Analogue of Flat Basis and 
Cohomological Intersection Numbers 
for General Hypergeometric Functions

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Dedicated to Professor Kazuhiko Aomoto on the occasion of his 60-th birthday

Abstract. The general hypergeometric functions of confluent type 
given by 1-dimensional integral are studied. To such functions, the 
rational de Rham cohomology group is associated and cohomological 
intersection numbers for a good basis are computed explicitly, using the 
property of the basis analogous to the flat basis of simple singularity of 
A-type.

1. Introduction

This paper concerns the explicit computation of intersection numbers for 
the de Rham cohomology classes associated with the general hypergeometric 
functions (GHF, for short) introduced in [1], [6] and [12]. According to [12], 
one can define, for any given partition \( \lambda \) of any positive integer \( n \), general 
hypergeometric functions as solutions of a holonomic system on a Zariski 
open set of the space of complex matrices \( M(r, n; \mathbb{C}) \) or by integrals of 
Euler-Laplace type of \((r - 1)\)-form. See Sec. 2 for the details. For the 
partition \( \lambda = (1, \ldots, 1) \), GHF, which was introduced by K. Aomoto [1] and 
I.M. Gelfand [6], gives a generalization of the famous Gauss hypergeometric 
function. In fact, Gauss hypergeometric function corresponds to the case 
\((r, n) = (2, 4)\). For the hypergeometric function of Aomoto and Gelfand, an 
intersection theory is developed in [4], [16] and the explicit computation 
of the cohomological intersection numbers is carried out for the de Rham 
cohomology classes represented by logarithmic forms in the case \( r = 2 \).

For partitions \( \lambda \) containing parts greater than or equal to 2, GHF gives 
generalizations to several variables of the classical hypergeometric functions

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of confluent type, say, Kummer’s confluent hypergeometric function, Bessel function, Hermite function and Airy function. The de Rham cohomology group associated to GHF is calculated explicitly in the case \( r = 2 \) in [10]. To compute the intersection numbers for this case, the definition of intersection numbers given in [19] can be applied. We choose a “good basis” for the de Rham cohomology group which turns out to be an analogue of flat basis for the Jacobi ring for the simple singularity of \( A \)-type ([20], [21]) at several points in \( \mathbb{P}^1 \). Using this good basis, we obtain the matrix of intersection numbers which is independent of the variables of the GHF as was the case for Aomoto-Gelfand hypergeometric function when the logarithmic form are taken as a basis of the cohomology group. The contents of this paper are as follows.

\( \S 2 \): General hypergeometric integral.
\( \S 3 \): Twisted de Rham cohomology.
\( \S 4 \): Cohomological intersection number.
\( \S 5 \): Main theorem.
\( \S 6 \): Invariance of intersection pairing by the group action.
\( \S 7 \): Flatness of the basis \( \varphi_i^{(k)} \).
\( \S 8 \): Proof of Theorem 5.1.

2. General Hypergeometric Integral

Let \((n_1, \ldots, n_l)\) be a partition of \( n \geq 3 \), namely a nonincreasing sequence of positive integers such that

\[
 n = \sum_{k=1}^{l} n_k.
\]

To this partition we associate the abelian complex Lie subgroup of dimension \( n \):

\[
 H = J(n_1) \times \cdots \times J(n_l),
\]

where \( J(n_k) \) is the Jordan group of size \( n_k \) defined by

\[
 J(n_k) = \left\{ h^{(k)} = \sum_{0 \leq i \leq n_k - 1} h_i^{(k)} A_i^{n_k} \mid h_0^{(k)} \neq 0, h_i^{(k)} \in \mathbb{C} \right\} \subset GL(n_k, \mathbb{C}),
\]
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$\Lambda_{n_k} = (\delta_{i+1,j})_{0 \leq i,j < n_k}$ being the shift matrix.

Let $Z$ be the set of $2 \times n$ complex matrices $z = (z^{(1)}, \ldots, z^{(l)}), z^{(k)} = (z_0^{(k)}, \ldots, z_{n_k-1}^{(k)}) \in M(2, n_k, \mathbb{C})$, satisfying the condition:

$$\begin{align*}
\det(z_0^{(k)}, z_1^{(k)}) \neq 0 \quad \text{for any } k \text{ such that } n_k \geq 2, \\
\det(z_0^{(k)}, z_0^{(k')}) \neq 0 \quad \text{for any } k \neq k'.
\end{align*}$$

The general hypergeometric integral (GHI) is defined as follows. Let $\tilde{H}$ be the universal covering group of $H$ and let $\chi : \tilde{H} \to \mathbb{C}^\times$ be a character of $\tilde{H}$, that is, a complex analytic homomorphism from $\tilde{H}$ to the complex torus $\mathbb{C}^\times$. Define the functions $\theta_i(x)$ of $x = (x_0, x_1, x_2, \ldots)$ by the generating function

$$\sum_{m=0}^{\infty} \theta_m(x)T^m = \log(x_0 + x_1T + x_2T^2 + \cdots).$$

Expanding the right hand side as

$$\log(x_0 + x_1T + x_2T^2 + \cdots) = \log x_0 + \log \left(1 + \frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right) = \log x_0 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right)^m,$$

we have

$$\theta_0(x) = \log x_0$$

and the weighted homogeneous polynomials in $x_1/x_0, x_2/x_0, \ldots$:

$$\theta_m(x) = \sum_{\lambda_1 + 2\lambda_2 + \cdots + m\lambda_m = m} (-1)^{\lambda_1 + \cdots + \lambda_m - 1} \frac{(\lambda_1 + \cdots + \lambda_m - 1)!}{\lambda_1! \cdots \lambda_m!} \times \left(\frac{x_1}{x_0}\right)^{\lambda_1} \cdots \left(\frac{x_m}{x_0}\right)^{\lambda_m}.$$

