Spherical Functions in a Certain Distinguished Model

By Keiji Takano

Abstract. We study unramified principal series representations of general linear groups over $p$-adic fields, distinguished with respect to the fixator of the Galois involution. We give a certain condition for the unramified principal series to be distinguished, and give a formula for spherical vectors in the distinguished models of $GL_n$, following the method of Kato and Hironaka. We give explicit results for $GL_2$ and $GL_3$.

0. Introduction

Let $E/F$ be a quadratic extension of $p$-adic fields, $G = GL_n(E)$ and $H = GL_n(F)$. Regard $H$ as the fixator of the Galois involution of $E/F$ acting on $G$. It is known ([F1]) that the symmetric variety $H \setminus G$ is multiplicity free; for any irreducible smooth representation $\pi$ of $G$, the dimension of the space of all $G$-morphisms of $\pi$ into the space $C^\infty(H \setminus G)$ of locally constant functions on $H \setminus G$ is at most one. If the above space of morphisms is non-zero, then $\pi$ is said to be $H$-distinguished, and we call the realization of $\pi$ in $C^\infty(H \setminus G)$ the $H$-distinguished model of $\pi$.

Distinguished representations are of particular importance for the connection with functoriality principle ([F1, 2]), and for the appearance in a certain Rankin-Selberg method ([R]; the quadratic extension version of the doubling method). Motivated by the latter, we study the unramified $H$-distinguished models and the spherical vectors therein. In the $GL_2$-case, an explicit formula for such spherical functions was already given by W. Banks ([B]). Also, several examples of spherical functions on $p$-adic symmetric varieties were studied by Y. Hironaka, S. Kato and F. Sato ([H1-4], [HS], [K]).

Now we summarize the contents of the present paper. Throughout this paper we assume that $E/F$ is unramified. Let $P$ be the Borel subgroup consisting of upper triangular matrices, $T$ be the maximal torus consisting

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of diagonal matrices, \( W \) be the Weyl group of \((G, T)\), identified with the subgroup consisting of permutation matrices, and \( w_0 \in W \) be the longest element, that is, the anti-diagonal monomial matrix. We use the Galois involution \( \theta \) on \( G \) twisted by \( w_0 \) (see the Notation section). Let \( H \) be the fixator of \( \theta \) in \( G \). It is isomorphic to \( GL_n(F) \). Let \( \mathcal{O}_E \) be the valuation ring of \( E \), \( \wp \) be a prime element of \( \mathcal{O}_E \), \( K = GL_n(\mathcal{O}_E) \) and \( B \) be the Iwahori subgroup consisting of elements of \( K \) whose entries below the diagonals belong to \( \wp \mathcal{O}_E \). In \( \S 1 \), we give a parametrization of \( H \setminus G/K \); let \( m \) be the integer part of \( n/2 \), and set

\[
\Lambda_m = \{ \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m ; \lambda_1 \leq \cdots \leq \lambda_m \leq 0 \}.
\]

For \( \lambda \in \Lambda_m \), put

\[
\wp_\lambda^0 = \text{diag}(\wp^{\lambda_1}, \cdots, \wp^{\lambda_m}, 1, \cdots, 1).
\]

We show that \( \{ \wp_\lambda^0 ; \lambda \in \Lambda_m \} \) gives a complete set of representatives of \( H \setminus G/K \) (see (1.4)).

In \( \S 2 \), we study \( H \)-distinguished models for unramified principal series. Let \( X_{\text{ur}}(T)_\theta \) be the set of all unramified characters on \( T \) which is trivial on \( T \cap H \). We show that if \( \chi \) is regular and the unramified principal series \( I(\chi) \) has an \( H \)-distinguished model, then \( \chi \in X_{\text{ur}}(T)_\theta \) for some \( w \in W \). At the same time we prove the uniqueness of the model, and determine the support of the \( H \)-invariant functionals (see (2.2)).

At the end of \( \S 2 \), we construct a non-zero \( H \)-invariant linear functional \( L_\chi \) on \( I(\chi) \) for \( \chi \in X_{\text{ur}}(T)_\theta \), using complex powers of relative \( P \)-invariants, as was proposed in [K]. We define the spherical function \( Q_\chi \) on \( H \setminus G/K \) by \( Q_\chi(g) = \langle L_\chi, \pi_\chi(g) \phi_{K, \chi} \rangle \). Here \( \phi_{K, \chi} \) is the unique \( K \)-fixed vector in \( I(\chi) \) such that \( \phi_{K, \chi}|_K \equiv 1 \). In \( \S 3 \), following the method of [H4] we prove a formula for \( Q_\chi \), which is the main result of this paper ((3.4));

**Theorem.** Let \( \chi \in X_{\text{ur}}(T)_\theta \) be regular and assume that \( c_w(\chi^{-1})c_{w^{-1}}(w \chi) \neq 0 \) for all \( w \in W \). Then, for \( \lambda \in \Lambda_m \),

\[
Q_\chi(\wp_\lambda^0) = \text{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}(w \chi)b_w(\chi)w \chi^{1/2}(w \chi)}{c_w(\chi^{-1})} \chi^{\delta_P/2}(\wp_\lambda^0).
\]
Here, \( c_w(\chi) \) is the usual \( c \)-function given by (14) of \( \S 3 \), \( b_w(\chi) \) is the factor determined by the functional equation of invariant functionals \( (3.3) \), \( \delta_P \) is the modulus of \( P \) and \( W_\theta = W \cap H \).

Note that the sum is taken over the little Weyl group \( W_\theta \), not over the full Weyl group \( W \) as in [H4]. The vanishing of terms associated to \( w/ \in W_\theta \) follows from the determination of the support of invariant functionals in \( \S 2 \).

If \( Q\chi(1) \neq 0 \), put \( \tilde{Q}\chi = Q\chi(1)^{-1} \cdot Q\chi \). Then the above formula can be rewritten as

\[
\tilde{Q}\chi(\varpi^\lambda_0) = \text{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}(w\chi)}{Q^{w\chi}(1)} \cdot w\chi\delta_P^{1/2}(\varpi^\lambda_0),
\]

provided that \( Q^{w\chi}(1) \neq 0 \) for all \( w \in W_\theta \) (see \( (3.5) \)). In \( \S 4 \), we compute the value \( Q\chi(1) \) directly for \( n = 2 \) and \( n = 3 \) and give the explicit formulae of \( \tilde{Q}\chi \) in these cases \( (4.1) \) and \( (4.3) \).

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Notation

Let \( E/F \) be a quadratic unramified extension of \( p \)-adic fields, with the absolute values \( | \cdot |_E, | \cdot |_F \) respectively. Let \( O_E \) be the valuation ring, \( k_E \) the residue class field, \( q_E \) the residue order, of \( E \). Similarly \( O_F, k_F \) and \( q_F \) are defined for \( F \). We may fix a prime element \( \varpi \) of \( E \) which is also a prime element of \( F \). We shall drop the subscript \( E \) and write \( O = O_E, q = q_E \), etc, when there is no fear of confusion. For \( x \in E \), the conjugate of \( x \) over \( F \) is denoted by \( \bar{x} \).

Let \( G \) be the group \( GL_n(E) \) and \( P, T, K, B \) be the subgroups of \( G \) given in the Introduction. Let \( P^- \) be the Borel subgroup opposite to \( P \) and \( N, N^- \) be the unipotent radical of \( P, P^- \) respectively. Put \( N_0 = N \cap B, T_0 = T \cap B, N_1^{-} = N^- \cap B \) and \( N_1 = w_0N_1^{-}w_0^{-1} \) where \( w_0 \) is the anti-diagonal monomial matrix in \( G \). The Iwahori factorization asserts that \( B = N_0T_0N_1^{-} \), uniquely decomposed in this order.
The Weyl group $W$ of $(G,T)$, isomorphic to the symmetric group of $n$-letters, is identified with the subgroup of $G$ consisting of permutation matrices. $W$ acts on quasi-characters $\chi$ of $T$ by $w\chi(t) = \chi(w^{-1}tw)$ for $w \in W, t \in T$. We say that $\chi$ is regular if $w\chi = \chi$ implies $w = 1$.

For $g = (g_{ij}) \in G$, we write $\tilde{g}$ for the matrix $(\tilde{g}_{ij})$. Define the involution $\theta$ on $G$ by

$$\theta(g) = w_0\tilde{g}w_0^{-1}$$

for $g \in G$.

Then $\theta$ leaves $K$ and $T$ stable, and $\theta(N) = N^-$. Let $H$ be the fixator of $\theta$ in $G$;

$$H = \{ h \in G ; \theta(h) = h \}.$$ 

Note that $H$ is isomorphic to $GL_n(F)$. We write $W_\theta$ for $W \cap H = \{ w \in W ; \theta(w) = w \}$, which is the centralizer of $w_0$ in $W$.

Any closed subgroups of $G$ and any homogeneous spaces of them are all regarded as totally disconnected Hausdorff spaces. For such a space $Y$, topological notions are used with respect to this Hausdorff topology unless otherwise stated. For a subset $Z$ of $Y$, the closure of $Z$ in $Y$ is denoted by $Z^c$. Let us write $C^\infty_c(Y)$ for the space of all locally constant $C$-valued functions on $Y$, $C^\infty_c(Y)$ for the subspace of those with compact support. Linear functionals on $C^\infty_c(Y)$ are called distributions on $Y$.

Fix a Haar measure $dg$ of $G$, normalized so that $\int_K dg = 1$. Also fix a left Haar measure $d\ell p$ of $P$ so that $\int_{P \cap K} d\ell p = 1$. Let $\delta_P$ be the modulus character of $P$. It is given explicitly by

$$\delta_P(\text{diag}(t_1,\ldots,t_n)) = \prod_{1 \leq i < j \leq n} |t_i t_j^{-1}| = \prod_{1 \leq i \leq n} |t_i|^{n-2i+1}$$

on $T$. On the space of all locally constant functions $f : G \to \mathbb{C}$ such that $f(pg) = \delta_P(p)f(g)$ ($p \in P, g \in G$) and that $P \backslash \text{supp}(f)$ is compact, there is a unique (up to constant) non-zero right $G$-invariant linear functional, which is denoted by $f \mapsto \hat{f}_P \backslash G f(\hat{g})d\hat{g}$. We may normalize this so that $\hat{f}_P \backslash G f(\hat{g})d\hat{g} = \int_K f(k)dk$.

