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Density of a Collection of Functions in N_Φ-Spaces

By Ha Huy Bang and Truong Van Thuong

Abstract. This paper presents sufficient conditions for a translation invariant subspace of $L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$ to be dense in $N_\Phi(\mathbb{R}^n)$.

Introduction

In the 1930s N.Wiener presented a necessary and sufficient condition under which a collection of functions generated by translating a single function to be complete in $L_1(\mathbb{R})$ and $L_2(\mathbb{R})$ [8]. R.A.Zalik proved later that to under some certain conditions the restriction to $\mathbb{R}$ of the family of functions $\{f(x+\alpha) : \alpha \in S\}$, where f is a function on $\mathbb{C}$ and S a sequence of distinct complex numbers, is complete in $L_p(\mathbb{R}^+)$ ([10], [11]). Recently, V.V.Volchkov has obtained some generalizations of N.Wiener's theorems in $L_p(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ [7].

Let φ be a function defined on $\mathbb{R}^n$ and a be a function defined on $\mathbb{Z}^n$. Their *semi-discrete convolution* [9] is defined by, for any $x \in \mathbb{R}^n$,

$$\varphi *' a(x) = \sum_{\alpha \in \mathbb{Z}^n} \varphi(x-\alpha)a(\alpha),$$

for which the series converges absolutely. Denote by $\ell_0(\mathbb{Z}^n)$ the space of all finitely supported functions on $\mathbb{Z}^n$ and by $S_0(\varphi)$ the image of $\ell_0(\mathbb{Z}^n)$ under $\varphi *'$.

A collection F of functions on $\mathbb{R}^n$ is called *shift invariant* [9] if for each $f \in F$, $\alpha \in \mathbb{Z}^n$ then $f(.+\alpha) \in F$. Then $S_0(\varphi)$ is a linear span of the integer translates of φ and is shift invariant. A set F is called *translation invariant* if

$$\tau_t : f \longrightarrow f(.+t)$$

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maps F into F for each $t \in \mathbb{R}^n$ and F is *dilation invariant* if

$$\sigma_h : f \longrightarrow f(h^{-1}.)$$

maps F into itself for each $h > 0$. Denote

$$U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi).$$

The problem of finding sufficient conditions on a collection of functions generated by dilating and translating of a single function to be dense in $L_p(\mathbb{R}^n)$ is studied by Kang Zhao [9]. The author showed that under some certain conditions on φ, then the spanU_h is dense in $L_p(\mathbb{R}^n)$.

A natural question arises under what conditions on the collection U_h and function φ, the spanU_h is dense in the space $N_\Phi(\mathbb{R}^n)$ generated by the concave function Φ [6]?

In the paper, we prove, in contrast with Orlicz spaces $L_\Phi(\mathbb{R}^n)$ (where the Young function Φ must satisfy the Δ_2-condition (see [3], [4])), the continuity of norm in any space $N_\Phi(\mathbb{R}^n)$, and give some sufficient conditions for a collection of functions generated by dilating and translating of a single function to be dense in $N_\Phi(\mathbb{R}^n)$. Besides some results similar to Kang Zhao's ones [9], a study of the geometrical properties of the spectrum of functions in $N_\Phi(\mathbb{R}^n)$ helps us to obtain certain new sufficient conditions for the density.

Main Results

Let $\mathcal{C}$ denote the family of all non-zero concave functions $\Phi : [0, +\infty) \to [0, +\infty)$, which are non-decreasing, unbounded and satisfy $\Phi(0) = 0$. For an arbitrary measurable function f and $\Phi \in \mathcal{C}$, we put $\Phi(\infty) := \lim_{x\to\infty} \Phi(x)$ and define

$$\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(t)\big)dt,$$

where $\lambda_f(t) = \mu(\{x : |f(x)| > t\})$, $t \geq 0$ and μ is a positive measure on $\mathbb{R}^n$. Let $N_\Phi(\mathbb{R}^n)$ be the space of all measurable functions f such that $\|f\|_{N_\Phi} < \infty$. Then $N_\Phi(\mathbb{R}^n)$ is a Banach space [6].