For example we have

$$\theta_0(x) = \log x_0.$$
\[\theta_1(x) = \frac{x_1}{x_0},\]
\[\theta_2(x) = \frac{x_2}{x_0} - \frac{1}{2} \left(\frac{x_1}{x_0}\right)^2\]
\[\theta_3(x) = \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right) \left(\frac{x_2}{x_0}\right) + \frac{1}{3} \left(\frac{x_1}{x_0}\right)^3\]
\[\theta_4(x) = \frac{x_4}{x_0} - \frac{1}{2} \left(\frac{x_2}{x_0}\right)^2 - \left(\frac{x_1}{x_0}\right) \left(\frac{x_3}{x_0}\right) + \left(\frac{x_1}{x_0}\right)^2 \left(\frac{x_2}{x_0}\right) - \frac{1}{4} \left(\frac{x_1}{x_0}\right)^4.\]

Then, the character \(\chi : \tilde{H} \to \mathbb{C}^\times\) is explicitly written as
\[\chi(h; \alpha) = \prod_{k=1}^{l} \exp \left(\sum_{i=0}^{n_k-1} \alpha_i^{(k)} \theta_i(h^{(k)})\right)\]
for appropriate complex constants \(\alpha = (\alpha^{(1)}, \ldots, \alpha^{(l)}) \in \mathbb{C}^n, \alpha^{(k)} = (\alpha_0^{(k)}, \ldots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}\). Define a biholomorphic map
\[\iota : \tilde{H} \to \prod_{k=1}^{l} \left(\mathbb{C}^\times \times \mathbb{C}^{n_k-1}\right) \subset \mathbb{C}^n\]
by
\[\iota(h) = (h_0^{(1)}, \ldots, h_{n_1-1}^{(1)}, \ldots, h_0^{(l)}, \ldots, h_{n_l-1}^{(l)})\]
for \(h = (h^{(1)}, \ldots, h^{(l)}) \in \tilde{H}\).

**Assumption.** For the character \(\chi(\cdot ; \alpha)\) of \(\tilde{H}_\lambda\), we assume
\[(2.2) \sum_{k=1}^{l} \alpha_0^{(k)} = 0.\]

For \(z \in Z\), we consider the \(n\) polynomials in \(t\):
\[tz = (tz_0^{(0)}, \ldots, tz_{n_1-1}^{(0)}, \ldots, tz_0^{(l)}, \ldots, tz_{n_l-1}^{(l)})\]
defined by the multiplication of matrices $t = (1, t)$ and $z^{(k)}_j$: 

$$tz^{(k)}_j = z^{(k)}_{0j} + tz^{(k)}_{1j}$$

and substitute these polynomials to the character $\chi(\cdot ; \alpha)$ to obtain the function $\chi(t^{-1}(tz); \alpha)$. By the assumption (2.2), $\chi(t^{-1}(tz); \alpha)$ is a multivalued function of $(t, z) \in \mathbb{P}^1 \times Z$ having the branch locus

$$\bigcup_{k=1}^{l} \{(t, z) \mid tz^{(k)}_0 = 0\}.$$

**Definition 2.1.** The general hypergeometric integral is defined by

$$F(z; \alpha) = \int_{\Delta(z)} \chi(t^{-1}(tz); \alpha) dt$$

where $\Delta(z)$ is some 1-dimensional cycle in $\mathbb{P}^1$ depending on $z \in Z$.

### 3. Twisted de Rham Cohomology

The hypergeometric integral is naturally regarded as a dual pairing of some cocycle of de Rham cohomology and the twisted cycle. We recall the definition of the de Rham cohomology.

For the moment we fix $z \in Z$, and consider the 1-form in $t$

$$\omega := d \log \chi(t^{-1}(tz); \alpha) = \left( \sum_{k=1}^{l} \sum_{j=0}^{n_k-1} \alpha^{(k)}_j \partial_t \theta_j(tz) \right) dt$$

obtained as the logarithmic derivative of $\chi(t^{-1}(tz); \alpha)$. The 1-form $\omega$ has poles at

$$p_k = -z^{(k)}_{00} / z^{(k)}_{01}, \quad (k = 1, \ldots, l)$$

of order $n_k$ and these poles are distinct each other by virtue of the assumption (2.1). Let $D$ be the divisor of the meromorphic 1-form $\omega$ in $\mathbb{P}^1$, i.e.,

$$D = \sum_{k=1}^{l} n_k p_k.$$
Let Ω • (∗D) be the sheaf of meromorphic 1-forms on ℙ¹ having poles at most on |D| = {p₁, . . . , p_l}. Consider the de Rham complex

\[ (Ω • (∗D), ∇_ω) : 0 → Ω^0(∗D) → Ω^1(∗D) → 0, \]

where ∇_ω is the connection defined by

\[ ∇_ω f = df + ω f, \quad f ∈ Ω^0(∗D). \]

The cohomology group of the complex of the global sections of the above complex of sheaves

\[ H^p(Γ(ℙ¹, Ω • (∗D)), ∇_ω) \]

is called the twisted rational de Rham cohomology group. We simply denote this group by \( H^p(Ω^•(∗D), ∇_ω) \).

In [10] we proved the following.