For a representation $(\pi,V)$ of $G$, we denote by $(\pi^*,V^*)$ the dual representation and by $(\hat{\pi},\hat{V})$ the smooth contragredient, that is, the smooth part of $(\pi^*,V^*)$. For a subgroup $U$ of $G$, the subspace of all $\pi(U)$-fixed vectors in $V$ is denoted by $V^U$. 
§1. Double Coset Decompositions

In this section first we recall the description of the double cosets $P \backslash G / H$ and prepare some properties of them for our later use. Then we give a parametrization of the double cosets $H \backslash G / K$ on which our spherical functions are defined.

Put $X = \{ x \in G; \theta(x)x = 1 \}$. $G$ acts on $X$ (from the right) by the $\theta$-twisted conjugation

$$(x, g) \mapsto x * g := \theta(g)^{-1}xg \text{ for } x \in X, g \in G.$$ 

Put $\tau(g) = 1 * g = \theta(g)^{-1}g$. By the Hilbert Theorem 90, $\tau$ induces a $G$-equivariant homeomorphism $H \backslash G \sim X$ (see e.g. [F1]). Similarly, the mapping $g \mapsto \tau(g^{-1})$ induces $G / H \sim X$. For each $v \in W \cap X$, we may fix an element $\eta_v \in G$ such that

$$\theta(\eta_v)\eta_v^{-1} = \tau(\eta_v^{-1}) = v$$ 

by the surjectivity of $\tau$. In particular we take $\eta_1 = 1$.

For $x \in G$ and $1 \leq i \leq n$, let $d_i(x)$ be the determinant of the upper left $i$ by $i$ block of $x$. Then for $p \in P$, $p' \in P^-$ and $x \in G$ we have

$$d_i(p'xp) = d_i(p')d_i(p)d_i(x).$$ 

So $d_i|_X$ gives a relative $P$-invariant polynomial function on $X$;

$$(2) \quad d_i(x * p) = d_i(\theta(p)^{-1}p))d_i(x) \text{ for all } p \in P, x \in X.$$ 

Note that $p \mapsto d_i(\theta(p)^{-1}p)$ is an $E$-rational character of $P$. Put $m = \lceil n/2 \rceil$, the integer part of $n/2$, and set

$$X^0 = \{ x \in X; d_i(x) \neq 0 \text{ for all } 1 \leq i \leq m \}.$$ 

Clearly $X^0$ is an open dense, $P$-stable subset of $X$ under the $*$-action.

(1.1) Lemma. $G$ decomposes into the disjoint union of the double cosets $P_\eta_vH$, where $v$ runs over $W \cap X$, and $P \cdot H$ is the unique open dense $(P, H)$-double coset in $G$. 

PROOF. The Bruhat decomposition for $G$ implies that

$$X = \bigcup_{w \in W} (P^{-w}P \cap X), \quad \text{and} \quad P^{-w}P \cap X \neq \emptyset \quad \text{if and only if} \quad w \in W \cap X.$$  

As in [F2, p.421], one has $P^{-v}P \cap X = v \cdot P$ for every $v \in W \cap X$. Pulling back through $\tau((\cdot)^{-1})$ we have the first assertion. To prove the second assertion, it is enough to see that $1 \cdot P = X^0$.

For $v \in W \cap X$ we define an involution $\theta_v$ on $G$ by

$$\theta_v(g) = v^{-1}\theta(g)v \quad \text{for} \quad g \in G.$$  

Put $P_v = P \cap v^{-1}P^v$. Then $\theta_v$ leaves $P_v$ stable. Let $R_v$ be the fixator of $\theta_v$ in $P_v$;

$$R_v = \{ r \in P ; v^{-1}\theta(r)v = r \}.$$  

(3)

$R_v$ is identified with the the stabilizer in $P \times H$ of the representative $\eta_v$ as follows;

$$\{ (p, h) \in P \times H ; p\eta_vh^{-1} = \eta_v \} = \{ (p, h) ; p \in P, \eta_v^{-1}p\eta_v = h \in H \} = \{ (p, \eta_v^{-1}p\eta_v) ; \theta_v(p) = p \in P \} = \{ (r, \eta_v^{-1}r\eta_v) ; r \in R_v \}.$$  

Regarding $R_v$ as a subgroup of $P \times H$ as above, the double coset $P\eta_vH$ is homeomorphic to $(P \times H)/R_v$. We have the following semi-direct product decomposition;

(4) 

$$R_v = (T \cap R_v) \ltimes (N \cap R_v).$$

Now, for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$, put

$$\varpi^\mu = \text{diag}(\varpi^{\mu_1}, \ldots, \varpi^{\mu_n}) \quad \text{and} \quad \varpi^{-\mu} = (\varpi^\mu)^{-1}.$$
Also, for $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$, put

$$\varpi_0^\lambda = \text{diag}(\varpi_{\lambda_1}, \cdots, \varpi_{\lambda_m}, 1, \cdots, 1), \quad \text{and} \quad \varpi_0^{-\lambda} = (\varpi_0^\lambda)^{-1}.$$ 

The following lemma is easily shown by a direct matrix calculation and the ultrametric inequality.

(1.2) Lemma. For $n \in \mathbb{N}_1$, $n' \in \mathbb{N}_1^-$ and $\mu \in \mathbb{Z}^n$ with $\mu_1 \leq \cdots \leq \mu_n$, one has

$$|d_i(n \varpi_\mu n')| = |d_i(\varpi_\mu)|.$$

For $\mu \in \mathbb{Z}^n$, $\varpi_\mu$ belongs to $X$ if and only if $\mu_{n-i+1} = -\mu_i$ for all $i$. If moreover $\mu_1 \leq \cdots \leq \mu_n$ is assumed, we must have $\mu_1 \leq \cdots \leq \mu_m \leq 0$. Now set

$$\Lambda_m = \{ \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m ; \lambda_1 \leq \cdots \leq \lambda_m \leq 0 \}.$$

Then we have

$$\{ \varpi_\mu \in X ; \mu \in \mathbb{Z}^n, \mu_1 \leq \cdots \leq \mu_n \} = \{ \tau(\varpi_0^\lambda) ; \lambda \in \Lambda_m \}.$$ 

The following corollary, which is similar to [H4, (2.2)], is important for our later use;

(1.3) Corollary. For $b \in B$ and $\lambda \in \Lambda_m$, one has

$$|d_i(\tau(\varpi_0^\lambda) \ast b)| = |d_i(\tau(\varpi_0^\lambda))| \neq 0.$$

Consequently, $B \varpi_0^{-\lambda} \subset P \cdot H$ holds for all $\lambda \in \Lambda_m$.

Proof. This follows directly from (1.2), using the Iwahori factorization for $B$. Note that we pull back through $\tau(\cdot)^{-1}$ to get $B \varpi_0^{-\lambda} \subset P \cdot H$. $\square$

In the rest of this section we give a parametrization of $H \backslash G/K$.

(1.4) Proposition. $G$ decomposes into the disjoint union of the double cosets $H \varpi_0^\lambda K$, where $\lambda$ runs over $\Lambda_m$. 

Proof. The assertion is equivalent to the decomposition
\[ X = \bigcup_{\lambda \in \Lambda_m} \tau(\varpi^\lambda_0) * K \] (disjoint union)
of \( X \) into \( K \)-orbits. First, by the Cartan decomposition for \( G \), one has
\[ X = \bigcup_{\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \atop \mu_1 \leq \cdots \leq \mu_n} (K \varpi^\mu K \cap X) \] (disjoint union)
and it is easy to see that \( K \varpi^\mu K \cap X \neq \emptyset \) for \( \mu_1 \leq \cdots \leq \mu_n \) if and only if \( \varpi^\mu \in X \). In this case we may replace \( \varpi^\mu \) by \( \tau(\varpi^\lambda_0) \), where \( \lambda \in \Lambda_m \) as above. We show that
\[ K \tau(\varpi^\lambda_0) K \cap X = \tau(\varpi^\lambda_0) * K \quad \text{for all } \lambda \in \Lambda_m. \]
Since \( k_1 \tau(\varpi^\lambda_0) k_2 = (\tau(\varpi^\lambda_0) k_2 \theta(k_1)) * \theta(k_1^{-1}) \), it is enough to show that, for \( \lambda \in \Lambda_m \),
\begin{equation}
\text{(*)} \quad \text{For any } k \in K \text{ with } \tau(\varpi^\lambda_0) k \in X, \text{ there is a } k' \in K \text{ such that } \tau(\varpi^\lambda_0) k = \tau(\varpi^\lambda_0) * k. \end{equation}
Put
\[ C_\lambda = \varpi^\lambda_0 K \varpi^{-\lambda} \cap \theta(\varpi^\lambda_0 K \varpi^{-\lambda}). \]
Then, one can show that \( C_\lambda \) is a \( \theta \)-stable subgroup contained in \( K \). In fact, if we understand that \( \lambda_{m+1} = \cdots = \lambda_n = 0 \), then \( C_\lambda \) is given by
\[ C_\lambda = \left\{ (c_{ij}) \in G \; ; \; \det(c_{ij}) \in \mathcal{O}^X, \; |c_{ij}| \leq \min(q^{-\lambda_i + \lambda_j}, q^{-\lambda_{n-i+1} + \lambda_{n-j+1}}) \right\} = \left\{ (c_{ij}) \in K \; ; \; |c_{ij}| \leq \min(q^{-\lambda_i + \lambda_j}, q^{-\lambda_{n-i+1} + \lambda_{n-j+1}}) \right\}. \]
Now \( (*) \) follows from the assertion
\begin{equation}
\text{(**)} \quad H^1(\{1, \theta\}, C_\lambda) = \{1\}. \end{equation}
Indeed, if \( \tau(\varpi^\lambda_0) k \in X, k \in K, \) then \( \varpi^\lambda_0 k \varpi^{-\lambda} = \theta(\varpi^\lambda_0 k \varpi^{-\lambda})^{-1} \in C_\lambda. \) So \( (** \) implies that there is a \( c \in C_\lambda \) such that \( \varpi^\lambda_0 k \varpi^{-\lambda} = \theta(c)^{-1} c \), which leads to the equation \( \tau(\varpi^\lambda_0) k = \tau(\varpi^\lambda_0) * k', \) with \( k' = \varpi^{-\lambda} c \varpi^\lambda_0 \in K. \)
To prove (**), first let $\rho_0 : K \to GL_n(k_E)$ be the mod-$\varpi$ map and $\tilde{\theta}$ be the involution on $GL_n(k_E)$ defined by $\tilde{\theta}(g) = w_0 g w_0^{-1}$, where bar denotes the Galois involution of $k_E/k_F$. Regard $\tilde{\theta}$ as a $k_F$-involution on $GL_n(k_E)$. It is clear that $\rho_0 \circ \theta = \tilde{\theta} \circ \rho_0$.

