th>
</tr>
</thead>
<tbody>
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<td>$H^2$</td>
<td>0</td>
<td>$M_l$</td>
</tr>
<tr>
<td>$E_y$</td>
<td>$-\delta_{xy}L_x$</td>
<td>$\delta_{yl}M_l$</td>
<td>$\delta_{lm}F_l^2$</td>
</tr>
<tr>
<td>$F_m$</td>
<td>$-\delta_{xl}F_l^2$</td>
<td>$F_l^3$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. $H^2 \times H^4 \to \mathbb{C}$.

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$E_x$</th>
<th>$F_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L_y$</td>
<td>0</td>
<td>$-\delta_{xy}$</td>
<td>0</td>
</tr>
<tr>
<td>$M_m$</td>
<td>0</td>
<td>0</td>
<td>$-\delta_{lm}$</td>
</tr>
<tr>
<td>$F_l^2$</td>
<td>$-1$</td>
<td>$-\delta_{xl}$</td>
<td>$F_l^3$</td>
</tr>
</tbody>
</table>

In Tables 1 and 2 we use the notation for $A, B$ in $L(A)$,

$$\delta_{AB} = \begin{cases} 1, & \text{if } A \subseteq B \\ 0, & \text{otherwise.} \end{cases}$$

and have relations

$$F_l^2 = -H^2 - 2(m_3^l - 1)M_l + \sum_{x \in L_3} L_x \quad \text{and} \quad F_l^3 = 2(m_3^l - 1).$$

Now we introduce some notations and rules of computations for easy to see. Let $v_X$ be some value associated to $X \in L(A)$, for example $p_X$, $\beta_X = \beta_X(\alpha)$ and their polynomial. We put

$$v^E = \sum_{x \in L_3} v_x E_x$$

and $v^F = \sum_{l \in L_2} v_l F_l^k$.

We have following remarks by the intersection pairing of Chow ring. Note that $k$-th powers of $E$ and $F$ are denote by $E^k$ and $F^k$ respectively. Furthermore we can check the following expressions

$$v'E \cdot v^E = -\sum_l \left\{ \left( \sum_{x \in l} v'_x \right) v_l \right\}$$

and $H \cdot v^E = -\sum_l v_l$. 

$$H \cdot v^F = -\sum_l v_l.$$
**Lemma 4.1.** We get following relations.

\[ H \cdot E = 0, \quad H^2 \cdot F = 0, \quad E^2 \cdot F = 0, \]

\[ H^3 = 1, \quad F^3 = 2(H - E) \cdot F^2. \]

**Remark.** We note that

\[ E^2 = - \sum_{x \in \mathcal{L}_3} L_x, \quad H \cdot F = \sum_{l \in \mathcal{L}_2} M_l, \quad E^3 = |\mathcal{L}_3|, \quad H \cdot F^2 = -|\mathcal{L}_2|. \]

### 4.2. The Chern classes of logarithmic 1-forms

Let \( \tau : X \rightarrow \mathbb{P}^3 \) be a blowing up of \( \mathbb{P}^3 \) along a singular set \( \mathcal{L} \) with a normal crossing divisor \( D = \tau^{-1} \cup_{H_i \in \mathcal{A}} H_i \). In this section we compute the Chern classes of \( \Omega^1_X(\log D) \) which implies Theorem 3.6, using the above intersection pairing. First we recall the following.