**PROPOSITION 3.1.** Let the parameters α in the connection form ω satisfy

\[
α_{nk}^{(k)} \begin{cases} 
≠ \mathbb{Z} & \text{if } n_k = 1, \\
≠ 0 & \text{if } n_k ≥ 2.
\end{cases}
\]

Then we have

1. \( H^i(Ω^•(∗D), ∇_ω) = 0, \quad (i ≠ 1), \)
2. \( H^1(Ω^•(∗D), ∇_ω) ≃ Γ(ℙ¹, Ω^1(D))/C · ω, \) where \( Ω^1(D) \) is the sheaf of meromorphic 1-forms η such that

\[ (η) + D ≥ 0, \]

3. \( \dim_C H^1(Ω^•(∗D), ∇_ω) = n - 2. \)

As a \( C \)-basis of the vector space \( Γ(ℙ¹, Ω^1(D)) \) we can take, for example, the 1-forms

\[
(tz_0^{(k)})^{-i} dt, \quad (k = 1, . . . , l; i = 2, . . . , n_k)
\]

\[
d log(tz_0^{(k)}) - d log(tz_0^{(k+1)}), \quad (k = 1, . . . , l - 1),
\]
which were chosen in [10], [19]. In this paper we take the following 1-forms as a basis.

\[
\begin{align*}
\phi^{(k)}_i &= d\theta_i(tz^{(k)}), & (k = 1, \ldots, l; i = 1, \ldots, n_k - 1) \\
\phi^{(k)}_0 &= d\theta_0(tz^{(k)}) - d\theta_0(tz^{(k+1)}), & (k = 1, \ldots, l - 1).
\end{align*}
\]

For later use, we also prepare the 1-form

\[
\phi^{(l)}_0 = d\theta_0(tz^{(l)}) - d\theta_0(tz^{(1)})
\]

Note that, by virtue of the conditions (2.2), the 1-form \(\omega\) is a linear combination of \(\phi^{(k)}_i\)'s listed in (3.2). The reason for the choice of the forms \(\phi^{(k)}_i\)'s will become clear in Sections 7 and 8.

4. Cohomological Intersection Number

We recall the definition of intersection numbers for the de Rham cohomology classes. For the details we refer to [19]. Consider two complexes of sheaves of meromorphic differential forms

\[
\begin{align*}
(\Omega^\bullet(D), \nabla_\omega) &: 0 \longrightarrow \Omega^0 \overset{\nabla_\omega}{\longrightarrow} \Omega^1(D) \longrightarrow 0, \\
(\Omega^\bullet(-D), \nabla_\omega) &: 0 \longrightarrow \Omega^0(-D) \overset{\nabla_\omega}{\longrightarrow} \Omega^1 \longrightarrow 0.
\end{align*}
\]

Then computing the associated hypercohomologies, we get the isomorphisms

\[
\begin{align*}
j_\omega &: \mathbb{H}^1(\mathbb{P}^1, (\Omega^\bullet(D), \nabla_\omega)) \longrightarrow \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C} \cdot \omega \\
k_\omega &: \mathbb{H}^1(\mathbb{P}^1, (\Omega^\bullet(-D), \nabla_\omega)) \longrightarrow \text{Ker}(\nabla_\omega : H^1(\mathbb{P}^1, \Omega^0(-D)) \rightarrow H^1(\mathbb{P}^1, \Omega^1)).
\end{align*}
\]

On the otherhand there exists an isomorphism

\[
\iota_\omega &: \mathbb{H}^\bullet(\mathbb{P}^1, (\Omega^\bullet(D), \nabla_\omega)) \longrightarrow \mathbb{H}^\bullet(\mathbb{P}^1, (\Omega^\bullet(-D), \nabla_\omega)).
\]
This follows from the following exact sequence of complexes of sheaves and from the fact that the complex represented by the third column is exact:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & \Omega^0(-D) & \to & \Omega^0 & \to & \bigoplus_{k=1}^l (\sum_{i=1}^{n_k} b_{ki}(t - p_k)^{i-1})_{p_k} & \to & 0 \\
\nabla_\omega & \downarrow & \nabla_\omega & \downarrow & \nabla_\omega & \downarrow & \\
0 & \to & \Omega^1 & \to & \Omega^1(D) & \to & \bigoplus_{k=1}^l (\sum_{i=1}^{n_k} c_{ki}(t - p_k)^{-i})_{p_k} & \to & 0 \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

where \(\pi\) is defined by taking the principal part of a meromorphic 1-form in \(\Omega^1(D)\) at each point \(p_k\) and \(\nabla_\omega\) is defined by applying \(\nabla_\omega\) to an element \(\sum_{i=1}^{n_k} b_{ki}(t - p_k)^{i-1}\) and then taking the principal part of the resulted germ of meromorphic 1-form at \(p_k\). Put \(i_\omega := k_\omega \circ i_\omega\).

Now consider the de Rham complex \((\Omega^\bullet(*D), \nabla_{-\omega})\) defined by the connection \(\nabla_{-\omega}\) with the connection form \(\omega\) which is dual to \(\nabla_\omega\):

\[
0 \to \Omega^0 \xrightarrow{\nabla_{-\omega}} \Omega^1(D) \to 0.
\]

Assuming the condition (3.1), we have

\[
H^p(\Omega^\bullet(*D), \nabla_{-\omega}) \simeq \begin{cases} 
\Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C} \cdot (-\omega) & \text{if } p = 1 \\
0 & \text{otherwise}
\end{cases}
\]

We define the intersection pairing between the de Rham cohomologies

\[
H^1(\Omega^1(*D), \nabla_\omega) \times H^1(\Omega^1(*D), \nabla_{-\omega}) \to \mathbb{C}
\]

as follows. Take \([\varphi^+] \in H^1(\Omega^\bullet(*D), \nabla_\omega)\) and \([\varphi^-] \in H^1(\Omega^\bullet(*D), \nabla_{-\omega})\) represented by the forms \(\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))\). Then \(i_\omega \circ j_\omega^{-1}([\varphi^+]) \in \text{Ker}(\nabla_\omega : H^1(\mathbb{P}^1, \Omega^0(-D)) \to H^1(\mathbb{P}^1, \Omega^1))\) and \([\varphi^-] \in \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C} \cdot (-\omega)\). Then by the Serre duality \(H^1(\mathbb{P}^1, \Omega^0(-D)) \times \Gamma(\mathbb{P}^1, \Omega^1(D)) \to H^1(\mathbb{P}^1, \Omega^1)\), we have an element of \(H^1(\mathbb{P}^1, \Omega^1)\), which is represented by
a global $(1,1)$-form by virtue of Dolbeault theorem. Integrating this 2-form over $\mathbb{P}^1$ we get a complex number, well defined for the classes $i_w \circ j_\omega^{-1}([\varphi^+]), [\varphi^-]$, which is denoted by $\langle [\varphi^+], [\varphi^-] \rangle$ and is called the intersection number of the classes $[\varphi^+]$ and $[\varphi^-]$.