The following property of $N_\Phi(\mathbb{R}^n)$ will be useful in the sequel.

THEOREM 1. *For every $f \in N_\Phi(\mathbb{R}^n)$,*

$$\lim_{t\to 0} \|f(.+t) - f\|_{N_\Phi} = 0. \tag{1}$$

PROOF. We shall begin with showing that the set A of all complex, measurable, simple functions with bounded support is dense in $N_\Phi(\mathbb{R}^n)$.

Fixed $f \in N_\Phi(\mathbb{R}^n)$. Without loss of generality we may assume that $f \geq 0$. As traditionally, for $m = 1, 2, \dots,$ and for $1 \leq k \leq m2^m$, we define

$$E_{m,k} = f^{-1}\big([\frac{k-1}{2^m}, \frac{k}{2^m})\big) \text{ and } F_m = f^{-1}\big([m, \infty]\big)$$

and put

$$s_m = \sum_{k=1}^{m2^m} \frac{k-1}{2^m} \chi_{E_{m,k}} + m\chi_{F_m}.$$

Then $E_{m,k}$ and F_m are measurable sets, $s_m \leq m$, $0 \leq s_1 \leq s_2 \leq \cdots \leq f$ and $s_m(x) \longrightarrow f(x)$ as $m \to \infty$, for every $x \in \mathbb{R}^n$.

Since $0 \leq s_m \leq f$, it follows that $s_m \in N_\Phi(\mathbb{R}^n)$ and $\mu(E_m) < \infty$, where $E_m = \{x \ : \ s_m(x) \neq 0 \}$.

It is easy to see that $s_m(x) \geq f(x) - 2^{-m}$ if $m \geq f(x)$, and $s_m(x) = m$ if $f(x) = \infty$. Hence, since $f \in N_\Phi(\mathbb{R}^n)$ and $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, we have for each $t > 0$

$$\lambda_{f-s_m}(t) = \mu(\{x : \ f(x) - s_m(x) > t\}) \to 0 \ \text{ as } \ m \to \infty.$$

On the other hand, $\lambda_{f-s_m} \leq \lambda_f$ and then $\Phi(\lambda_{f-s_m}) \leq \Phi(\lambda_f)$. Therefore, the dominated convergence theorem shows that

$$\lim_{m\to\infty} \|f - s_m\|_{N_\Phi} = \lim_{m\to\infty} \int_0^\infty \Phi(\lambda_{f-s_m}(t))dt = 0.$$

Further, since $\mu(E_m) < \infty$, there exists a ball B_m such that

$$\|s_m\|_\infty \Phi(\mu(E_m \setminus B_m)) \to 0 \ \text{ as } m \to \infty.$$

We define

$$s'_m(x) = \begin{cases} s_m(x) & \text{if } x \in B_m \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_m. \end{cases}$$

Then

$$\begin{aligned} \|s_m - s'_m\|_{N_\Phi} &= \int_0^\infty \Phi(\lambda_{s_m - s'_m}(t))dt \\ &= \int_0^{\|s_m\|_\infty} \Phi(\mu\{x \in \mathbb{R}^n \setminus B_m \ : \ s_m(x) > t\})dt \\ &\le \|s_m\|_\infty \Phi(\mu(E_m \setminus B_m)). \end{aligned}$$

Thus we have proved

$$\lim_{m\to\infty} \|f - s'_m\|_{N_\Phi} = 0$$

as was to be shown.

Therefore, to prove the theorem, it suffices to show (1) for any $f \in A$. Assume on the contrary that, there exist $\{t_k\} \subset \mathbb{R}^n$, $|t_k| \to 0$ and $\varepsilon > 0$ such that

$$\|f(.+t_k) - f\|_{N_\Phi} \ge \varepsilon, \qquad \forall k \ge 1. \tag{2}$$

Since $f \in L^1_{\ell oc}(\mathbb{R}^n)$, we have for each $K_\ell = [-\ell, \ell]^n$