**Proposition 4.2 ([11]).** The \( i \)-th Chern class \( c_i \) of \( \Omega^1_X \) is given by

\[ c_1 = -4H + 2E + F \]

\[ c_2 = 6H^2 - F^2 - 2H \cdot F \]

\[ c_3 = -4H^3 - 2E^3 + 2H \cdot F^2. \]

The Chern polynomial of \( \bigoplus_{x \in \mathcal{L}_3} \mathcal{O}_{E_x} \) is

\[ C_t\left( \bigoplus_{x \in \mathcal{L}_3} \mathcal{O}_{E_x} \right) = 1 + Et + E^2t^2 + E^3t^3 \]

and one of \( \bigoplus_{l \in \mathcal{L}_2} \mathcal{O}_{F_l} \) is

\[ C_t\left( \bigoplus_{l \in \mathcal{L}_2} \mathcal{O}_{F_l} \right) = 1 + Ft + F^2t^2 + F^3t^3. \]

**Proof.** We rewrite calculations in [11, pp. 621–624] with our notations. (cf. [10]) \( \square \)
Proposition 4.3. Let $D_i$ be the proper transform of $H_i$ by $\tau$ for $H_i \in \mathcal{A}$. Denote by $h_i$ the $i$-th Chern classes of $\bigoplus_{H_i \in \mathcal{A}} \mathcal{O}_{D_i}$. Then we have

\[
\begin{align*}
    h_1 &= nH - pE - pF \\
    h_2 &= \left(\frac{n+1}{2}\right)H^2 + \left(\frac{p+1}{2}\right)E^2 + \left(\frac{p+1}{2}\right)F^2 \\
    & \quad - \{ (nH - pE) + (H - E) \} \cdot pF \\
    h_3 &= \left(\frac{n+2}{3}\right) - \left(\frac{p+2}{3}\right)E^3 - \left(\frac{p+2}{3}\right)F^3 \\
    & \quad + \{ (nH - pE) + 2(H - E) \} \cdot \left(\frac{p+1}{2}\right)E^2.
\end{align*}
\]

We recall a rule of notations, for example,

\[
\left(\frac{p+1}{2}\right)E^2 = \sum_{x \in \mathcal{L}_3} \left(\frac{p+1}{2}\right)E_x^2.
\]

Proof. We denote

\[
\mathbb{D} := \sum_{H_i \in \mathcal{A}} D_i, \quad \mathbb{D}^{(k)} := \sum_{H_i \in \mathcal{A}} D_i^k.
\]

Then we have its Chern polynomial

\[
C_t(\bigoplus_{H_i \in \mathcal{A}} \mathcal{O}_{D_i}) = \prod_{H_i \in \mathcal{A}} (1 + D_i t + D_i^2 t^2 + D_i^3 t^3)
\]

\[
= 1 + \left( \sum_i D_i \right) t + \left( \sum_{i \leq j} D_i \cdot D_j \right) t^2 + \left( \sum_{i \leq j \leq k} D_i \cdot D_j \cdot D_k \right) t^3
\]

\[
= 1 + \mathbb{D} t + \frac{1}{2} \left\{ \mathbb{D}^2 + \mathbb{D}^{(2)} \right\} t^2 + \frac{1}{6} \left\{ \mathbb{D}^3 + 3 \mathbb{D} \cdot \mathbb{D}^{(2)} + 2 \mathbb{D}^{(3)} \right\} t^3.
\]

We shall compute its coefficients which are the Chern classes $h_i$. we have

\[
D_i = H - \sum_{x \in \mathcal{L}_3, x \subset H_i} E_x - \sum_{l \in \mathcal{L}_2, l \subset H_i} F_l.
\]
Since the cardinality of $\mathcal{A}$ is $n$ and $p_X$ is the number of elements of $\mathcal{A}$ including $X$ for $X \in L$, we get

$$\sum_{H_i \in \mathcal{A}} H^k = n H^k$$

$$\sum_{H_i \in \mathcal{A}} \sum_{x \subset H_i} E^k_x = \sum_x p_x E^k_x = p E^k, \quad \sum_{H_i \in \mathcal{A}} \sum_{l \subset H_i} F^k_l = \sum_l p_l F^k_l = p F^k.$$ 