Put $M_\lambda = \rho_0(C_\lambda)$. We observe that $M_\lambda$ is the group of $k_F$-rational points of a Zariski-connected group over $k_F$. Let $l$ be the largest number such that $\lambda_i \neq 0$ and assume that $\lambda$ is of the form

$$\lambda_1 = \cdots = \lambda_{i_1} < \lambda_{i_1+1} = \cdots = \lambda_{i_1+i_2} < \cdots < \lambda_{i_1+\cdots+i_k-1+1} = \cdots = \lambda_l < 0.$$ 

Then, $M_\lambda$ consists of all matrices of the form

$$\begin{pmatrix} p_1 & x & 0 \\ g & \cdot & \cdot \\ 0 & y & p_2 \end{pmatrix}$$

where $p_1 \in GL_1(k_E)$ is upper quasi-triangular of type $(i_1, \cdots, i_k)$, $p_2 \in GL_l(k_E)$ is lower quasi-triangular of type $(i_k, \cdots, i_1)$, $g \in GL_{n-2l}(k_E)$ and $x, y \in \text{Mat}_{l,n-2l}(k_E)$. So $M_\lambda$ is a semi-direct product of rational points of Zariski-connected groups

$$\left\{ \begin{pmatrix} p_1 & 0 \\ g & p_2 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 & x & 0 \\ \cdot & \cdot & \cdot \\ 0 & y & 1 \end{pmatrix} \right\}.$$ 

Put $C_\lambda^{(1)} = \ker(\rho_0) \cap C_\lambda$. Then, one has an exact sequence of 1-cohomology sets;

$$H^1(\{1, \theta\}, C_\lambda^{(1)}) \longrightarrow H^1(\{1, \theta\}, C_\lambda) \longrightarrow H^1(\{1, \tilde{\theta}\}, M_\lambda).$$

The last set is trivial by Lang’s theorem. So (**), following from

(***)

$$H^1(\{1, \theta\}, C_\lambda^{(1)}) = \{1\}.$$ 

To prove (***) put $K(N) = 1 + \varpi^N \text{Mat}_n(\mathcal{O})$ for each integer $N \geq 1$ and define $\rho_N : K(N) \to \text{Mat}_n(k_E)$ by

$$\rho_N(1 + \varpi^N a) = a \mod \varpi \quad \text{for } a \in \text{Mat}_n(\mathcal{O}).$$

Then $\rho_N$ is a homomorphism onto the additive group of $\text{Mat}_n(k_E)$ and

$$K(N+1) = \ker(\rho_N), \quad K(1) = \ker(\rho_0).$$

Note that $K(N)$ is normal in $K$ and is $\theta$-stable. Define the involution $\tilde{\theta}$ on the additive group of $\text{Mat}_n(k_E)$ by the same as before. Then the relation $\rho_N \circ \theta = \tilde{\theta} \circ \rho_N$ holds. Put
\( C^{(N)}_\lambda = C_\lambda \cap K(N) \) and \( A^{(N)}_\lambda = \rho_N(C^{(N)}_\lambda) \). Then \( A^{(N)}_\lambda \) is an additive subgroup of \( \text{Mat}_n(k_E) \) over \( k_F \). As before one has an exact sequence
\[
H^1(\{1, \theta\}, C^{(N+1)}_\lambda) \longrightarrow H^1(\{1, \theta\}, C^{(N)}_\lambda) \longrightarrow H^1(\{1, \tilde{\theta}\}, A^{(N)}_\lambda).
\]
By the additive version of the Hilbert Theorem 90, the last term vanishes. So the vanishing of \( H^1(\{1, \theta\}, C^{(N)}_\lambda) \) follows from that of \( H^1(\{1, \theta\}, C^{(N+1)}_\lambda) \). Inductively, \( (***) \) follows from the vanishing of \( H^1(\{1, \theta\}, C^{(N)}_\lambda) \) for some large \( N \).

Now, if \( N \geq -\lambda_1 \), one has \( C_\lambda \supset K(N) \), thus \( C^{(N)}_\lambda = K(N) \). By the argument exactly as in [PR, pp. 292–294], one can show that
\[
H^1(\{1, \theta\}, K(N)) = \{1\}
\]
for any integer \( N \), hence the proof is completed. \( \square \)

\section{Invariant Functionals on Unramified Principal Series}

Let \( X_{\text{ur}}(T) \) be the set of all unramified quasi-characters of \( T \). We regard \( \chi \in X_{\text{ur}}(T) \) also as a quasi-character of \( P \) by letting \( \chi|_N \equiv 1 \). For \( \chi \in X_{\text{ur}}(T) \) let \((\pi_\chi, I(\chi))\) be the unramified principal series attached to \( \chi \). Thus \( I(\chi) \) is the space of all locally constant \( C \)-valued functions \( \varphi \) on \( G \) satisfying
\[
\varphi(pg) = \chi(p)\delta_P(p)^{1/2}\varphi(g) \quad \text{for} \quad p \in P, \ g \in G,
\]
and \( \pi_\chi \) is the right translation of \( G \) on \( I(\chi) \).

Let \( \lambda, \rho \) be respectively the left, right translations of \( G \) on \( C^{\infty}_c(G) \). For \( \chi \in X_{\text{ur}}(T) \), let \( D_\chi(G) \) be the space of all distributions \( D \) on \( G \) satisfying
\[
\langle D, \lambda(p)f \rangle = \chi(p)^{-1}\delta_P(p)^{1/2}\langle D, f \rangle \quad \text{for all} \quad p \in P, \ f \in C^{\infty}_c(G),
\]
and \( D_\chi(G)^H \) be the space of all \( D \in D_\chi(G) \) satisfying
\[
\langle D, \rho(h)f \rangle = \langle D, f \rangle \quad \text{for all} \quad h \in H, \ f \in C^{\infty}_c(G).
\]
Define \( p_\chi : C^{\infty}_c(G) \to I(\chi) \) as usual by
\[
(p_\chi(f))(g) = \int_P \chi^{-1}(p)\delta_P(p)^{1/2} f(pg) d\ell_p
\]
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for $f \in C_c^\infty(G)$, $g \in G$. As is shown in [H4, (1.2)], the dual map $p^*_\chi$ of $p_\chi$ gives rise to a right $G$-isomorphism $I(\chi)^* \sim \mathcal{D}_\chi(G)$. Therefore we have

\[(7) \quad \text{Hom}_H(I(\chi), \mathbb{C}) = (I(\chi)^*)^H \xrightarrow{p^*_\chi} \mathcal{D}_\chi(G)^H.\]

By this we may regard $H$-invariant linear functionals as $P \times H$-relatively invariant distributions on $G$. From now on we study the space $\mathcal{D}_\chi(G)^H$ by the standard method, so called the Bruhat theory for $P\backslash G/H$.

For any locally closed subset $\Omega$ of $G$ satisfying $P\Omega H \subset \Omega$, define $\mathcal{D}_\chi(\Omega)^H$ to be the space of all distributions on $\Omega$ having the same $P \times H$-equivariance as those in $\mathcal{D}_\chi(G)^H$ (i.e., the relations (5) and (6)). Put $\Omega_v = P\eta_v H$ for $v \in W \cap X$, where $\eta_v$ is as in §1. Recall that $G = \bigcup_{v \in W \cap X} \Omega_v$ (disjoint union) and $\Omega_v \simeq (P \times H)/R_v$, where $R_v$ is defined by (3).

\[(2.1) \text{ Lemma.} \quad \text{Let } \chi \in X_{ur}(T) \text{ and } v \in W \cap X.\]

(i) The modulus character $\delta_v$ of $R_v$ is trivial on $N \cap R_v$, and on $T \cap R_v$ it is given by

$$\delta_v(t) = \delta_P(t)^{1/2} \quad \text{for all } t \in T \cap R_v.$$

(ii) One has

$$\dim \mathcal{D}_\chi(\Omega_v)^H = \begin{cases} 1 & \text{if } \chi|_{T \cap R_v} \equiv 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. (i) By the semi-direct product decomposition (4) of $R_v$, $\delta_v$ is directly computed as the Jacobian of the adjoint action of $T \cap R_v$ on $N \cap R_v$. Let $x = (x_{ij}) \in N \cap R_v$. Since $N \cap R_v \subseteq N \cap v^{-1}Nv$, we may only look at the entries $x_{ij}$ with $i < j$, $v(i) > v(j)$. (Here and henceforth we regard elements of $W$ also as permutations of indices.) Moreover since $\theta_v(x) = x$, we must have $x_{n-v(i)+1,n-v(j)+1} = x_{ij}$. In particular if $i = n - v(i) + 1$ and $j = n - v(j) + 1$, then $x_{ij} \in F$. Similarly, for $t = \text{diag}(t_1, \cdots, t_n) \in T \cap R_v$, $\delta_v(t) = $
we have $t_{n-v(i)+1} = \bar{t}_i$ and in particular, $t_i \in F^\times$ if $n - v(i) + 1 = i$. Now, $\delta_v(t)$ is computed as;

$$
\delta_v(t) = \prod |t_i t_j^{-1}|_F \text{ (product over } i = n - v(i) + 1 < j = n - v(j) + 1) \\
\times \left( \prod |t_i t_j^{-1}| \right)^{1/2} \text{ (} i < j, v(i) > v(j) \text{ and,} \\
i \neq n - v(i) + 1 \text{ or } j \neq n - v(j) + 1 \right) \\
= \prod_{i < j, v(i) > v(j)} |t_i t_j^{-1}|^{1/2}
$$

since $| \cdot |_F = | \cdot |^{1/2}$. Comparing with (1), it is enough to see that

$$
\prod_{i < j, v(i) < v(j)} |t_i t_j^{-1}| = 1 \text{ for } t = \text{diag}(t_1, \cdots, t_n) \in T \cap R_v.
$$