$$\int_{K_\ell} |f(x+t_k) - f(x)|dx \to 0 \qquad \text{as } k \to \infty.$$

Therefore, by Theorem D [2, p. 93], there exists a subsequence $\{t_{k_j}\}$, we still denote by $\{t_k\}$ such that $f(.+t_k) \to f$ a.e. on K_ℓ. Hence, there exists a subsequence, denoted again by $\{t_k\}$ such that $f(.+t_k) \to f$ a.e. on $\mathbb{R}^n$. Define

$$g_m(x) = \inf_{k\ge m} |f(x+t_k)|, \ x \in \mathbb{R}^n$$

then $\{g_m\}$ is a nondecreasing sequence and $g_m \to |f|$ a.e. By the result in [6], we have

$$\lambda_{g_m}(t) \to \lambda_{|f|}(t) \ \text{ as } m \to \infty, \ \text{for every } t > 0.$$

Since $\Phi \in \mathcal{C}$,

$$\Phi(\lambda_{|f|}(t)) = \lim_{m\to\infty} \Phi(\lambda_{|g_m|}(t)) \le \varliminf_{k\to\infty} \Phi(\lambda_{|f(.+t_k)|}(t)),\ t > 0. \tag{3}$$

It follows from $\Phi \in \mathcal{C}$ that $\Phi(a+b) \le \Phi(a) + \Phi(b)$ for $a, b \ge 0$. Observing that, for any $f, g \in N_\Phi(\mathbb{R}^n)$ and $t > 0$ we have $\lambda_{f+g}(2t) \le \lambda_f(t) + \lambda_g(t)$, then

$$\Phi(\lambda_{|f(.+t_k)-f|}(2t)) \le \Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t)).$$

Hence,

$$0 \le \big[\Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t))\big] - \Phi(\lambda_{|f(.+t_k)-f|}(2t)),\ \forall t > 0,\ \forall k \ge 1.$$

It is easy to check that

$$\lim_{t\to 0} \|f(.+t)\|_{N_\Phi} = \|f\|_{N_\Phi}.$$

Applying Fatou's lemma to the sequence

$$\{[\Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t))] - \Phi(\lambda_{|f(.+t_k)-f|}(2t))\},$$

we obtain

$$\begin{aligned} &\int_0^\infty \varliminf_{k\to\infty} \big[[\Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t))] - \Phi(\lambda_{|f(.+t_k)-f|}(2t))\big]dt \\ &\quad \le \varliminf_{k\to\infty} \int_0^\infty \big[[\Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t))] - \Phi(\lambda_{|f(.+t_k)-f|}(2t)\big]dt \\ &\quad = 2\int_0^\infty \Phi(\lambda_{|f|}(t))dt - \frac{1}{2}\varlimsup_{k\to\infty} \int_0^\infty \Phi(\lambda_{|f(.+t_k)-f|}(t))dt. \end{aligned} \tag{4}$$

Since $|t_k| \to 0$ and the support of f is bounded, there exists a ball B such that

$$\begin{aligned} \lambda_{|f(.+t_k)-f|}(t) &= \mu(\{x \in \mathbb{R}^n : |f(x+t_k) - f(x)| > t\}) \\ &= \mu(\{x \in B : |f(x+t_k) - f(x)| > t\}) \end{aligned}$$

for all $k \geq 1$ and $t > 0$. Therefore, taking account of $f(.+t_k) \to f$ a.e. on $\mathbb{R}^n$ and $\mu(B) < \infty$, we have

$$\lim_{k\to\infty} \lambda_{|f(.+t_k)-f|}(t) = 0$$

and then

$$\lim_{k\to\infty} \Phi(\lambda_{|f(.+t_k)-f|}(t)) = 0. \tag{5}$$

Combining (3) and (5), we get for any $t > 0$

$$\begin{aligned} 2\Phi(\lambda_{|f|}(t)) &= \lim_{k\to\infty} \Phi(\lambda_{|g_k|}(t)) + \Phi(\lambda_{|f|}(t)) - \lim_{k\to\infty} \Phi(\lambda_{|f(.+t_k)-f|}(2t)) \\ &\leq \underline{\lim}_{k\to\infty} \big[\Phi(\lambda_{|f(.+t_k)|}(t)) + \Phi(\lambda_{|f|}(t)) - \Phi(\lambda_{|f(.+t_k)-f|}(2t))\big]. \end{aligned} \tag{6}$$