Then the first Chern class $h_1 = \mathbb{D}$ is

$$\mathbb{D} = n H - p E - p F$$

and by Lemma 4.1 we have

$$\mathbb{D}^2 = n^2 H^2 + p^2 E^2 + p^2 F^2 - 2(n H - p E) \cdot p F$$

$$\mathbb{D}^3 = n^3 - p^3 E^3 - p^3 F^3 + 3(n H - p E) \cdot p^2 F^2.$$ 

Secondly we shall compute $\mathbb{D}^{(k)}$ ($k = 2, 3$). By Tables 1 and 2 we obtain

$$D_i^2 = \left\{ H - \sum_{x \subset H_i} E_x - \sum_{l \subset H_i} F_l \right\}^2$$

$$= H^2 + \sum_{x \subset H_i} E^2_x + \sum_{l \subset H_i} F^2_l - 2H \cdot \left( \sum_{l \subset H_i} F_l \right)$$

$$+ 2 \left( \sum_{x \subset H_i} E_x \right) \cdot \left( \sum_{l \subset H_i} F_l \right),$$

$$D_i^3 = \left\{ H - \sum_{x \subset H_i} E_x - \sum_{l \subset H_i} F_l \right\}^3$$

$$= H^3 - \sum_{x \subset H_i} E^3_x - \sum_{l \subset H_i} F^3_l + 3H \cdot \left( \sum_{l \subset H_i} F^2_l \right)$$

$$- 3 \left( \sum_{x \subset H_i} E_x \right) \cdot \left( \sum_{l \subset H_i} F^2_l \right).$$
We compute a sum of their last terms for all $H_i$ as follows. Since $E_x \cdot F_l = 0$ for $x \not\subset l$ we can see
\[
E \cdot F^k = \left( \sum_x E_x \right) \cdot \left( \sum_l F_l^k \right) = \sum_l \sum_{x \subset l} E_x \cdot F_l^k
\]
and then
\[
\sum_{H_i \in A} \left( \sum_{x \subset H_i} E_x \right) \cdot \left( \sum_{l \subset H_i} F_l^k \right) = \sum_{H_i \in A} \sum_{l \subset H_i} \sum_{x \subset l} E_x \cdot F_l^k
\]
\[
= \sum_l p_l \sum_{x \subset l} E_x \cdot F_l^k = E \cdot pF^k.
\]
Consequently we obtain
\[
\mathbb{D}^{(2)} = nH^2 + pE^2 + pF^2 - 2(H - E) \cdot pF
\]
\[
\mathbb{D}^{(3)} = n - pE^3 - pF^3 + 3(H - E) \cdot pF^2
\]
Using Lemma 4.1 we can compute the Chern classes $h_i$. □

Therefore we can obtain the Chern classes of logarithmic 1-forms.

**Theorem 4.4.**

\[
c_1(\Omega^1_X (\log D)) = (n - 4)H - (p - 3)E - (p - 2)F
\]
\[
c_2(\Omega^1_X (\log D)) = \left( \frac{n - 3}{2} \right)H^2 + \left( \frac{p - 2}{2} \right)E^2 + \left( \frac{p - 1}{2} \right)F^2
\]
\[
- (n - 3)H \cdot (p - 2)F + (p - 2)E \cdot (p - 2)F
\]
\[
c_3(\Omega^1_X (\log D)) = \left( \frac{n - 2}{3} \right) - \left( \frac{p - 1}{3} \right)E^3 - \left( \frac{p}{3} \right)F^3
\]
\[
+ (n - 2)H \cdot \left( \frac{p - 1}{2} \right)F^2 - (p - 1)E \cdot \left( \frac{p - 1}{2} \right)F^2.
\]

**Proof.** We have
\[
C_t(\Omega^1_X (\log D)) = C_t(\Omega^1_X) \cdot C_t(\bigoplus_{H_i \in A} \mathcal{O}_{D_i}) \cdot C_t(\bigoplus_{x \in \mathcal{L}_3} \mathcal{O}_{E_x}) \cdot C_t(\bigoplus_{l \in \mathcal{L}_2} \mathcal{O}_{F_l}).
\]
Therefore the above propositions give direct calculations of Chern classes of $\Omega^1_X(\log D)$, with relations Lemma 4.1. □

Now we notice that the equality $(nH - pE) \cdot F^2 = (H - E) \cdot pF^2$ holds when $\mathcal{L} = \mathcal{L}$.