For the 1-forms $\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ and $\omega \in \Gamma(\mathbb{P}^1, \Omega^1(D))$, we set

$$\varphi^+ = g^+(t)dt, \quad \varphi^- = g^-(t)dt, \quad \omega = h(t)dt.$$ 

Put

$$\frac{\varphi^+ \star \varphi^-}{\omega} := \frac{g^+(t)g^-(t)}{h(t)}dt.$$ 

By following carefully the argument in [19], we see the following, the proof of which we omit.

**Proposition 4.1.** The intersection number of the cohomology classes $[\varphi^+] \in H^1(\Omega^1(*D), \nabla_\omega)$ and $[\varphi^-] \in H^1(\Omega^1(*D), \nabla_{-\omega})$ with the representatives $\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ is given by summing up the residues of the form at each point of $|D|$:

$$\langle [\varphi^+], [\varphi^-] \rangle = 2\pi \sqrt{-1} \sum_{k=1}^l \text{Res}_{t=p_k} \frac{\varphi^+ \star \varphi^-}{\omega}.$$ 

### 5. Main Theorem

As in Section 3, we consider the elements of $\Gamma(\mathbb{P}^1, \Omega^1(D))$:

$$(5.1) \quad \varphi^{(1)}_0, \ldots, \varphi^{(1)}_{n_1-1}, \ldots, \varphi^{(l)}_0, \ldots, \varphi^{(l)}_{n_l-1}.$$ 

If one omits one of $\varphi^{(1)}_0, \ldots, \varphi^{(l)}_0$, the $n-1$ remaining 1-forms give a $\mathbb{C}$-basis of $\Gamma(\Omega^1(D))$. The classes in $H^1(\Omega^*(*D), \nabla_\omega)$ and in $H^1(\Omega^*(*D), \nabla_{-\omega})$ represented by the 1-form $\varphi^{(k)}_i$ is denoted by $[\varphi^{(k)^+}_i]$ and $[\varphi^{(k)^-}_i]$ respectively. Although we can obtain a basis of $H^1(\Omega^*(*D), \nabla_\omega)$ by omitting one of the classes $[\varphi^{(k)}_{i_{nk-1}}] (k = 1, \ldots, l)$, in order to present the matrix of intersection numbers $\langle ([\varphi^{(k)}_i]^+), [\varphi^{(k')^-}_j] \rangle$ in a symmetric manner, we compute these numbers for the classes given by the forms (5.1).
Introduce a series of polynomials $e_0(x) = 1, e_1(x), e_2(x), \ldots$ of $x = (x_1, x_2, \ldots)$ by using the generating function

$$(1 + x_1T + x_2T^2 + \cdots)^{-1} = \sum_{k=0}^{\infty} e_k(x)T^k$$

and put

$$\beta^{(k)} = (1, \beta_1^{(k)}, \ldots, \beta_{n_k-1}^{(k)}):= \left(\frac{\alpha_{n_k-1}^{(k)}}{\alpha_{n_k-1}^{(k)}}, \frac{\alpha_{n_k-2}^{(k)}}{\alpha_{n_k-1}^{(k)}}, \ldots, \frac{\alpha_0^{(k)}}{\alpha_{n_k-1}^{(k)}}\right),

(k = 1, \ldots, l).$$

**Theorem 5.1.** The matrix of intersection numbers

$$I = (I_{kk'})_{k,k'=1,\ldots,l}, I_{k,k'} = (\langle \varphi_i^{(k)}, \varphi_j^{(k')} \rangle)_{0 \leq i < n_k, 0 \leq j < n_{k'}}$$

is symmetric and have the form

$$I = \begin{pmatrix}
I_{11} & I_{12} & 0 & \ldots & 0 & I_{1l} \\
I_{21} & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & I_{l-1,l} \\
I_{l1} & 0 & \ldots & 0 & I_{l,l-1} & I_{ll}
\end{pmatrix}$$

where

$$I_{kk} = \frac{2\pi \sqrt{-1}}{\alpha_{n_k-1}^{(k)}} \begin{pmatrix}
e_0(\beta^{(k)}) & \cdots & e_0(\beta^{(k)}) \\
e_1(\beta^{(k)}) & \cdots & e_1(\beta^{(k)}) \\
\vdots & \ddots & \vdots \\
e_0(\beta^{(k)}) & \cdots & e_{n_k-1}(\beta^{(k)}) \\
\end{pmatrix}
+ \delta_{nk+1,1} \frac{2\pi \sqrt{-1}}{\alpha_{n_k+1}^{(k+1)}} \begin{pmatrix}1 \end{pmatrix}$$
\[ I_{k-1,k} = \frac{2\pi \sqrt{-1}}{\alpha_{n_{k-1}}^{(k)}} \left( \begin{array}{c} -1 \\ \end{array} \right) \quad (k = 1, \ldots, l) \]

Here when \( k = 1 \), we understand \((k - 1, k)\) as \((l, 1)\) by convention.