Since (4) and (6), we have

$$2\int_0^\infty \Phi(\lambda_{|f|}(t))dt \leq 2\int_0^\infty \Phi(\lambda_{|f|}(t))dt - \frac{1}{2}\overline{\lim}_{k\to\infty} \int_0^\infty \Phi(\lambda_{|f(.+t_k)-f|}(t))dt.$$

Hence

$$\int_0^\infty \Phi(\lambda_{|f(.+t_k)-f|}(t))dt \to 0 \quad \text{as } k \to \infty,$$

i.e., $\lim_{k\to\infty} \|f(.+t_k) - f\|_{N_\Phi} = 0$, which contradicts (2). The proof is complete. □

The two following lemmas are based on Theorem 1 and can be proved in a similar way to that of Lemma 2.1 and Lemma 2.2 [9].

Lemma 1. *Let $\varphi \in N_\Phi(\mathbb{R}^n)$. Assume that $\frac{1}{h}$ is an integer larger than 1. If $\varphi \in \overline{\text{span} U_h}$, where $U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)$, then $\overline{\text{span} U_h}$ is translation invariant.*

Denote by $\mathbb{R}^*$ the abelian group of all nonzero real numbers with the operation of ordinary multiplication and $\text{dist}(\varphi, S)_{N_\Phi} = \inf\{\|\varphi - f\|_{N_\Phi},\ f \in S\}$.

Lemma 2. *Let $\varphi \in N_\Phi(\mathbb{R}^n)$ and let G be a subgroup of $\mathbb{R}^*$. If*

$$\lim_{h\in G,\ h\to 0} \operatorname{dist}(\varphi, \sigma_h S_0(\varphi))_{N_\Phi} = 0,$$

then $\overline{\bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)}$ is translation invariant, for any sequence $\{h_j\} \subset G$ with $\lim_{j\to\infty} h_j = 0$.

Definition 1. A measure μ on $\mathbb{R}^n$ is said to be admissible if for any permutation $(m_1, \ldots, m_n)$ of $(1, \ldots, n)$, $1 \le k \le n-1$ and any ball B in the k-dimensional space of the variables $x_{m_1}, \ldots, x_{m_k}$ then

$$\mu(B \times \mathbb{R}^{n-k}) = \infty.$$

In the sequel we assume that μ is admissible. Further, we also assume that the σ-algebra $\mathcal{B}$ of subsets of $\mathbb{R}^n$ has the following property: If $E \in \mathcal{B}$ and $\mu(E) = +\infty$ then there exists some set $F \in \mathcal{B}$ such that $F \subset E$ and $0 < \mu(F) < \infty$. The last property is a necessary and sufficient condition so that $M_\Phi(\mathbb{R}^n)$ is normed, where we denote $M_\Phi(\mathbb{R}^n)$ the space of measurable functions g such that

$$\|g\|_{M_\Phi} = \sup\left\{\frac{1}{\Phi(\mu(E))}\int_E |g(x)|dx :\ E \subset \mathbb{R}^n,\ 0 < \mu(E) < \infty\right\} < \infty.$$

Then $M_\Phi(\mathbb{R}^n)$ is a Banach space and $N_\Phi^*(\mathbb{R}^n) = M_\Phi(\mathbb{R}^n)$ [6].

The *spectrum* of a function g, denoted by $\operatorname{sp}(g)$, is defined to be the support of $\hat{g}$, the Fourier transform of g. According to the method of the proof of Theorem 1 [1] we obtain the following result.

Lemma 3. *Let $f \in N_\Phi(\mathbb{R}^n), f(x) \not\equiv 0$ and let $\xi^0 \in \operatorname{sp}(f)$ be an arbitrary point. Then the restriction of $\hat{f}$ to any neighbourhood of ξ^0 cannot concentrate on any finite number of hyperplanes.*

Proof. We little sketch the proof. Without loss of generality, we shall prove the result for functions f with bounded spectrum and $\xi^0 = 0$.