**Definition 4.1.** We define $R \cdot F^2$ by an equation

$$(nH - pE) \cdot F^2 = (H - E) \cdot pF^2 + R \cdot F^2.$$

**Lemma 4.5.**

$$R \cdot F^2 = \sum_{l \in \mathcal{L}_2} \sum_{x \notin \mathcal{L}_3} (p_x - p_l)$$

**Proof.** Since $n - p_l = \sum_{x \in \mathcal{L}_3, x \subset l} (p_x - p_l)$ for a line $l \in \mathcal{L}_2$, we get

$$n - \sum_{x \in \mathcal{L}_3, x \subset l} p_x = (1 - \sum_{x \in \mathcal{L}_3} 1)p_l + \sum_{x \notin \mathcal{L}_3} (p_x - p_l).$$

Then taking a sum of these for all $l \in \mathcal{L}_2$, we get the above expression. □

For some value $v_X$ associated to $X \in \mathcal{L}$, we write

$$R \cdot vF^2 = \sum_{l \in \mathcal{L}_2} v_l \sum_{x \notin \mathcal{L}_3} (p_x - p_l).$$

In particular if $\mathcal{L} = \mathcal{L}$, we get $R \cdot F^2 = 0$ namely

$$(nH - pE) \cdot vF^2 = (H - E) \cdot vpF^2.$$  

Since $-2\binom{p}{3} + p\binom{p-1}{2} - \binom{p-1}{2} = \binom{p-1}{3}$, the following is obtained.

**Corollary 4.6.**

$$c_3(\Omega^1_X(\log D)) = \binom{n-2}{3} - \binom{p-1}{3}E^3 + (H - E) \cdot \binom{p-1}{3}F^2$$

$$- (H - R) \cdot \binom{p-1}{2}F^2.$$
By the way, Gauss-Bonnet formula says that
\[ c_3(\Omega_X^1(\log D)) = -\chi(A). \]
Therefore, when \( \mathcal{L} = L \), the above corollary implies Theorem 3.6. Note that Theorem 3.6 can be checked by topological direct argument with combinatorics.

4.3. Hodge number

The invertible sheaf \( V_\alpha \) is given by
\[
V_\alpha = \pi^* \mathcal{O}_\mathbb{P}(\nu(\alpha)) \otimes \mathcal{O}_X \left( \sum_{x \in \mathcal{L}_3} \beta_x(\alpha) E_x + \sum_{l \in \mathcal{L}_2} \beta_l(\alpha) F_l \right)
\]

\[ = -\nu H + \beta E + \beta F. \]

We trace the analogue of Defintion 4.1 in order to get a relation similar to Lemma 4.5.

**DEFINITION 4.2.** We define \( \mathbb{R}_\alpha \cdot \mathbb{F}^2 \) by the equation
\[
(\nu H - \beta E) \cdot \mathbb{F}^2 = (H - E) \cdot \beta \mathbb{F}^2 + \mathbb{R}_\alpha \cdot \mathbb{F}^2.
\]

In the same way of Lemma 4.5 we get

**LEMMA 4.7.**
\[
\mathbb{R}_\alpha \cdot \mathbb{F}^2 = -\sum_{l \in \mathcal{L}_2} \sum_{x \in \mathcal{L}_3} \sum_{x \subset l} (\alpha_x - \alpha_l) + (H - E) \cdot \mathbb{F}^2 + \epsilon \mathbb{F}^2.
\]