As is well-known (e.g. [M, p.289]),

$$
\prod_{i < j, v(i) < v(j)} |t_i t_j^{-1}| = \delta_P(t)^{1/2} \delta_P(vtv^{-1})^{1/2}.
$$

If $t \in T \cap R_v$ then $vtv^{-1} = w_0 \bar{t}_0 w_0^{-1}$, thus

$$
\delta_P(t)^{1/2} \delta_P(vtv^{-1})^{1/2} = \delta_P(t)^{1/2} \delta_P(w_0 \bar{t}_0 w_0^{-1})^{1/2} = \delta_P(t)^{1/2} \delta_P(\bar{t})^{-1/2} = 1.
$$

(ii) Recall the following criterion for the existence of relative invariant distributions on homogeneous spaces ([BZ, (1.21)]); let $G_1$ be a totally disconnected locally compact group, $H_1$ be a closed subgroup of $G_1$, $\omega$ be a quasi-character of $G_1$. Then, there is a non-zero distribution $D$ on $G_1/H_1$ satisfying $\langle D, \lambda(g)f \rangle = \omega(g) \langle D, f \rangle$ for all $g \in G_1$, $f \in C_c^\infty(G_1/H_1)$ if and only if

$$
\omega|_{H_1} = \delta_{G_1|H_1} \cdot \delta_{H_1}^{-1}.
$$

Here, $\delta_{G_1}$, $\delta_{H_1}$ are the modulus characters of $G_1$, $H_1$ respectively. If such a non-zero distribution $D$ exists, it is unique up to constant multiples. Applying this to $G_1 = P \times H$, $H_1 = R_v$ and $\omega = \chi^{-1} \delta_P^{1/2} \times 1$, (ii) is a direct consequence of (i). □
For each \( v \in W \cap X \) put
\[
X_{ur}(T)_{\theta, v} = \{ \chi \in X_{ur}(T) ; \chi|_{T \cap R_v} \equiv 1 \} \quad \text{and} \quad X_{ur}(T)_{\theta} = X_{ur}(T)_{\theta, 1}.
\]

(2.2) **Proposition.**

(i) If \( D_\chi(G)^H \neq (0) \), then \( \chi \in X_{ur}(T)_{\theta, v} \) for some \( v \in W \cap X \).

(ii) If \( \chi \) is regular, then \( D_\chi(G)^H \) is at most one dimensional. If \( \chi \) is regular and \( D_\chi(G)^H \neq (0) \), then \( v \in W \cap X \) in (i) is uniquely determined, and for any non-zero \( D \in D_\chi(G)^H \), the support \( \text{supp}(D) \) of \( D \) is given by the closure \( \Omega_v^\ell \) of \( \Omega_v \).

(iii) If \( \chi \) is regular and \( D_\chi(G)^H \neq (0) \), then there is a \( w \in W \) such that \( w \chi \in X_{ur}(T)_{\theta} \).

**Proof.** (i) follows immediately from (2.1) (ii).

(ii) Assume that \( \chi \) is regular and \( D_\chi(G)^H \neq (0) \). The uniqueness of \( v \in W \cap X \) such that \( \chi \in X_{ur}(T)_{\theta, v} \) follows immediately from the regularity of \( \chi \). Thus, if \( \chi \in X_{ur}(T)_{\theta, v} \), \( \Omega_v \) is the unique \( P \times H \)-orbit such that \( D_\chi(\Omega_v^\ell)^H \neq (0) \), hence one has \( D_\chi(\Omega_v^\ell)^H = (0) \) for any \( P \times H \)-stable subset \( \Omega \) of \( G \) such that \( \Omega \cap \Omega_v = \emptyset \).

Now since \( \Omega_v \) is open in its closure \( \Omega_v^\ell \), we have the following two exact sequences;

\[
0 \longrightarrow D_\chi(\Omega_v^\ell)^H \xrightarrow{\text{ext}} D_\chi(G)^H \xrightarrow{\text{res}} D_\chi(G - \Omega_v^\ell)^H,
\]

\[
0 \longrightarrow D_\chi(\Omega_v^\ell - \Omega_v)^H \xrightarrow{\text{ext}} D_\chi(\Omega_v^\ell)^H \xrightarrow{\text{res}} D_\chi(\Omega_v)^H.
\]

As was noticed above, one has \( D_\chi(G - \Omega_v^\ell)^H = D_\chi(\Omega_v^\ell - \Omega_v)^H = (0) \), thus

\[
D_\chi(G)^H \xleftarrow{\text{ext}} D_\chi(\Omega_v^\ell)^H \xrightarrow{\text{res}} D_\chi(\Omega_v)^H.
\]

Since \( D_\chi(G)^H \) is non-zero and \( D_\chi(\Omega_v)^H \) is one dimensional, all three spaces in (8) are isomorphic and one dimensional. This completes the proof of (ii).

(iii) Let \( \chi \) be regular, \( D_\chi(G)^H \neq (0) \) and \( \chi \in X_{ur}(T)_{\theta, v} \). Put \( v_0 = w_0 v \).

Then \( v_0^2 = 1 \), so \( v_0 \) is a product of disjoint transpositions. Put
\[
\chi_i(x) = \chi(\text{diag}(1, \ldots, 1, x, 1, \ldots, 1)).
\]
If \( i \) is an index such that \( v_0(i) = i \), then \( \chi \in X_{\text{ur}}(T)_{\theta, v} \) implies that \( \chi_i | F^\times \equiv 1 \). Since \( \chi_i \) and \( E/F \) are unramified, one must have \( \chi_i \equiv 1 \). By the regularity of \( \chi \), one can conclude that there is no pair \( \{i, j\} \) of indices such that \( i = v_0(i) \neq j = v_0(j) \), so \( v_0 \) is a product of \( m = [n/2] \)-disjoint transpositions, say,

\[
v_0 = (i_1, j_1) \cdots (i_m, j_m), \quad i_1 < \cdots < i_m \quad \text{and} \quad i_k < j_k \text{ for all } k.
\]

Take \( w \in W \) so that

\[
w(i_k) = k, \quad w(j_k) = n - k + 1 \quad \text{for all } k.
\]

Then one has \( wv_0w^{-1} = w_0 \), that is, \( v = \theta(w)^{-1}w \). This shows that \( w^{-1}(T \cap H)w = T \cap R_v \), hence \( w\chi|_{T \cap H} \equiv 1 \).

**Remark.** Actually we do not need the assumption that \( \chi \) is unramified in (2.2) (i) and (ii). Essentially the same statement as (2.2) (i) is proved in [JLR].

According to the above proposition, to study unramified principal series \( I(\chi) \) with \( (I(\chi)^*)^H \neq (0) \), we may restrict ourselves (at least generically) to the case \( \chi \in X_{\text{ur}}(T)_{\theta} \). In the following we construct a non-zero \( H \)-invariant linear functional \( L_\chi \) on \( I(\chi) \) explicitly for \( \chi \in X_{\text{ur}}(T)_{\theta} \).

Write \( \chi \in X_{\text{ur}}(T) \) as

\[
(9) \quad \chi(\text{diag}(t_1, \cdots, t_n)) = \prod_{i=1}^{n} |t_i|^{s_i}, \quad s_i \in \mathbb{C}.
\]

If \( \chi \in X_{\text{ur}}(T)_{\theta} \), we may assume that \( s_{n-i+1} = -s_i \) for all \( i \) since \( \chi \) is trivial on \( T \cap H \). So \( \chi \in X_{\text{ur}}(T)_{\theta} \) is of the form

\[
(10) \quad \chi(\text{diag}(t_1, \cdots, t_n)) = \prod_{i=1}^{m} |t_it_{n-i+1}^{-1}|^{s_i}
\]

for \( s_1, \cdots, s_m \in \mathbb{C} \). Define a \( \mathbb{C} \)-valued function \( \Delta_\chi \) on \( P \cdot H \) by

\[
(11) \quad \Delta_\chi(g) = \prod_{i=1}^{m} |d_i (\theta(g)g^{-1})|^{s_i} \quad \text{for } g \in P \cdot H,
\]
where $s'_1, \cdots , s'_m \in \mathbb{C}$ are related with $s_1, \cdots , s_m \in \mathbb{C}$ by

$$
\begin{align*}
(12) \quad \begin{cases}
  s'_i = s_i - s_{i+1} - 1 & \text{for } i < m, \\
  s'_m = s_m - \frac{n-2m+1}{2}.
\end{cases}
\end{align*}
$$

If $\text{Re}(s'_i) > 0$ for all $i$, $\Delta_\chi$ can be extended to a continuous function on $G$ (but not locally constant on $G$ in general). For $p \in P$ (with diagonal entries $t_1, \cdots , t_n$), $g \in G$ and $h \in H$ we have

$$
\Delta_\chi(pgh) = \left( \prod_{i=1}^m |t_i|^{s'_i - \sum_{j=1}^m s'_j} |\bar{t}_{n-i+1}|^{\sum_{j=1}^m s'_j} \right) \Delta_\chi(g) \quad \text{by (2)}
$$

$$
= \chi^{-1} \delta_p^{1/2}(p) \Delta_\chi(g) \quad \text{by (10) and (12)}.
$$

For $\chi \in X_{ur}(T)_\theta$ such that $\text{Re}(s'_i) > 0$, define a linear functional $L_\chi$ on $I(\chi)$ by

$$
\langle L_\chi, \varphi \rangle = \oint_{P \setminus G} \Delta_\chi(\dot{g}) \varphi(\dot{g}) d\dot{g} = \int_K \Delta_\chi(k) \varphi(k) dk
$$

for $\varphi \in I(\chi)$. As in [H4, Remark(1.1)], the functional $L_\chi$, which is initially defined for $\text{Re}(s'_i) > 0$, is analytically continued to whole of $(s_1, \cdots , s_m) \in \mathbb{C}^m$. By the right $H$-invariance of $\Delta_\chi$, $L_\chi$ belongs to $(I(\chi)^*)_H$.