Assume on the contrary that there exist a neighbourhood $U \ni 0$ and hyperplanes $H_1, \ldots, H_m$ such that the restriction of $\hat{f}(\xi)$ to U concentrates on $H_1, \dots, H_m$. Without loss of generality we may assume that $0 \in H_j, j = 1, \dots, m$. Then H_j can be defined by the equation

$$a_{j1}\xi_1 + \cdots + a_{jn}\xi_n = 0,$$

where $(a_{j1}, \dots, a_{jn})$ is a unit vector in $\mathbb{R}^n$.

We put for each $j = 1, \dots , m$

$$G_j = \mathbb{R}^n \backslash (\bigcup_{i \neq j} H_i)$$

Then G_j is open. For any $\psi(\xi) \in C_0^\infty(G_j)$, the distribution $\psi(\xi)\hat{f}(\xi)$ concentrates on the hyperplane H_j. We introduce the transformation

$$x = (x_1, \cdots , x_n) \quad \rightleftharpoons \quad (y_1, \cdots , y_n) = y,$$

where $y_1, \cdots , y_n$ are the coordinates of x in the new rectangular system of coordinates, which is chosen such a way that the hyperplane

$$a_{j1}x_1 + \cdots + a_{jn}x_n = 0$$

will be transformed into the hyperplane $y_j = 0$. The coordinate transformation

$$x_k = \sum_{s=1}^{n} \alpha_{k,s} y_s, \quad k = 1, \cdots , n$$

is defined by a real orthogonal matrix $A = (\alpha_{k,s})$ and $|\det A| = 1$.

Put $g(y) = (F^{-1}\psi * f)(x)$. Then $\|g\|_{N_\Phi} = \|F^{-1}\psi * f\|_{N_\Phi}$, supp$\hat{g}$ is compact and, clearly, the Fourier transform of $g(y)$ will concentrate on the hyperplane $\xi_j = 0$. By an argument analogous to that used for the proof of Theorem 1 [1], we see that $g(y)$ does not depend on y_j.

Since $g \in N_\Phi(\mathbb{R}^n)$, we get

$$\int_0^\infty \Phi(\lambda_g(t))dt < \infty. \tag{7}$$

We shall show that $g(y) \equiv 0$. Actually, assume on the contrary that $g(y^0) \neq 0$ for some point y^0. Because $g(y) = F^{-1}(\psi\hat{f})(x)$ is continuous, there exist a number $\varepsilon > 0$ and a neighbourhood V of y^0 such that $|g(y)| > \varepsilon$ for all $y \in V$. Hence, since $g(y)$ does not depend on y_j, we get

$$\lambda_g(\varepsilon) = \mu(\{\ y \in \mathbb{R}^n :\ |g(y)| > \varepsilon\ \}) = \infty.$$

From $\lambda_g(t)$ is a nonincreasing function, $\lambda_g(t) = +\infty$ on the interval $[0, \varepsilon]$. Since $\Phi(t)$ is nondecreasing and unbounded, it follows that $\Phi(\lambda_g(t)) = +\infty$ on $[0, \varepsilon]$, which contradicts (7). Thus, we get $g(y) \equiv 0$, i.e., $\psi(\xi)\hat{f}(\xi) \equiv 0$. Since $\psi(\xi) \in C_0^\infty(G_j)$ is arbitrarily chosen, we get $\hat{f}(\xi) \equiv 0$ on the hyperplane H_j. So $\hat{f}(\xi)$ must concentrate on the planes $H_i \cap H_j,\ i, j = 1, \ldots, m,\ i \neq j$.

We put for $i, j = 1, \ldots, m,\ i \neq j$

$$G_{ij} := \mathbb{R}^n \setminus \cup\{\ H_k \cap H_\ell\ :\ (k, \ell) \neq (i, j),\ k \neq \ell\}.$$

Then G_{ij} is open. For any $\psi(\xi) \in C_0^\infty(G_{ij})$, the distribution $\psi(\xi)\hat{f}(\xi)$ concentrates on the plane $H_i \cap H_j$.