Note that \( (\nu H - \beta E) \cdot \nu \mathbb{F}^2 = (H - E) \cdot \beta \nu \mathbb{F}^2 + \mathbb{R}_\alpha \cdot \nu \mathbb{F}^2 \) for a value \( \nu_l \) associated to \( l \). If \( \mathcal{L} = L \) then
\[
\mathbb{R}_\alpha \cdot \mathbb{F}^2 = (H - E) \cdot \mathbb{F}^2 + \epsilon \mathbb{E} \cdot \mathbb{F}^2.
\]
And we can see easily
\[(H - \mathbb{R}_\alpha) \cdot F^2 + (H - \mathbb{R}_\alpha^*) \cdot F^2 = (H - \mathbb{R}) \cdot F^2.\]

**Proposition 4.8.**
\[
\chi(X, \mathcal{V}_\alpha) = -\left(\nu - \frac{1}{3}\right) + \left(\beta\right) \mathbb{E}^3 - (H - \mathbb{E}) \cdot \left(\beta\right) F^2 + (H - \mathbb{R}_\alpha) \cdot \left(\beta\right) F^2.
\]

**Proof.** For a line bundle \(L\), it is well known that \(\chi(X, L) = \frac{1}{6}L^3 + \frac{1}{4}c_1L^2 + \frac{1}{12}(c_1^2 + c_2)L + \frac{1}{24}c_1c_2\), where \(c_i = c_i(X)\). Then the straight calculation leads the expression above. \(\square\)

**Proposition 4.9.**
\[
\chi(X, \Omega^1_X(\log D) \otimes \mathcal{V}_\alpha) = (\nu^* - 1) \left(\nu - \frac{1}{2}\right) - \beta^* \left(\frac{\beta}{2}\right) \mathbb{E}^3 + (H - \mathbb{E}) \cdot \beta^* \mathbb{F}^2 - (H - \mathbb{R}_\alpha) \cdot \beta \beta^* \mathbb{F}^2 + (\mathbb{R} - \mathbb{R}_\alpha) \cdot \left(\beta\right) F^2.
\]

**Proof.** Denote \(i\)-th Chern class of \(\Omega^1_X(\log D)\) by \(d_i\). Note that the rank of \(\Omega^1_X(\log D)\) is 3. We can check the following by using that Euler characteristic can be written in terms of Chern classes (see [10].);

\[
\chi(\Omega^1_X(\log D) \otimes \mathcal{V}_\alpha) = \chi(\Omega^1_X(\log D)) + 3\chi(\mathcal{V}_\alpha) - 3\chi(\mathcal{O}_X) + \xi
\]

here \(\xi = \frac{1}{2}(d_1^2 - 2d_2 + d_1c_1 + d_1\mathcal{V}_\alpha)\mathcal{V}_\alpha\). By the straight calculation we can get that

\[
\chi(\Omega^1_X(\log D)) = (n - 1)H^3, \quad \chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2 = H^3
\]

and this proposition. \(\square\)

We unify last terms in the above propositions.
Definition 4.3. We define $E^{p,q}(\alpha)$ by

$$E^{p,q}(\alpha) = (H - \mathbb{R}\alpha) \cdot \left( \frac{\beta^*}{p} \right) \left( \frac{\beta}{q} \right) F^2 + (H - \mathbb{R}\alpha^*) \cdot \left( \frac{\beta^*}{p-1} \right) \left( \frac{\beta}{q} \right) F^2.$$ 

Note that $E^{p,q}(\alpha) = E^{q,p}(\alpha^*)$ and

$$\sum_{p+q=3} E^{p,q}(\alpha) = (H - \mathbb{R}) \cdot \left( \frac{p-1}{2} \right) F^2.$$ 

Therefore we get the Hodge numbers as follows.