(2.3) Proposition. If $\chi \in X_{ur}(T)_\theta$ is regular, then $(I(\chi)^*)_H$ is one dimensional. In fact, $L_\chi$ defined above gives a non-zero $H$-invariant linear functional on $I(\chi)$, which is unique up to constant multiples.

Proof. By (2.2)(ii), it is enough to see that $L_\chi \neq 0$. Taking $\lambda = 0$ in (1.3), we have $\Delta_\chi(b) = 1$ for all $b \in B$. Therefore, $\langle L_\chi, p_\chi(ch_B) \rangle = \text{vol}(B)$, which is non-zero. □
§3. A Formula for Our Spherical Functions

Let \( \chi \in X_{\text{ur}}(T)_\theta \) and assume that \( \chi \) is regular. We have observed in §2 that \( \dim(I(\chi)^*)^H = 1 \). By the Frobenius reciprocity,

\[
(I(\chi)^*)^H \simeq \text{Hom}_G(I(\chi), C^\infty(H\setminus G)),
\]

hence there is a unique realization of \( I(\chi) \) in \( C^\infty(H\setminus G) \). Let \( L_\chi \) be as in (13) of §2, and define

\[
Q_\chi(g) = \langle L_\chi, \pi_\chi(g)\phi_{K,\chi} \rangle
\]

where \( \phi_{K,\chi} \) is the unique element of \( I(\chi) \) such that \( \phi_{K,\chi}(k) = 1 \) for all \( k \in K \). The function \( Q_\chi \) on \( G \) is then the unique (up to constant) right \( K \)-invariant function in the realization of \( I(\chi) \) in \( C^\infty(H\setminus G) \). By the description (1.4) of \( H\setminus G/K \), \( Q_\chi \) is completely determined by their values at \( \varpi_\lambda^{\delta^{1/2}} \in G \), for \( \lambda \in \Lambda_m \). In this section, following [H4] we give an expression of \( Q_\chi \) as a linear combination of quasi-characters \( w_\chi \delta^{1/2}_P \), where \( w \) varies in the little Weyl group \( W_\theta = W \cap H \).

For \( \chi \in X_{\text{ur}}(T) \), identify \( I(\chi)^\sim \) with \( I(\chi^{-1}) \) by the natural pairing

\[
\langle \langle \phi, \psi \rangle \rangle = \iint_{P \setminus G} \phi(\hat{g})\psi(\hat{g})d\hat{g} \quad \left( = \int_K \phi(k)\psi(k)dk \right)
\]

for \( \phi \in I(\chi), \psi \in I(\chi^{-1}) \). Let \( p_B : I(\chi) \to I(\chi)^B \) be defined by

\[
p_B(\phi)(g) = \text{vol}(B)^{-1} \int_B \phi(gb)db
\]

and \( p_B^*(\ell) := \ell \circ p_B \) for \( \ell \in I(\chi)^* \). Then \( p_B \) is the identity on \( I(\chi)^B \) and \( p_B^*(\ell) \) is fixed by \( B \) for all \( \ell \in I(\chi)^* \). Since \( B \) is open and compact, \( p_B^*(\ell) \) is a smooth linear form on \( I(\chi) \). By the above identification \( I(\chi)^\sim = I(\chi^{-1}) \), \( p_B^*(\ell) \) is regarded as an element of \( I(\chi^{-1})^B \) so that \( \langle p_B^*(\ell), \phi \rangle = \langle \ell, p_B(\phi) \rangle = \langle \langle \phi, p_B^*(\ell) \rangle \rangle \) for all \( \phi \in I(\chi) \).

(3.1) Lemma. Let \( \ell \in I(\chi)^* \). As an element of \( I(\chi^{-1})^B \), \( p_B^*(\ell) \) is given by

\[
p_B^*(\ell) = \sum_{w \in W} \text{vol}(BwB)^{-1} \langle \ell, \phi_{w,\chi} \rangle \phi_{w,\chi^{-1}}.
\]
Here, \( \phi_{w, \chi} = p_{\chi}(\text{ch}_{BwB}) \) for \( w \in W \).

**Proof.** \( \{ \phi_{w, \chi}^{-1} \}_{w \in W} \) forms a basis of \( I(\chi^{-1})^B \) (see [C, (2.1)]). Write

\[
p_B^*(\ell) = \sum_{w \in W} a_w \phi_{w, \chi}^{-1} \quad (a_w \in \mathbb{C}).
\]

Then, for \( w \in W \), taking \( \langle \langle \phi_{w, \chi}, \cdot \rangle \rangle \) on both sides,

\[
\langle \langle \phi_{w, \chi}, p_B^*(\ell) \rangle \rangle = a_w \text{vol}(BwB).
\]

On the other hand,

\[
\langle p_B^*(\ell), \phi_{w, \chi} \rangle = \langle \ell, p_B(\phi_{w, \chi}) \rangle = \langle \ell, \phi_{w, \chi} \rangle
\]

by definition. Thus \( a_w = \text{vol}(BwB)^{-1} \langle \ell, \phi_{w, \chi} \rangle \). \( \square \)

Let \( \chi \in X_{\text{ur}}(T) \) be regular and for \( w \in W \), let \( T_w^\chi : I(\chi) \to I(w^w \chi) \) be the standard intertwining operator ([C, §3]) and \( c_w(\chi) \) be defined by the relation \( T_w^\chi(\phi_K, \chi) = c_w(\chi) \phi_K, w^w \chi \). If \( \chi \) is of the form (9), then \( c_w(\chi) \) is given by

\[
c_w(\chi) = \prod_{i<j, w(i) > w(j)} \frac{1 - q^{-s_i + s_j - 1}}{1 - q^{-s_i + s_j}}
\]

(see [C, (3.1), (3.3)]).

(3.2) **Lemma** [H4, Prop.(1.6),(1.7)]. For \( w \in W \) and a regular \( \chi \in X_{\text{ur}}(T) \), assume that \( c_w(\chi^{-1})c_{w^{-1}}(w^w \chi) \neq 0 \). Define an intertwining map \( \widetilde{T}_w^\chi : I(\chi)^* \to I(w^w \chi)^* \) by

\[
\widetilde{T}_w^\chi = \frac{c_w(\chi^{-1})}{c_{w^{-1}}(w^w \chi)} \cdot (T_w^{-1})^*.
\]

Then,

(i) \( \widetilde{T}_w^\chi \) is an extension of \( T_w^{-1} : I(\chi^{-1}) \to I(w^w \chi^{-1}) \), regarding \( I(\chi^{-1}) \subset I(\chi)^* \), \( I(w^w \chi^{-1}) \subset I(w^w \chi)^* \) and,

(ii) \( p_B^* \circ \widetilde{T}_w^\chi = T_w^{-1} \circ p_B^* \).
Now let $\chi \in X_{\text{ur}}(T)_{\theta}$ be regular and $L_\chi \in (I(\chi)^*)^H$ be defined by (13). Observe that for $w \in W$, $w\chi \in X_{\text{ur}}(T)_{\theta,v}$ where $v = \theta(w)w^{-1}$. In particular $w\chi \in X_{\text{ur}}(T)_{\theta}$ if and only if $w \in W_\theta$, so $L_{w\chi} \in (I(w\chi)^*)^H$ is defined for $w \in W_\theta$ as before. For each $w \in W$, we choose $L_{w\chi} \in (I(w\chi)^*)^H$ as follows:

\[
\begin{cases}
  \text{If } w \in W_\theta, & L_{w\chi}^{(w)} = L_{w\chi} \\
  \text{If } w \notin W_\theta \text{ and } (I(w\chi)^*)^H \neq \langle 0 \rangle, & \text{then fix a non-zero } L_{w\chi}^{(w)} \in (I(w\chi)^*)^H \text{ arbitrarily.} \\
  \text{If } w \notin W_\theta \text{ and } (I(w\chi)^*)^H = \langle 0 \rangle, & L_{w\chi}^{(w)} = 0.
\end{cases}
\]

Then, by (2.2)(ii),

\[Q_{\chi}(\varpi_0^\lambda) = \langle \pi_{\chi}^* (\varpi_0^{-\lambda})L_\chi, \phi_{K,\chi} \rangle = \langle \pi_{\chi}^* (\varpi_0^{-\lambda})L_\chi, p_B(\phi_{K,\chi}) \rangle = \langle p_B^* \left( \pi_{\chi}^* (\varpi_0^{-\lambda})L_\chi \right), \phi_{K,\chi} \rangle,
\]

(15)

Here, $c_w(\chi)$ is given by (14), $b_w(\chi)$ is determined by the functional equation of invariant functionals in the above lemma, and $W_\theta = W \cap H$. 