By an argument analogous to the previous one, we obtain $\psi(\xi)\hat{f}(\xi) \equiv 0$. Since $\psi \in C_0^\infty(G_{ij})$ is arbitrarily chosen, we see that $\hat{f}(\xi)$ must concentrate on $H_i \cap H_j \cap H_\ell,\ i, j, \ell = 1, \ldots, m,\ i \neq j \neq \ell$.

Repeating the above arguments $(k-3)$ times more, we deduce that the distribution $\hat{f}(\xi)$ concentrates on $\bigcap_{i=1}^{m} H_i$ and then, by the same way, we get $\hat{f}(\xi) \equiv 0$, which contradicts $f(x) \not\equiv 0$. The proof is complete. □

The following lemma, which will be used in the sequel, is the analogy for the space $N_\Phi(\mathbb{R}^n)$ of Theorem 9.3 [5], and has a similar proof.

Lemma 4. *Let $f \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$ and $g \in M_\Phi(\mathbb{R}^n)$. If $f * g = 0$ then*

$$\mathrm{sp}(g) \subset Z(f) := \{t \in \mathbb{R}^n\ :\ \hat{f}(t) = 0\ \}.$$

Theorem 2. *Let Y be a translation invariant subspace of $L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$. If for each $\xi \in Z(Y) := \bigcap_{f \in Y} \{t \in \mathbb{R}^n :\ \hat{f}(t) = 0\}$ there is a neighbourhood V of ξ such that $V \cap Z(Y)$ is contained in a finite number of hyperplanes, then Y is dense in $N_\Phi(\mathbb{R}^n)$.*

Proof. Assume on the contrary that Y is not dense in $N_\Phi(\mathbb{R}^n)$. Then, since $(N_\Phi(\mathbb{R}^n))^* = M_\Phi(\mathbb{R}^n)$ (Theorem 4.3 [6]) and the Hahn-Banach theorem, there exists a non-zero function $g \in M_\Phi(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)g(-x)dx = 0 \text{ for all } f \in \overline{Y}.$$

Since Y is a translation invariant subspace, we have

$$\int_{\mathbb{R}^n} f(y-x)g(x)dx = 0 \text{ for all } f \in Y.$$

In other words, $f * g = 0$. By Lemma 4, we obtain

$$\mathrm{sp}(g) \subset \{t \in \mathbb{R}^n : \hat{f}(t) = 0\} \text{ for all } f \in Y.$$

Hence, $\mathrm{sp}(g) \subset Z(Y)$.

The hypothesis that for each $\xi \in \mathrm{sp}(g)$ there is a neighbourhood V of ξ such that $\mathrm{sp}(g) \cap V$ is contained in a finite number of hyperplanes and Lemma 3 imply that $g = 0$, which contradicts $g \neq 0$. The proof is complete. □

Corollary 1. *Let Y be a translation invariant subspace of $L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$. If $Z(Y)$ is contained in a finite number of hyperplanes, then Y is dense in $N_\Phi(\mathbb{R}^n)$.*

Theorem 3. *Let $\varphi \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$ with $\hat{\varphi}(0) \neq 0$ and let $\frac{1}{h}$ be an integer larger than 1. If $\varphi \in \overline{\mathrm{span} U_h}$, then $\overline{\mathrm{span} U_h} = N_\Phi(\mathbb{R}^n)$.*

Proof. For any $g \in (N_\Phi(\mathbb{R}^n))^* = M_\Phi(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} f(x)g(x)dx = 0$$

for all $f \in \overline{\mathrm{span} U_h}$, we will prove that $g = 0$. By virtue of Lemma 1, we get

$$\int_{\mathbb{R}^n} \sigma_h^j \varphi(y-x)g(x)dx = 0, \quad \forall j \geq 1, \quad \forall y \in \mathbb{R}^n.$$

Note that the Fourier transform of $\sigma_h^j\varphi(x)$ is $h^{jn}\hat{\varphi}(h^j t)$. It follows from Lemma 4 that

$$\operatorname{sp}(g) \subset Z^*(\varphi) := \bigcap_{j=1}^{\infty}\{t \in \mathbb{R}^n : \hat{\varphi}(h^j t) = 0\}.$$