**Remark 5.2.** The intersection numbers computed in the above theorem are independent of the variables \( z \in Z \) of general hypergeometric functions. This fact relies on the choice of representatives of the cohomology classes. Here we took \( \varphi_i^{(k)} \) as representatives, which, as will be seen in Sec. 7, can be regarded as an analogue of the flat basis of the Jacobi ring for the simple singularity of \( A \)-type at each point \( p_k \) of \(|D|\). As for the flat basis, we refer the reader to [20], [21].

**6. Invariance of Intersection Numbers by the Group Action**

Let us consider the action of \( G = GL(2, \mathbb{C}) \) and of \( H \) on \( Z \) defined by

\[ \rho_{g,h} : Z \longrightarrow Z, \quad z \mapsto gz h \]

and let \( X \) be the subset of \( Z \) consisting of the matrices

\[ x = (x^{(1)}, \ldots, x^{(l)}), \quad x^{(k)} \in M(2, n_k, \mathbb{C}) \]

with

\[ x^{(k)} = \left( \begin{array}{cccc} x_0^{(k)} & x_1^{(k)} & \cdots & x_{n_{k-1}}^{(k)} \\ 1 & 0 & \cdots & 0 \end{array} \right) \]

satisfying

1. \( x_0^{(1)}, \ldots, x_0^{(l)} \) are distinct complex numbers,
2. \( x_1^{(k)} \neq 0 \) for \( k \) such that \( n_k \geq 2 \),
3. in the case \( l = 1 \), \( x_0^{(1)}, x_2^{(1)} \) are fixed to arbitrary prescribed numbers and \( x_1^{(1)} \) to an arbitrary prescribed nonzero number, say, \( x_0^{(1)} = 0, x_1^{(1)} = 1, x_2^{(1)} = 0 \),
4. in the case \( l = 2 \), \( x_0^{(1)}, x_0^{(2)} \) are fixed to arbitrary prescribed distinct numbers and \( x_1^{(1)} \) to an arbitrary prescribed nonzero number, say, \( x_0^{(1)} = 0, x_1^{(1)} = 1, x_2^{(1)} = 1 \),
5. in the case \( l \geq 3 \), three among \( x_0^{(1)}, \ldots, x_0^{(l)} \), say \( x_0^{(1)}, x_0^{(2)}, x_0^{(3)} \), are fixed to some prescribed 3 distinct numbers.
Note that $X$ is a closed submanifold of $Z$ of dimension $n - 3$.

**Proposition 6.1.** The subset $X$ gives a realization of the quotient space $G \setminus Z/H$:

$$
X \longrightarrow G \setminus Z/H \\
x \mapsto [x]
$$

is a homeomorphism.

By the proposition, we see that for any $z \in Z$ there are $g \in G$ and $h \in H$ such that

$$
x = gzh \in X.
$$

The forms $\varphi_i^{(k)} \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ depend on $z \in Z$. When we want to make apparent the dependence of these forms on $z$ we write $\varphi_i^{(k)}(z)$ instead of writing $\varphi_i^{(k)}$. We want to reduce the computation of the intersection numbers for $\varphi_i^{(k)}(z)$ to those for $\varphi_i^{(k)}(x)$ with $x \in X$. The first step is the following.

**Lemma 6.2.** The 1-forms $\varphi_i^{(k)}$ and $\omega$ are invariant under the action of $H$.

**Proof.** Since $\omega$ is a linear combination of $\varphi_i^{(k)}$'s, it suffices to show that $\varphi_i^{(k)}$ are invariant under the action of $H$. We prove in the case $i \geq 1$, since the case $i = 0$ is similarly proved. In this case, $\varphi_i^{(k)}(z) = d_t(\theta_i(tz^{(k)}))$. By the definition of the functions $\theta_i(x)$, we have

$$
\theta_i(\iota(hh')) = \theta_i(\iota(h)) + \theta_i(\iota(h')) \quad (h, h' \in J(n_k)).
$$

Thus

$$
\theta_i(tz^{(k)}h^{(k)}) = \theta_i(tz^{(k)}) + \theta_i(\iota(h^{(k)})).
$$

Taking the exterior derivative of the both sides with respect to $t$, we get

$$
d(\theta_i(tz^{(k)}h^{(k)})) = d(\theta_i(tz^{(k)})).
$$

This implies the invariance $\varphi_i^{(k)}(z) = \varphi_i^{(k)}(zh) \quad (h \in H)$. □

Next we consider the action of $G$ on $Z$. 

Lemma 6.3. We have

\[(\varphi_i^{(k)}(z), \varphi_i^{(k)}(g_z)), \quad (g \in G).\]

Proof. Consider the projective transformation

\[P_g : \mathbb{P}^1 \ni t \mapsto s := t \cdot g = \frac{b + dt}{a + ct} \in \mathbb{P}^1 \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.\]

In view of Proposition 4.1, the intersection number for \(\psi^+, \psi^- \in \Gamma(\mathbb{P}^1, \Omega_1(D))\) satisfies

\[(\psi^+, [\psi^-]) = \langle [P_g^* \psi^+], [P_g^* \psi^-] \rangle.\]

On the other hand, for the forms \(\varphi_i^{(k)}(z)\), we have

\[P_g^* \varphi_i^{(k)}(z) = \varphi_i^{(k)}(g(z)).\]

In fact, for the case \(i \geq 1\),

\[P_g^* \varphi_i^{(k)}(z) = P_g^* d(\theta_i(sz^{(k)})) = d(\theta_i((1, t \cdot g)z^{(k)})) = d(\theta_i((1, t)g z^{(k)})) = \varphi_i^{(k)}(g z).\]

The case \(i = 0\) can be shown similarly. Combining (6.2) and (6.3), we have the desired identity (6.1). □

Summing up we have shown the following.