**Proof.** By definition,
Let \( \{ f_{w,\chi^{-1}} \}_{w \in W} \) be the Casselman basis of \( I(\chi^{-1})^B \) (see [C, p.402]) and write
\[
p_B^* \left( \pi_\chi^*(\varpi_0^{-\lambda})L_\chi \right) = \sum_{w \in W} \alpha_w \cdot f_{w,\chi^{-1}},
\]
regarding \( p_B^* \left( \pi_\chi^*(\varpi_0^{-\lambda})L_\chi \right) \) as an element of \( I(\chi^{-1})^B \). Applying \( T_\chi^w (\cdot) (1) \) on both sides we have
\[
\alpha_w = T_\chi^w \left( p_B^* \left( \pi_\chi^*(\varpi_0^{-\lambda})L_\chi \right) \right) (1)
\]
\[
= p_B^* \left( \tilde{T}_\chi^w \left( \pi_\chi^*(\varpi_0^{-\lambda})L_\chi \right) \right) (1) \quad \text{by (3.2)(ii)}
\]
\[
= p_B^* \left( \pi_{w,\chi}^*(\varpi_0^{-\lambda})\tilde{T}_\chi^w (L_\chi) \right) (1)
\]
\[
= b_w(\chi)p_B^* \left( \pi_{w,\chi}^*(\varpi_0^{-\lambda})L_{\chi}^{(w)} \right) (1) \quad \text{by (3.3)}.
\]
Using (3.1) here for \( \ell = \pi_{w,\chi}^*(\varpi_0^{-\lambda})L_{\chi}^{(w)} \), this is equal to
\[
b_w(\chi) \sum_{w' \in W} \text{vol}(Bw'B)^{-1} \langle \pi_{w,\chi}^*(\varpi_0^{-\lambda})L_{\chi}^{(w)}, \phi_{w',w,\chi} \rangle \phi_{w',w,\chi}^{-1} (1)
\]
\[
= b_w(\chi) \text{vol}(B)^{-1} \langle L_{\chi}^{(w)}, \pi_{w,\chi}^*(\varpi_0^{-\lambda})\phi_{1,w,\chi} \rangle.
\]
Returning to (15),
\[
Q_{\chi}(\varpi_0^\lambda) = \text{vol}(B)^{-1} \sum_{w \in W} b_w(\chi)\langle L_{\chi}^{(w)}, \pi_{w,\chi}^*(\varpi_0^\lambda)\phi_{1,w,\chi} \rangle
\]
\[
\cdot \langle \langle \phi_{K,\chi}, f_{w,\chi^{-1}} \rangle \rangle.
\]
Here, we have
\[
\langle L_{\chi}^{(w)}, \pi_{w,\chi}^*(\varpi_0^\lambda)\phi_{1,w,\chi} \rangle = \langle L_{\chi}^{(w)}, p_{w,\chi}(\text{ch}_{B\varpi_0^{-\lambda}}) \rangle
\]
\[
= \begin{cases} 
\text{vol}(B) \cdot w_\chi \delta^{1/2}(\varpi_0^\lambda) & \text{if } w \in W_\theta, \\
0 & \text{otherwise}.
\end{cases}
\]
Indeed, if \( w \in W_\theta \), replacing \( w \chi \) by \( \chi \) it is enough to see this for \( w = 1 \). By definition,

\[
\langle L_\chi, p_\chi(\text{ch}_{Bw_0^{-\lambda}}) \rangle = \int_B \Delta_\chi(bw_0^{-\lambda}) db = \int_B \prod_{i=1}^m |d_i \left( \tau(\omega_0^\lambda) * b^{-1} \right) |^s_i db
\]

\[
= \text{vol}(B) \cdot \prod_{i=1}^m |d_i \left( \tau(\omega_0^\lambda) \right) |^s_i \quad \text{by (1.3)}
\]

\[
= \text{vol}(B) \cdot \chi^{\delta^{1/2}(\omega_0^\lambda)} \quad \text{by (10) and (12)}.
\]

On the other hand, if \( w \not\in W_\theta \) and \( L_\chi^{(w)} \neq 0 \), then the support of the distribution \( p_\chi^{(w)}(L_\chi^{(w)}) \) is \( \Omega_\chi^{(w)} = (P_{\eta_0}H)^{cd} \) with \( v = \theta(w)^{-1}w \neq 1 \), by (2.2)(ii).

Since \( \Omega_\chi^{(w)} \cap \Omega_1 = \emptyset \) for \( v \neq 1 \) by (1.1) and \( B\omega_0^{-\lambda} \subset \Omega_1 \) by (1.3), we have

\[
\text{supp} \left( p_\chi^{(w)}(L_\chi^{(w)}) \right) \cap B\omega_0^{-\lambda} = \emptyset , \quad \text{hence} \quad \langle L_\chi^{(w)}, p_\chi(\text{ch}_{B\omega_0^{-\lambda}}) \rangle = 0.
\]

Now (16) reduces to

\[
Q_\chi(\omega_0^\lambda) = \sum_{w \in W_\theta} b_w(\chi) \langle \langle \phi_{K,\chi}, f_{w,\chi^{-1}} \rangle \rangle^w \chi^{\delta^{1/2}(\omega_0^\lambda)}.
\]

By the formula for zonal spherical functions in [C, (4.2)], it is known that

\[
\langle \langle \phi_{K,\chi}, f_{w,\chi^{-1}} \rangle \rangle = \text{vol}(B) w_0 B \cdot \frac{c_{w_0}(w_0w^{-1})}{c_w(\chi^{-1})}.
\]

Finally, if \( w \in W_\theta \) and \( \chi \in X_{ur}(T)_\theta \), then one has \( w_0w^{-1} = w(\omega_0^{-1}) = w_0 \chi \). This completes the proof. \( \square \)

For a regular \( \chi \in X_{ur}(T)_\theta \), assume that

\[
Q_w(\chi)(1) \neq 0 \quad \text{for all} \quad w \in W_\theta.
\]

Then, for each \( w \in W_\theta \), applying both sides of (3.3) to \( \phi_{K,w}\chi \), one has

\[
\langle \overline{T}_w(L_\chi), \phi_{K,w}\chi \rangle = \frac{c_w(\chi^{-1})}{c_{w^{-1}}(\chi)} \cdot \langle L_\chi, T_w(\phi_{K,w}\chi) \rangle = c_w(\chi^{-1}) \langle L_\chi, \phi_{K,w}\chi \rangle
\]

\[
= b_w(\chi) \langle L_w\chi, \phi_{K,w}\chi \rangle.
\]
Thus
\[ b_w(\chi) = \frac{Q_{\chi}(1)}{Q_{w\chi}(1)} \cdot c_w(\chi^{-1}). \]

Put \( \tilde{Q}_\chi = Q_{\chi}(1)^{-1} \cdot Q_\chi \). Then we have

(3.5) Corollary. For a regular \( \chi \in X_{ur}(T)_\theta \) such that \( c_w(\chi^{-1})c_w^{-1}(w\chi) \neq 0 \) for all \( w \in W \) and that \( Q_{w\chi}(1) \neq 0 \) for all \( w \in W_\theta \),

\[ \tilde{Q}_\chi(\varpi^\lambda_0) = \text{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}(w\chi)}{Q_{w\chi}(1)} \cdot w_0 \delta_P^{1/2}(\varpi_0^\lambda). \]

§4. Explicit Computations for \( n = 2 \) and \( 3 \)

In this section we give an explicit formula of \( \tilde{Q}_\chi \) for \( n = 2 \) and \( n = 3 \). By (3.5) it is enough to compute \( Q_{\chi}(1) \) for \( \chi \in X_{ur}(T)_\theta \).

(I) \( n=2 \)

In this case, \( W_\theta = W = \{ 1, w_0 \} \). Write \( \chi \in X_{ur}(T)_\theta \) as

\[ \chi \left( \begin{array}{cc} t_1 & 0 \\ 0 & t_2 \end{array} \right) = |t_1|^s|t_2|^{-s}, \quad s \in \mathbb{C}. \]

Then \( \chi \) is regular if and only if \( q^{-s} \neq \pm 1 \), which we assume below. The function \( \Delta_\chi \) defined by (11) is then of the form

\[ \Delta_\chi(g) = |d_1(\theta(g)g^{-1})|^{s-\frac{1}{2}}. \]

Note that \( d_1(x) \) is just the \((1,1)\)-entry of the matrix \( x \). By the decomposition \( K = B \cup Bw_0B \) (disjoint),

\[ Q_{\chi}(1) = \int_K \Delta_\chi(k)dk = \int_B \Delta_\chi(b)db + \int_{Bw_0B} \Delta_\chi(y)dy. \]

The integral over \( B \) is \( \text{vol}(B) \) by (1.3). Also, by the Iwahori factorization,

\[ \int_{Bw_0B} \Delta_\chi(y)dy = \text{vol}(Bw_0B) \int_{O} \Delta_\chi \left( w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx \]
where $dx$ is the additive Haar measure of $\mathcal{O}$ normalized so that $\text{vol}(\mathcal{O}) = 1$.

Now the $(1, 1)$-entry of $\theta \left( w_0 \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \right) \left( w_0 \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \right)^{-1}$ is $1 - x\bar{x}$, by a direct calculation. We have to compute the integral

\[(18) \quad \int_{|x|<1} |1 - x\bar{x}|^{s-\frac{1}{2}} dx = \int_{|x|=1} |1 - x\bar{x}|^{s-\frac{1}{2}} dx\]

but the latter integral, over $|x| = 1$, is already computed in [FH, pp. 705–706] as follows (see also (4.2) of this section); the volume of the set

$$\{x \in \mathcal{O}; |x| = 1, |1 - x\bar{x}| = 1\}$$

is $1 - q_F^{-1} - 2q^{-1}$, and for $i \geq 1$ the volume of the set

$$\{x \in \mathcal{O}; |x| = 1, |1 - x\bar{x}| = q^{-i}\}$$

is $q_F^{-i}(1 - q^{-1})$, both are with respect to our additive Haar measure. Thus (18) is computed as

\[(19) \quad q^{-1} + (1 - q_F^{-1} - 2q^{-1}) + \sum_{i \geq 1} q_F^{-i}(1 - q^{-1})q^{-i(s-\frac{1}{2})} = 1 - q_F^{-1} - q^{-1} + (1 - q^{-1}) \cdot \frac{q^{-s}}{1 - q^{-s}} \quad \text{for Re}(s) > 0.\]

Since $\text{vol}(B) = (q + 1)^{-1}$, $\text{vol}(Bw_0B) = q(q + 1)^{-1}$, returning to (17) we have

$$Q_\chi(1) = \frac{1}{q+1} \cdot (1 + q \times (19)) = \frac{1}{q+1} \cdot \left( q - q_F + (q - 1) \cdot \frac{q^{-s}}{1 - q^{-s}} \right)$$

$$= \frac{q - q_F}{q+1} \cdot \frac{1 + q^{-s-\frac{1}{2}}}{1 - q^{-s}}.$$

So $Q_{w,\chi}(1) \neq 0$ for all $w \in W_\theta$ if and only if $q^{-s} \neq -q^{\pm 1/2}$.

(4.1) Theorem. For $n = 2$, assume that $\chi \in X_{ur}(T)_\theta$ is of the form

$$\chi(\text{diag}(t_1, t_2)) = |t_1|^s |t_2|^{-s}, \quad s \in \mathbb{C}, \quad q^{-s} \neq \pm 1, -q^{1/2}, \pm q^{-1/2}. \quad \text{Then } \tilde{Q}_\chi \text{ is given by}$$

$$\tilde{Q}_\chi \left( \begin{array}{cc} \omega & 0 \\
0 & 1 \end{array} \right) = \frac{q_F}{q_F - 1} \cdot \left( \frac{1 - q^{-s-\frac{1}{2}}}{1 + q^{-s}} \cdot q^{-\lambda(s+\frac{1}{2})} + \frac{1 - q^{s-\frac{1}{2}}}{1 + q^s} \cdot q^{\lambda(s-\frac{1}{2})} \right)$$
for \( \lambda \in \mathbb{Z}, \lambda \leq 0 \).