Theorem 4.10. 

$$h^{p,q}(\alpha) = \left( \frac{\nu^* - 1}{p} \right) \left( \frac{\nu - 1}{q} \right) - \left( \frac{\beta^*}{p} \right) \left( \frac{\beta}{q} \right) F^3$$

$$+ (H - \mathbb{R}) \cdot \left( \frac{\beta^*}{p} \right) \left( \frac{\beta}{q} \right) F^2 - E^{p,q}(\alpha).$$

Proof. We know

$$\chi(X, \mathcal{V}_\alpha) = \sum_i (-1)^i h^{0,i}(\alpha)$$

$$\chi(X, \Omega^1_X(\log D) \otimes \mathcal{V}_\alpha) = \sum_i (-1)^i h^{1,i}(\alpha).$$

Recall that $H^i(U, V_\alpha)$ vanishes for $\alpha$ is generic and $i$ is not $N = 3$. So we have $h^{p,q}(\alpha) = 0$ when $p + q \neq 3$. We obtain

$$h^{0,3}(\alpha) = -\chi(X, \mathcal{V}_\alpha)$$

$$h^{1,2}(\alpha) = \chi(X, \Omega^1_X(\log D) \otimes \mathcal{V}_\alpha)$$

$$h^{3,0}(\alpha) = \overline{h^{0,3}(\alpha^*)} = -\chi(X, \mathcal{V}_{\alpha^*})$$

$$h^{2,1}(\alpha) = \overline{h^{1,2}(\alpha^*)} = \chi(X, \Omega^1_X(\log D) \otimes \mathcal{V}_{\alpha^*}).$$

Therefore two propositions above induce this theorem. $\square$

Remark. We compare Theorem 4.10 with Collary 4.6. For a singular set $\mathcal{L}$, denote the description of $c_3(\Omega^1_X(\log D))$ in Collary 4.6 by $-\chi_{\mathcal{L}}$ and
one of $h^{p,q}(\alpha)$ in Theorem 4.10 by $h^{p,q}(\alpha)_L$. Note that they have four terms respectively. Then we have
\[
-\chi_L = \sum_{p+q=3} h^{p,q}(\alpha)_L.
\]
Furthermore the sum of $i$-th terms of $h^{p,q}(\alpha)_L$ is $i$-th terms of $-\chi_L$ for $1 \leq i \leq 4$.

5. Minimal Singular Sets and Examples

Recall the blowing up for an arrangement. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^3$, $\mathcal{L}$ be a singular set of $\mathcal{A}$ and $\tau: X \to \mathbb{P}^3$ be a blowing up along $\mathcal{L}$ such that the total transform $D$ of $\cup_{H \in \mathcal{A}} H$ is a normal crossing divisor.

First minimal $\mathcal{L}_2$ consists of $k$-lines, $k > 2$, where $k$-line $l$ is an element of $L_2$ such that the number of hyperplanes in $\mathcal{A}$ including $l$ is $k$, we call a singular line.

Secondly we shall find minimal $\mathcal{L}_3$. Denote the description of $c_3(\Omega^1_X(\log D))$ in Collary 4.6 by $-\chi_L$. We take any $x_0 \in \mathcal{L}$ and put $\mathcal{L}' = \mathcal{L} - \{x_0\}$. The difference $d(x_0) = -\chi_L + \chi_{\mathcal{L}'}$ of them for $\mathcal{L}$ and $\mathcal{L}'$ can be write explicitly
\[
d(x_0) = \left(\frac{p_{x_0} - 1}{3}\right) - \sum_{l \in \mathcal{L}_{2, x_0}} \left(\frac{p_l - 1}{3}\right) - \sum_{l \in \mathcal{L}_{2, x_0}} (p_{x_0} - p_l) \left(\frac{p_l - 1}{2}\right).
\]
On the other hand it is well-known fact that Gauss-Bonnet formula
\[
c_3(\Omega^1_X(\log D)) = -\chi(U, \mathbb{C}).
\]
Since $U = X \setminus D = \mathbb{P}^3 \setminus \cup_{H \in \mathcal{A}} H$ we can see that if $\mathcal{L}$ and $\mathcal{L}'$ are singular sets of $\mathcal{A}$ then $d(x_0) = 0$.