Proposition 6.4. The intersection number \(\langle [\varphi_i^{(k)}^+], [\varphi_j^{(k)}^-] \rangle\) is invariant by the action of \(G \times H\) on \(Z\), namely we have

\[\langle [\varphi_i^{(k)}^+(z)], [\varphi_j^{(k)}^-(z)] \rangle = \langle [\varphi_i^{(k)}^+(\rho_{g,h}(z))], [\varphi_j^{(k)}^-(\rho_{g,h}(z))] \rangle \quad \text{for all} \quad (g, h) \in G \times H.\]
7. Flatness of the Basis $\varphi^{(k)}_i$

As is seen in Section 6, for the aim of computing intersection numbers for the forms $\varphi^{(k)}_i$'s, it is sufficient to consider $\varphi^{(k)}_i(x)$ for $x \in X$. In this section we fix $x \in X$ and write simply $\varphi^{(k)}_i$ for $\varphi^{(k)}_i(x)$. We look into in detail the property of these forms which permit us to regard these forms as analogues of flat basis of the Jacobi ring of simple singularity of $A$-type.

Let $x \in X$ be as in Section 6. Note that the pole divisor of the 1-form $\omega = d \log \chi(t;x;\alpha)$ is

$$D = \sum_{k=1}^{l} n_k p_k, \quad p_k = -x^{(k)}_0.$$

We consider the forms

$$\varphi^{(k)}_0, \ldots, \varphi^{(k)}_{n_k-1},$$

having poles at $p_k$. Take a local coordinate $u$ at $p_k$ defined by

$$u = \frac{1}{x^{(k)}_1(t + x^{(k)}_0)}$$

and put

$$y_i = x^{(k)}_i / x^{(k)}_1, \quad (i = 1, \ldots, n_k - 1)$$

Note that $y_1 = 1$. Then the forms (7.1) are expressed as

$$\begin{align*}
\varphi^{(k)}_i &= d\left(\theta_i(1, y_1 u^{-1}, \ldots, y_{n_k-1} u^{-n_k+1})\right), \quad (i = 1, \ldots, n_k - 1) \\
\varphi^{(k)}_0 &= d \log(u) - d \log(u - p_{k+1} + p_k).
\end{align*}$$

This situation motivates to introduce the polynomials $h_m(u)$ in $u^{-1}$ depending on the parameters $(y_1, y_2, \ldots), y_1 = 1$, by substituting

$$x_0 = 1, x_1 = y_1 u^{-1}, x_2 = y_2 u^{-1}, \ldots \quad (y_1 = 1).$$
in the functions $\theta_m(x)$ ($m = 1, 2 \ldots$):

$$h_m(u) = \theta_m(1, y_1 u^{-1}, y_2 u^{-1}, \ldots)$$

$$= \sum_{\lambda_1 + 2\lambda_2 + \cdots + m\lambda_m = m} (-1)^{\lambda_1 + \cdots + \lambda_m - 1}(\lambda_1 + \cdots + \lambda_m - 1)!$$

$$\times \frac{y_1^{\lambda_1} \cdots y_m^{\lambda_m}}{\lambda_1! \cdots \lambda_m!} u^{-(\lambda_1 + \cdots + \lambda_m)}.$$  \hspace{1cm} (7.3)

Note that $h_m(u)$ is a polynomial of $u^{-1}$ of degree $m$ without constant term whose top term is

$$( -1 )^{m+1} u^{-m} / m$$

and the coefficients of $u^{-1}$ is equal to $y_m$. Consider a Laurent series in $u$: $f = -u^{-1}(1 + s_1 u + s_2 u^2 + \cdots)$

with parameters $s = (s_1, s_2, \ldots)$. Then the power $f^m$ is a Laurent series in $u$ whose principal part $(f^m)_-$ is a polynomial of $u^{-1}$ of degree $m$ with the top term $(-1)^m u^{-m}$. Note that the coefficients of $u^{-1}$ of $(f^m)_-$ has the form

$$( -1 )^m m s_{m-1} + ( \text{a polynomial in } s_1, \ldots, s_{m-2} ).$$

Then the property we want to establish for $h_m(u)$ is the following.

**Proposition 7.1.** Determine $s_1, s_2, \ldots$ by the condition:

$$(7.4) \hspace{1cm} y_m = \text{the coefficient of } u^{-1} \text{ of } - \frac{1}{m} f^m, \quad (m = 1, 2, \ldots).$$

Then the identities

$$(7.5) \hspace{1cm} h_m(u) = -\frac{1}{m} (f^m)_- \quad (m = 1, 2, \ldots)$$

hold as polynomials in $u^{-1}$.

To prove the proposition, it is convenient to use the Schur functions $p_0(t), p_1(t), p_2(t), \ldots$ defined by the generating function:

$$\exp(t_1 T + t_2 T^2 + \cdots) = \sum_{m=0}^{\infty} p_m(t) T^m,$$
where $p_0(t) = 1$. For the parameters $s = (s_1, s_2, \ldots)$ in $f$, we define $t = (t_1, t_2, \ldots)$ by

$$s_m = p_m(t), \quad (m = 1, 2, \ldots).$$

Then

$$f^m = (-1)^m u^{-m}(1 + s_1 u + s_2 u^2 + \cdots)^m$$
$$= (-1)^m u^{-m} \exp(t_1 u + t_2 u^2 + \cdots)^m$$
$$= (-1)^m u^{-m} \sum_{k=0}^{\infty} p_k(mt) u^k.$$

Hence we have

$$(f^m)_- = (-1)^m \sum_{k=1}^{m} p_{m-k}(mt) u^{-k}.$$

The condition (7.4) is then written as

$$(7.6) \quad y_m = \frac{(-1)^{m+1}}{m} p_{m-1}(mt), \quad (m = 1, 2, \ldots).$$

Putting the expression (7.6) into (7.3), we see that $h_m(u)$ is written as

$$h_m(u) = (-1)^{m+1} \sum_{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m} \frac{(\lambda_1 + \cdots + \lambda_m - 1)!}{\lambda_1! \cdots \lambda_m!}$$
$$\times (p_0(t))^{\lambda_1} \left(\frac{1}{2p_1(2t)}\right)^{\lambda_2} \cdots \left(\frac{1}{mp_{m-1}(mt)}\right)^{\lambda_m} u^{-(\lambda_1+\cdots+\lambda_m)}.$$

Thus the verification of the identity (7.5) is reduced to showing the following identities for the Schur functions.