**Proof.** For our choice of \( \chi \),

\[
c_{w_0}(\chi) = \frac{1 - q^{-2s-1}}{1 - q^{-2s}}
\]

(see (14)). Therefore, \( c_{w_0}(\chi^{-1})c_{w_0^{-1}}(w_0 \chi) \neq 0 \) holds if and only if \( q^{-s} \neq \pm q^{-1/2} \). Also, by the above computation of \( Q_\chi(1) \),

\[
\frac{c_{w_0}(\chi)}{Q_\chi(1)} = \frac{q + 1}{q - q F} \cdot \frac{(1 - q^{-2s-1})(1 - q^{-s})}{(1 - q^{-2s})(1 + q^{-s-\frac{1}{2}})} = q + 1 \cdot \frac{1 - q^{-s-\frac{1}{2}}}{q - q F} \cdot \frac{1 + q^{-s}}{}
\]

which proves the above formula, by (3.5). □

**Remark.** If we replace \( \lambda (\leq 0) \) by \(-\lambda (\geq 0)\), then (4.1) coincides with the formula given by Banks [B].

(II) \( n=3 \)

Put \( w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then \( W = \{ 1, w_1, w_2, w_1 w_2, w_2 w_1, w_0 \} \), \( W_\theta = \{ 1, w_0 \} \). Write \( \chi \in X_{ur}(T)_{\theta} \) as

\[
\chi \left( \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} \right) = |t_1|^s |t_3|^{-s}, \quad s \in \mathbb{C}.
\]

Then \( \chi \) is regular if and only if \( q^{-s} \neq \pm 1 \), which again we assume below.

The function \( \Delta_{\chi} \) is of the form

\[
\Delta_{\chi}(g) = |d_1(\phi(g)g^{-1})|^{s-1}.
\]

As in the case \( n = 2 \), we use \( K = \cup_{w \in W} BwB \) (disjoint union) and the Iwahori factorization \( B = N_1^{-1} T_0 N_0 \) to compute \( Q_\chi(1) \);

\[
Q_\chi(1) = \int_K \Delta_{\chi}(k) dk = \sum_{w \in W} \int_{BwB} \Delta_{\chi}(g) dg
\]

\[
= \sum_{w \in W} \text{vol}(BwB) \int_{N_1^{-1}} \int_{N_0} \Delta_\chi(wn'n) dn' dn.
\]

(20)
Here the Haar measures $dn'$, $dn$ of $N_1^-$, $N_0$ are normalized so that $\text{vol}(N_1^-) = \text{vol}(N_0) = 1$. We shall compute the six integrals in (20).

(II-1) $w = 1$. By (1.3),

$$\int_{N_1^-} \int_{N_0} \Delta_{\chi}(n'n)dn'dn = 1.$$  \hfill (21)

(II-$w_1$) $w = w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 \end{pmatrix}$. In this case, using

$$w_1 \begin{pmatrix} 1 & 0 & * \\ 1 & * & 1 \\ 1 & 0 & 1 \end{pmatrix} w_1^{-1} \subset N, \quad w_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} w_1^{-1} \subset N$$

we obtain

$$\int_{N_1^-} \int_{N_0} \Delta_{\chi}(w_1n'n)dn'dn$$

$$= \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{\mathcal{O}} \Delta_{\chi} \left( w_1 \begin{pmatrix} 1 & 0 & 0 \\ \varpi x & 1 & \varpi y \\ 0 & \varpi y & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right) dxdydz.$$  \hfill (22)

Here and henceforth, the additive Haar measures are normalized so that $\text{vol}(\mathcal{O}) = 1$. By a direct calculation of the matrix inside $\Delta_{\chi}$, this is equal to

$$\int_{\mathcal{O}} \int_{\mathcal{O}} \int_{\mathcal{O}} | -z + \varpi \bar{z} \bar{x} + \varpi \bar{y} - \varpi^2 \bar{x}y |^{s-1} dxdydz$$

$$= \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{\mathcal{O}} | (\varpi \bar{y} - z) + \varpi \bar{x} (\bar{z} - \varpi y) |^{s-1} dxdydz$$

$$= \int_{\mathcal{O}} \int_{\mathcal{O}} |z + \varpi \bar{x} \bar{z}|^{s-1} dxdz \quad \text{(by replacing } z \text{ by } z + \varpi \bar{y})$$

$$= \int_{\mathcal{O}} |z|^{s-1} dz \quad \text{(since } |z| > |\varpi \bar{x} \bar{z}| \text{ for all } x, z \in \mathcal{O})$$

$$= (1 - q^{-1}) \cdot \frac{1}{1 - q^{-s}}.$$  \hfill (22)
(II-w_2) \ w = w_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \ As \ in \ (II-w_1), \ we \ may \ write

\begin{align*}
\int_{N_1^-} \int_{N_0} \Delta_\chi(w_2n'n)dn'dn \\
= \int_{O^3} \Delta_\chi \left( w_2 \begin{pmatrix} 1 & \varpi x & 1 \\ \varpi y & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & z & 1 \end{pmatrix} \right) \ dxdydz,
\end{align*}

and \ by \ a \ matrix \ calculation \ inside \ \Delta_\chi, \ this \ is \ equal \ to

\begin{align*}
\int_{O^3} |\bar{z} - \varpi x + \varpi yz - \varpi^2 \bar{x}y|^{s-1} dxdydz \\
= \int_{O^2} |\bar{z} + \varpi yz|^{s-1} dydz \quad (by \ z \rightsquigarrow z + \varpi \bar{x}) \\
= (22).
\end{align*}

(II-w_1w_2) \ w = w_1w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \ We \ may \ change \ the \ order \ of \ the \ Iwahori \ factorization \ to \ make \ computations \ easier. \ By

\begin{align*}
w_1w_2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} w_2^{-1}w_1^{-1} \subset N, \quad w_1w_2 \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 1 \end{pmatrix} w_2^{-1}w_1^{-1} \subset N
\end{align*}

we \ have

\begin{align*}
\int_{N_1^-} \int_{N_0} \Delta_\chi(w_1w_2n'n)dn'dn \\
= \int_{O^3} \Delta_\chi \left( w_1w_2 \begin{pmatrix} 1 & 0 & x \\ 1 & y & 0 \end{pmatrix} \begin{pmatrix} 1 & \varpi z & 1 \\ 0 & 0 & 1 \end{pmatrix} \right) \ dxdydz.
\end{align*}
Computing the matrix inside $\Delta_\chi$, this is equal to
\[
\int_{O^3} \left| -xy + \varpi xz - y + \varpi \bar{z} \right|^s dxdydz
\]
\[
= \int_{O^2} \left| y + xy \right|^s dxdy \quad \text{(by } y \mapsto -y + \varpi \bar{z})
\]
\[
= \int_{O^2} \left| y + xy \right|^s dxdy \quad \text{(by } x \mapsto (\bar{y}^{-1}y)x)
\]
\[
= \int_{O} |1 + x|^s dx \cdot \int_{O} |y|^s dy
\]
\[
= (1 - q^{-1})^2 \cdot \frac{1}{(1 - q^{-s})^2}.
\]

(II-$w_2w_1$) $w = w_2w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. As in (II-$w_1w_2$), we have
\[
\int_{N^{-1}_1} \int_{N_0} \Delta_\chi(w_2w_1n'n)dn'dn
\]
\[
= \int_{O^3} \Delta_\chi \left( w_2w_1 \begin{pmatrix} 1 & x & y \\ 1 & 0 & 1 \\ 1 & 0 & \varpi z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right) dxdydz
\]
\[
= \int_{O^3} \left| -xy + x + \varpi \bar{y} \bar{z} - \varpi z \right|^s dxdydz
\]
\[
= \int_{O^2} \left| x - xy \right|^s dxdy \quad \text{(by } x \mapsto x + \varpi \bar{z})
\]
\[
= (23).
\]

(II-$w_0$) $w = w_0$. Since $w_0N^{-1}_1w_0^{-1} \subset N$,
\[
\int_{N^{-1}_1} \int_{N_0} \Delta_\chi(w_0n'n)dn'dn
\]
\[
= \int_{O^3} \Delta_\chi \left( w_0 \begin{pmatrix} 1 & x & y \\ 1 & 0 & z \\ 1 & 0 & 1 \end{pmatrix} \right) dxdydz
\]
\[
= \int_{O^3} |\bar{y}(xz - y) - \bar{x}z + 1|^s dxdydz
\]
\[
= \int_{O^3} |\bar{x}z(\bar{y} - 1) + (1 - \bar{y})|^s dxdydz \quad \text{(by } y \mapsto (\bar{x}^{-1}x)y)\).
We divide $y \in \mathcal{O}$ into $\{|y| < 1\}$ and $\{|y| = 1\}$. The integral over $\{|y| < 1\}$ is easy to compute; since $|\bar{y} - 1| = 1$ for $|y| < 1$,

$$\int_{|y|<1} \int_{x,z \in \mathcal{O}} |\bar{y}z(\bar{y} - 1) + (1 - y\bar{y})|^{s-1} dxdydz$$

$$= \int_{|y|<1} \int_{x,z \in \mathcal{O}} |\bar{y}z + (1 - y\bar{y})|^{s-1} dxdydz$$

$$= \int_{|y|<1} \int_{|x|<1} \int_{z \in \mathcal{O}} |\bar{y}z + (1 - y\bar{y})|^{s-1} dxdydz$$

$$+ \int_{|y|<1} \int_{|x|=1} \int_{z \in \mathcal{O}} |\bar{y}z + (1 - y\bar{y})|^{s-1} dxdydz.$$  

The integrand in the first term is 1. Replacing $z$ by $\bar{x}^{-1}z - \bar{x}^{-1}(1 - y\bar{y})$ in the second, the above is equal to

$$q^{-2} + q^{-1}(1 - q^{-1}) \cdot \int_{z \in \mathcal{O}} |z|^{s-1} dz = q^{-2} + q^{-1}(1 - q^{-1})^2 \cdot \frac{1}{1 - q^{-s}}.$$