It follows from $\varphi \in L_1(\mathbb{R}^n)$ and $\hat{\varphi}(0) \neq 0$ that for each $t \in \mathbb{R}^n$, $\hat{\varphi}(h^j t) \neq 0$ when j is sufficiently large. Hence, $\operatorname{sp}(g) = \emptyset$, i.e., $g = 0$. By the Hahn-Banach theorem, we have $\overline{\operatorname{span} U_h} = N_\Phi(\mathbb{R}^n)$. The proof is complete. □

REMARK 1. If $\Phi(t) = t$, i.e., $N_\Phi(\mathbb{R}^n) = L_1(\mathbb{R}^n)$, then it was shown in [9] that the condition $\hat{\varphi}(0) \neq 0$ is necessary for the density of the $\operatorname{span} U_h$ in $L_1(\mathbb{R}^n)$.

THEOREM 4. *Let $\varphi \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$ and let $\frac{1}{h}$ be an integer > 1. Suppose $\varphi \in \overline{\operatorname{span} U_h}$. If for each $\xi \in Z^*(\varphi)$ there is a neighbourhood V of ξ such that $Z^*(\varphi) \cap V$ is contained in a finite number of hyperplanes, then* $\operatorname{span} U_h$ *is dense in $N_\Phi(\mathbb{R}^n)$.*

PROOF. Assume on the contrary; then there exists a non-zero function $g \in M_\Phi(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)g(x)dx = 0 \text{ for all } f \in \overline{\operatorname{span} U_h}.$$

By virtue of Lemma 1, $\overline{\operatorname{span} U_h}$ is translation invariant, and hence,

$$\int_{\mathbb{R}^n} \sigma_h^j\varphi(y-x)g(x)dx = 0, \forall j \geq 1,\ \forall y \in \mathbb{R}^n.$$

Therefore, since Lemma 4, $\operatorname{sp}(g) \subset Z^*(\varphi)$. By the hypothesis and Lemma 3, we get $g = 0$, a contradiction. The proof is complete. □

REMARK 2. In the above theorems, if we replace the condition $\varphi \in \overline{\operatorname{span} U_h}$ by $\lim\limits_{h\to 0, h\in G} \operatorname{dist}(\varphi, \sigma_h S_0(\varphi))_{N_\Phi} = 0$, then U_h is dense in $N_\Phi(\mathbb{R}^n)$.

COROLLARY 2. *Let $\varphi \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$, $\frac{1}{h}$ be an integer > 1 and $\varphi \in \overline{\operatorname{span} U_h}$. If $Z^*(\varphi)$ is contained in a finite number of hyperplanes then* $\operatorname{span} U_h$ *is dense in $N_\Phi(\mathbb{R}^n)$.*

By an argument analogous to that used for the proof of Proposition 6.1 [9], we obtain the following results:

Corollary 3. *Let* $\varphi \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n), \hat{\varphi}(0) \neq 0$ *and* $\frac{1}{h}$ *be an integer larger than 1. If* $\varphi \in \overline{\sigma_h S_0(\varphi)}$*, then*

$$\lim_{j\to\infty} \operatorname{dist}(f, \sigma_h^j S_0(\varphi))_{N_\Phi} = 0, \qquad \forall f \in N_\Phi(\mathbb{R}^n).$$

Corollary 4. *Let* $\varphi \in L_1(\mathbb{R}^n) \cap N_\Phi(\mathbb{R}^n)$ *and* $\frac{1}{h}$ *be an integer larger than 1. If* $Z^*(\varphi)$ *is contained in a finite number of hyperplanes and* $\varphi \in \overline{\sigma_h S_0(\varphi)}$*, then*

$$\lim_{j\to\infty} \operatorname{dist}(f, \sigma_h^j S_0(\varphi))_{N_\Phi} = 0, \qquad \forall f \in N_\Phi(\mathbb{R}^n).$$

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Ha Huy Bang
Institute of Mathematics
P.O. Box 631, Bo Ho
Hanoi, Vietnam
E-mail: hhbang@thevinh.ncst.ac.vn

Truong Van Thuong
Department of Mathematics
Hue University
32 Le Loi, Hue, Vietnam