**Lemma 7.2.** We have the identities

$$\frac{1}{m} p_{m-k}(mt) = \sum_{\lambda_1+2\lambda_2+\cdots+m\lambda_m=k} \frac{(\lambda_1 + \cdots + \lambda_m - 1)!}{\lambda_1! \cdots \lambda_m!}$$
$$\times (p_0(t))^{\lambda_1} \left(\frac{1}{2p_1(2t)}\right)^{\lambda_2} \cdots \left(\frac{1}{mp_{m-1}(mt)}\right)^{\lambda_m}$$

$$(7.7)$$
for \( m = 1, 2, \ldots \) and \( k = 1, 2, \ldots, m \).

**Proof.** The proof is carried out by induction on \( m \) and \( k \). In the case \( m = 1 \) or the case \( k = 1 \), the identities (7.7) trivially hold. Assume that (7.7) holds for \( m \) replaced by \( 1, 2, \ldots, m - 1 \). Moreover, for \( m \) fixed, the identity (7.7) holds for \( k \) replaced by \( 1, 2, \ldots, k - 1 \) We will prove (7.7) still holds for the case where \( k \) is replaced by \( k + 1 \). We may assume \( k \geq 2 \). In this case the possible \( n \)-tuple of indices \( \lambda = (\lambda_1, \ldots, \lambda_n) \) appearing in the sum of the right hand side of (7.7) satisfies \( \lambda_n = 0 \). Differentiate the both sides of the identity (7.7). Then we get

\[
\text{L.H.S} = p_{m-k-1}(mt).
\]

and

\[
\text{R.H.S} = \sum_{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m, \lambda_1+\cdots+\lambda_m=k} \frac{(k-1)!}{\lambda_1! \cdots \lambda_m!} \prod_{i \neq j} \frac{\left(\frac{1}{2}p_{i-1}(it)\right)^{\lambda_i}}{\lambda_i!} \prod_{1 \leq i < j} \frac{\left(\frac{1}{2}p_{j-1}(jt)\right)^{\lambda_j-1}}{(\lambda_j-1)!} \prod_{j \leq i \neq m-1} \frac{\left(\frac{1}{2}p_{i-1}(it)\right)^{\lambda_i}}{\lambda_i!}.
\]

We want to show that this right hand side is equal to

\[
(7.8) \quad m \sum_{\nu_1+2\nu_2+\cdots+m\nu_m=m, \nu_1+\cdots+\nu_m=k+1} \frac{k!}{\nu_1! \cdots \nu_m!} \prod_{1 \leq i \leq m} \left(\frac{1}{2}p_{i-1}(it)\right)^{\nu_i}.
\]
Now we fix the indices $\nu = (\nu_1, \cdots, \nu_m)$ such that $\nu_1 + 2\nu_2 + \cdots + m\nu_m = m, \nu_1 + \cdots + \nu_m = k + 1$. Then, in the sum R.H.S, the contribution to the coefficients of $\prod_i \left( \frac{1}{\pi} p_{i-1}(it) \right)^{\nu_i}$ comes from the following cases of indices $\lambda$ and $\mu$. Take any index $1 \leq \alpha, \beta \leq m - 1$ such that $\alpha + \beta \leq m - 1$. If $\alpha < \beta$, we put

$$\lambda = (\nu_1, \ldots, \nu_{\alpha - 1}, \nu_{\beta - 1}, \ldots, \nu_m)$$

$$\mu = (0, \ldots, 1, \ldots, 1, \ldots, 0), \quad j = \alpha + \beta.$$

If $\alpha = \beta$, we put

$$\lambda = (\nu_1, \ldots, \nu_{\alpha - 2}, \ldots, \nu_m)$$

$$\mu = (0, \ldots, 2, \ldots, 0), \quad j = 2\alpha.$$

Summing up all the contribution, we have

$$\frac{(k - 1)!}{\nu_1! \cdots \nu_m!} \left\{ \sum_{1 \leq \alpha < \beta, \alpha + \beta \leq m - 1} (\alpha + \beta)\mu_\alpha\mu_\beta + \sum_{1 \leq \alpha, 2\alpha \leq m - 1} 2\alpha\mu_\alpha(\mu_\alpha - 1) \right\}$$

$$= \frac{(k - 1)!}{\nu_1! \cdots \nu_m!} \left\{ \sum_{1 \leq \alpha, \beta \leq m - 1} \alpha\mu_\alpha\mu_\beta - \sum_{1 \leq \alpha \leq m - 1} \alpha\mu_\alpha \right\}$$

$$= \frac{(k - 1)!}{\nu_1! \cdots \nu_m!} km$$

Thus R.H.S is written as (7.8) as is desired. $\square$

As a corollary, we have

**Corollary 7.3.** In the above situation, we have

$$\varphi_i^{(k)}(x) = -(\partial f \cdot f^{-1})_i du \quad (i = 1, \ldots, n_k - 1).$$
8. Proof of Theorem 5.1