Next, the integral over $\{|y| = 1\}$ is;

$$\int_{|y|=1} \int_{x,z \in \mathcal{O}} |\bar{y}z(\bar{y} - 1) + (1 - y\bar{y})|^{s-1} dxdydz$$

$$= \sum_{i=1}^{\infty} \int_{|y|=1} \int_{|x|=q^{-i}} \int_{|z|\leq q^{-i}} q^i \cdot |\bar{y}z(\bar{y} - 1) + (1 - y\bar{y})|^{s-1} dxdydz$$

$$= \sum_{i=1}^{\infty} q^{-i}(1 - q^{-1}) \int_{|y|=1} \int_{|z|\leq q^{-i}} q^i \cdot |\bar{y}z(\bar{y} - 1) + (1 - y\bar{y})|^{s-1} dydz$$

$$= (1 - q^{-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^j \cdot \left\{ \left( \int_{|y| = 1, |\bar{y} - 1| = q^{-j}} \int_{|z|\leq q^{-i-j}} |z + (1 - y\bar{y})|^{s-1} dydz \right) + \left( \int_{|y| = 1, |\bar{y} - 1| > q^{-i-j}} \int_{|z|\leq q^{-i-j}} |z + (1 - y\bar{y})|^{s-1} dydz \right) \right\}.$$
In the former integral in \( \{ \cdots \} \), we may replace \( z \) by \( z - (1 - y\bar{y}) \). In the latter, the integrand is \( |1 - y\bar{y}|^{s-1} \). So, this is written as

\[
(26) \quad (1-q^{-1}) \sum_{i,j \geq 0} q^j \left\{ v_{i,j}(1-q^{-1}) \cdot \frac{q^{-i-j}}{1-q^{-s}} + q^{-i-j} \sum_{k=0}^{i-1} v'_{k,j} q^{-(k+j)(s-1)} \right\}
\]

where, for \( i, j, k \geq 0 \),

\[
v_{i,j} = \text{vol} \left( \{ y \in O : |y| = 1, |\bar{y} - 1| = q^{-j}, |1 - y\bar{y}| \leq q^{-i-j} \} \right), \\
v'_{k,j} = \text{vol} \left( \{ y \in O : |y| = 1, |\bar{y} - 1| = q^{-j}, |1 - y\bar{y}| = q^{-k-j} \} \right),
\]

both measured by the additive Haar measure as before. Note that if \( |y| = 1, |\bar{y} - 1| = q^{-j} \) then \( |1 - y\bar{y}| = |1 - \bar{y} + \bar{y}(1 - y)| \leq q^{-j} \).

**Lemma.**

(i)

\[
v_{0,0} = 1 - 2q^{-1}, \quad v_{0,j} = q^{-j}(1-q^{-1}) \quad (j \geq 1), \\
v_{i,0} = q^{-\frac{i}{2}} \quad (i \geq 1), \quad v_{i,j} = q^{-\frac{i}{2} - j}(1-q^{-\frac{1}{2}}) \quad (i, j \geq 1).
\]

(ii)

\[
v'_{0,0} = 1 - q^{-\frac{1}{2}} - 2q^{-1}, \quad v'_{0,j} = q^{-j}(1-q^{-\frac{1}{2}}) \quad (j \geq 1), \\
v'_{k,0} = q^{-\frac{k}{2}}(1-q^{-\frac{1}{2}}) \quad (k \geq 1), \quad v'_{k,j} = q^{-\frac{k}{2} - j}(1-q^{-\frac{1}{2}})^2 \quad (i, j \geq 1).
\]

**Proof.** Put \( U^{(m)} = 1 + \varpi^m O \) for \( m \geq 1 \) and \( U^{(0)} = O^\times \). At first, it is easy to see that \( v_{0,j} \) is computed as \( \text{vol}(U^{(j)}) - \text{vol}(U^{(j+1)}) \), so the first two of (i) are easy. To compute \( v_{i,j} \) for \( i \geq 1 \), set

\[
U_{i,j} = \{ y \in O : |y| = 1, |\bar{y} - 1| \leq q^{-j}, |1 - y\bar{y}| \leq q^{-i-j} \}
\]

for \( i, j \geq 0 \). Then, \( U_{i,j} \) is a multiplicative subgroup of \( O^\times \), lying between \( U^{(j)} \) and \( U^{(i+j)} \). Moreover it is easy to observe that

\[
U_{i,j} = \bigcup_{\varepsilon \in U^{(j)} / U^{(i+j)}} \varepsilon \cdot U^{(i+j)}.
\]
Since \( E/F \) is unramified, the norm map \( N_{E/F} : E^\times \to F^\times \) gives a surjection \( U^{(m)} \to U_F^{(m)} \) for each \( m \), hence induces \( U^{(m)}/U^{(n)} \to U_F^{(m)}/U_F^{(n)} \) for \( m \leq n \). Here \( U_F^{(m)} = U^{(m)} \cap F \). Now the volume of \( U_{i,j} \) is computed as:

\[
\text{vol}(U_{i,j}) = \# \ker[U^j/U^{(i+j)}] \times \text{vol}(U^{(i+j)})
\]

\[
= q^{-i-j} \frac{\#(U^j/U^{(i+j)})}{\#(U_F^j/U_F^{(i+j)})} = \begin{cases} q^{-\frac{i}{2}}(1 + q^{-\frac{j}{2}}) & (j = 0), \\ q^{-\frac{i}{2} - j} & (j \geq 1). \end{cases}
\]

Using this, \( v_{i,j} \) is computed as \( \text{vol}(U_{i,j}) - \text{vol}(U_{i-1,j+1}) \) for \( i \geq 1 \) so (i) follows immediately. Also, (ii) follows from (i) since \( v'_{k,j} = v_{k,j} - v_{k+1,j} \). □

Applying (4.2) and by an earnest computation, (26) is equal to

\[
(27) \quad -q^{-1}(1 - q^{-1})^2 + \frac{(1 - q^{-1})^2(1 - q^{-1} + q^{-s} - \frac{1}{2})}{(1 - q^{-s})^2} - q^{-\frac{3}{2}}(1 + q^{-\frac{1}{2}})
\]

\[
+ \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s}}.
\]

Returning to (24),

\[
(24) = (25) + (27)
\]

\[
(28) \quad = -q^{-\frac{3}{2}} + \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s}} + \frac{(1 - q^{-1})^2(1 - q^{-1} + q^{-s} - \frac{1}{2})}{(1 - q^{-s})^2}.
\]

Finally, since \( \text{vol}(BwB) = \text{vol}(B) \times q^{\ell(w)} \), returning to (20),

\[
Q_X(1) = \text{vol}(B) \times \{ (21) + 2 \times q \times (22) + 2 \times q^2 \times (23) + q^3 \times (28) \}
\]

\[
= \text{vol}(B) \left\{ (1 - q^{3/2}) + (2q(1 - q^{-1}) + q^2(1 - q^{-1})) \cdot \frac{1}{1 - q^{-s}} \right.
\]

\[
+ \left. (2q^2(1 - q^{-1})^2 + q^3(1 - q^{-1})^2(1 - q^{-1} + q^{-s} - \frac{1}{2})) \right\}
\]

\[
\cdot \frac{1}{(1 - q^{-s})^2}
\]

which is simplified as

\[
(29) \quad Q_X(1) = \text{vol}(B)(q^3 - q_F^3) \frac{(1 - q^{-s} - 1)(1 + q^{-s} - \frac{1}{2})}{(1 - q^{-s})^2}.
\]
Note that \( Q_w \chi(1) \neq 0 \) for all \( w \in W_\theta \) if and only if \( q^{-s} \neq q^{\pm 1}, -q^{\pm 1/2} \).

(4.3) Theorem. For \( n = 3 \), assume that \( \chi \in X_{ar}(T)_{\theta} \) is of the form
\[
\chi(\text{diag}(t_1, t_2, t_3)) = |t_1|^s |t_3|^{-s}, \ s \in \mathbb{C}, q^{-s} \neq \pm 1, q^{\pm 1}, -q^{1/2}, \pm q^{-1/2}.
\]
Then, \( \tilde{Q}_\chi \) is given by
\[
\tilde{Q}_\chi(\varpi_\lambda^0) = \frac{qq_F}{qF - 1} \left( \frac{(1 - q^{-s})(1 - q^{-s + \frac{1}{2}})}{1 - q^{-2s}} q^{-\lambda(s+1)} \right)
+ \left( \frac{(1 - q^{s-1})(1 - q^{s - \frac{1}{2}})}{1 - q^{2s}} q^{\lambda(s-1)} \right)
\]
for \( \lambda \in \mathbb{Z}, \lambda \leq 0 \).

Proof. First, by (14) it is known that \( c_w(\chi^{-1})c_w^{-1}(w \chi) \neq 0 \) for all \( w \in W \) if and only if \( q^{-s} \neq q^{-1}, \pm q^{-1/2} \). Also, for our choice of \( \chi \),
\[
c_w(\chi) = \left( \frac{1 - q^{-s-1}}{1 - q^{-s}} \right)^2 \cdot \frac{1 - q^{-2s-1}}{1 - q^{-2s}}.
\]
Therefore, by (29),
\[
\frac{c_w(\chi)}{Q_\chi(1)} = \frac{1}{\text{vol}(B)q^3(1 - q^{-3})} \cdot \frac{(1 - q^{-1})^2(1 - q^{-s-1})^2(1 - q^{-2s-1})}{(1 - q^{-s-1})(1 + q^{-s - \frac{1}{2}})(1 - q^{-s})(1 - q^{-2s})}
= \frac{1}{\text{vol}(Bw_0B)qF - 1} \cdot \frac{(1 - q^{-s-1})(1 - q^{-s - \frac{1}{2}})}{1 - q^{-2s}}
\]
which proves the above formula, by (3.5). \( \square \)

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Spherical Functions in a Certain Distinguished Model


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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka
560-0043, Japan

Current address
Akashi College of Technology
Akashi, Hyogo
674-8501, Japan
E-mail: takano@akashi.ac.jp