By the explicit form above, if $x_0$ with $p_{x_0} = 3$ or if $x_0$ is included in only one singular line then $d(x_0) = 0$. If $x_0$ is included in two or more singular lines then $d(x_0) \neq 0$, we call a singular point. Consequently the minimal singular set of $\mathcal{A}$ consists of all singular lines and all points included in two or more singular lines.
5.1. 2-generic arrangements

An arrangement of hyperplanes is called to be the $p$-generic, if $p_X = k$ for all $X \in L_k(A), k \leq p$. Let $A$ be 2-generic arrangement of hyperplanes in $\mathbb{P}^3$. Then $A$ has no singular lines. We get the combinatorial Formula

$$\binom{n}{3} = \sum_{x \in L_3} \binom{p_x}{3}.$$ 

The topological Euler characteristic is

$$-\chi(A) = \binom{n-2}{3} - \sum_{x \in L_3} \binom{p_x-1}{3},$$

and the Hodge numbers are

$$h_{p,q}(\alpha) = \binom{\nu^* - 1}{p} \binom{\nu - 1}{q} - \sum_{x \in \mathcal{L}_3} \binom{\beta_x^*}{p} \binom{\beta_x}{q}.$$ 

5.2. An arrangement without singular points

We assume that an arrangement $A$ has no singular points. Namely the singular set of $A$ is the set of singular lines. We have the combinatorial Formula

$$\binom{n}{3} = \sum_{l \in L_2} \left\{ \binom{p_l}{3} + (n - p_l) \binom{p_l}{2} \right\}.$$ 

The topological Euler characteristic is

$$-\chi(A) = \binom{n-2}{3} - \sum_{l \in L_2} \left\{ \binom{p_l-1}{3} - (n - p_l - 1) \binom{p_l-1}{2} \right\},$$

and the Hodge numbers are

$$h_{p,q}(\alpha) = \binom{\nu^* - 1}{p} \binom{\nu - 1}{q} - \sum_{l \in \mathcal{L}_2} \binom{\beta_l^*}{p} \binom{\beta_l}{q}$$

$$- \sum_{l \in \mathcal{L}_2} \left\{ (\nu - \beta_l - 1) \binom{\beta_l^*}{p} \binom{\beta_l}{q-1} ight\}$$

$$+ (\nu^* - \beta_l^* - 1) \binom{\beta_l^*}{p-1} \binom{\beta_l}{q} \right\}.$$
5.3. Generalized Ceva’s arrangement

We take five points of \( \mathbb{P}^3 \) in general position and the arrangement \( C_3 \) of ten hyperplanes determined by any three points is natural generalization of Ceva’s configuration in \( \mathbb{P}^2 \) (cf. [16]). We can choose the homogeneous coordinate \([z_1, z_2, z_3, z_4]\) such that \( C_3 \) is defined by the equation

\[
z_1 z_2 z_3 z_4 \prod_{i<j} (z_i - z_j) = 0.
\]

This arrangement has 15 points and 25 lines. We have combinatorial data;

- the number of 2-lines is 15,
- the number of 3-lines is 10 (that are singular lines),
- the number of 4-points is 10,
- the number of 6-points is 5 (that are singular points).

Here \( i \)-point \( x \) is in \( L_3 \) with \( p_x = i \) and \( j \)-line \( l \) is in \( L_2 \) with \( p_l = j \). We can take the singular set \( \mathcal{L} \) consisting of 6-points and 3-lines. Therefore we obtain \( -\chi(\mathcal{A}) = 6 \).

Now we take \( \alpha = (a, a, \ldots, a) \) such that \( a \) is a rational number, \( 0 < a < 1 \), and \( 10a \) is integer. If \( 6a \) is not integer, then \( \alpha \) is generic. We can compute

\[
(h^{3,0}(\alpha), h^{2,1}(\alpha), h^{1,2}(\alpha), h^{0,3}(\alpha)),
\]

is called the Hodge type of generic \( \alpha \). Just there are only the following cases;

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References


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