In view of the invariance of the intersection numbers \(\langle [\varphi_i^{(k)}]^+, [\varphi_j^{(k')}^-] \rangle\) by the action \(G \times H\) (Sec. 6), it is sufficient to prove the theorem for \(z \in X\). In this case the flatness of the basis \(\varphi_i^{(k)}\)'s plays a crucial role. Recall that

\[
\langle [\varphi_i^{(k)}^+], [\varphi_j^{(k')}^-] \rangle = 2\pi \sqrt{-1} \sum_{k=1}^{l} \text{Res}_{t=p_k} \frac{\varphi_i^{(k)} \ast \varphi_j^{(k')}}{\omega}.
\]

Take the local coordinate \(u\) at \(p_k\) as in (7.2) and choose the Laurent series \(f\) at \(u = 0\) of the form

\[
f = -u^{-1}(1 + s_1 u + s_2 u^2 + \cdots)
\]
as in Section 7. Then Corollary 7.3 says that, at \(u = 0\), the 1-forms \(\varphi_i^{(k)}\) can be expressed as

\[
\varphi_i^{(k)} = -(\partial f \cdot f^{i-1})_\omega, \quad (i = 1, \ldots, n_k - 1).
\]

Similarly the 1-form \(\omega\) is expressed as

\[
\omega = \alpha_0^{(k)} d \log u + \sum_{m=1}^{n_k-1} \alpha_{m}^{(k)} \varphi_m^{(k)} + (1\text{-form holomorphic at } u = 0)
\]

\[
= -\sum_{m=0}^{n_k-1} \alpha_{m}^{(k)} (\partial f \cdot f^{m-1})_\omega + (1\text{-form holomorphic at } u = 0).
\]

Then we can prove the following.

**Lemma 8.1.** We have

\[
\text{Res}_{u=0} \frac{\varphi_i^{(k')} \ast \varphi_j^{(k')}}{\omega} = \begin{cases} 
\frac{1}{\alpha_{n_k-1}^{(k)}} e_{i+j-n_k+1}(\beta^{(k)}) & k' = k \\
\frac{1}{\alpha_j^{(k)}} & k' = k-1, n_k = 1, (i, j) = (0, 0), \\
0 & \text{otherwise.}
\end{cases}
\]
Proof. We prove only the case \( k' = k \) and \( n_k \geq 2, i \geq 1, j \geq 1 \). Using the expression (8.1) for \( \varphi^{(k)}_i \), we have

\[
\text{Res}_{u=0} \frac{\varphi^{(k)}_i \ast \varphi^{(k)}_j}{\omega} = - \text{Res}_{u=0} \frac{(\partial f \cdot f^{i-1})_-(\partial f \cdot f^{j-1})_-}{\sum_{m=0}^{n_k-1} \alpha^{(k)}_m (\partial f \cdot f^{m-1})_- + (\text{holo. function at } u = 0)} du
\]

\[
= - \frac{1}{\alpha^{(k)}_{n_k-1}} \text{Res}_{u=0} \frac{\partial f \cdot f^{i+j-n_k}}{1 + \beta_1 f^{-1} + \ldots + \beta_{n_k-1} f^{-(n_k-1)}} du
\]

\[
= - \frac{1}{\alpha^{(k)}_{n_k-1}} \text{Res}_{u=0} \partial f \cdot f^{i+j-n_k} \sum_{m=0}^{\infty} e_m(\beta) f^{-m} du
\]

\[
= \frac{1}{\alpha^{(k)}_{n_k-1}} e^{i+j-n_k+1}(\beta).
\]

Here we have used the fact

\[
\text{Res}_{u=0} \partial f \cdot f^{i+j-n_k-m} du = \begin{cases} -1 & i + j - n_k - m = -1 \\ 0 & \text{otherwise} \end{cases}
\]

For the other cases \( i = 0 \) or \( j = 0 \) or \( n_k = 1 \), the assertion is similarly proved. □

The above computation in the proof of Lemma 8.1 shows that

\[
\text{Res}_{u=0} \frac{\varphi^+ \ast \varphi^-}{\omega} = 0
\]

if the sum of the orders of pole of \( \varphi^+ \) and \( \varphi^- \) at \( u = 0 \) is less than or equal to \( n_k \). This remark implies the following.

Lemma 8.2.

\[
\operatorname{Res}_{u=0} \frac{\varphi^{(k)}_i \ast \varphi^{(k')}_j}{\omega} = 0 \quad \text{if} \quad |k - k'| \geq 2, (k, k') \neq (1, l), (l, 1)
\]

\[
\operatorname{Res}_{u=0} \frac{\varphi^{(k-1)}_i \ast \varphi^{(k)}_j}{\omega} = \begin{cases} -1/\alpha^{(k)}_{n_k-1}, & (i, j) = (0, n_k - 1) \\ 0 & \text{otherwise} \end{cases}
\]
When $k = 1$, we understand the second formula as that for the case $(k - 1, k) = (l, 1)$.

Combining these lemmas we have the following lemma which complete the proof of Theorem 5.1.

**Lemma 8.3.** We have the following equality.

\[
\langle [\varphi_i^{(k)^+}], [\varphi_j^{(k)^-}] \rangle = \frac{2\pi \sqrt{-1}}{\alpha_{nk-1}^{(k)}} e_{i+j-nk+1}(\beta^{(k)}) + \frac{2\pi \sqrt{-1}}{\alpha_0^{(k)}} \delta_{nk+1,1} \delta_{i,0} \delta_{j,0},
\]

\[
\langle [\varphi_i^{(k-1)^+}], [\varphi_j^{(k)^-}] \rangle = -\frac{2\pi \sqrt{-1}}{\alpha_{nk-1}^{(k)}} \delta_{i,0} \delta_{j,nk-1},
\]

\[
\langle [\varphi_i^{(k)^+}], [\varphi_j^{(k')^-}] \rangle = 0 \quad \text{if} \quad |k - k'| \geq 2, (k, k') \neq (1, l), (l, 1).
\]

In the second equality, we used the same convention as in Lemma 8.2.

**References**


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