A Base Point Free Theorem of Reid Type

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Abstract. Assume that $X$ is a normal projective variety over $\mathbb{C}$, of dimension $\leq 3$, and that $(X, \Delta)$ is a log variety that is weakly Kawamata log terminal. Let $L$ be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$ for some $a \in \mathbb{N}$. Then $Bs|mL| = \emptyset$ for every $m \gg 0$.

Introduction

We generally use the notation and terminology of [Utah]. Let $X$ be a normal projective variety over $\mathbb{C}$. Let $(X, \Delta)$ be a log variety which is log canonical. We assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Let $\Theta = \sum_{i=1}^{s} \Theta_i$ be a reduced divisor with only simple normal crossings on an $n$-dimensional non-singular complete variety over $\mathbb{C}$. We denote $\text{Strata}(\Theta) := \{ \Gamma \mid 1 \leq k \leq n, 1 \leq i_1 < i_2 < \cdots < i_k \leq s, \Gamma \text{ is an irreducible component of } \Theta_{i_1} \cap \Theta_{i_2} \cap \cdots \cap \Theta_{i_k} \neq \emptyset \}$.

First we recall the famous base point free theorem of Kawamata and Shokurov.

THEOREM 0. (Theorem 3-1-1 of [KMM]) Assume that $(X, \Delta)$ is weakly Kawamata log terminal (wklt). Let $L$ be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is ample for some $a \in \mathbb{N}$. Then $Bs|mL| = \emptyset$ for every $m \gg 0$.

We note that, if $aL - (K_X + \Delta)$ is nef and big but not ample, there exists a counterexample (cf. Remark 3-1-2 of [KMM]).

In [Rd], Reid introduced the notion of “log big” and suggested that, if $aL - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$, then the theorem holds.

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Definition. (due to Reid [Rd]) Let $f: Y \to X$ be a log resolution such that $K_Y = f^*(K_X + \Delta) + \sum a_jE_j$ (where $a_j \geq -1$). Let $L$ be a Cartier divisor on $X$. $L$ is called nef and log big on $(X, \Delta)$ if $L$ is nef and big and $(L |_{f(\Gamma)})^{\dim f(\Gamma)} > 0$ for any member $\Gamma$ of $\text{Strata}(\sum a_j=-1 E_j)$.

Remark. The definition of the notion of “nef and log big” does not depend on the choice of the log resolution $f$ (Claim of [Fu]).

Now we state the main theorem of this paper.

Main Theorem. Assume that $\dim X \leq 3$ and that $(X, \Delta)$ is weakly Kawamata log terminal (wklt). Let $L$ be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$ for some $a \in \mathbb{N}$. Then $\text{Bs}|mL| = \emptyset$ for every $m \gg 0$.

Remark. In [Fu0], the author proved the theorem under the conditions that $X$ is non-singular, $\Delta$ is a simple normal crossing divisor and $\dim X$ is arbitrary.

1. Preliminaries

We collect some results needed for the proof of Main Theorem.

Proposition 1. (Reid Type Vanishing, cf. [Fu]) Assume that $(X, \Delta)$ is wklt. Let $D$ be a $\mathbb{Q}$-Cartier integral Weil divisor. If $D - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$, then $H^i(X, \mathcal{O}_X(D)) = 0$ for every $i > 0$.

Proof. The proof is analogous to that of the main theorem of [Fu]. But, when we read [Fu], we must take notice of the facts below.

Fact 1. For real numbers $a$ and $b$, $[a + b] \geq [a] + [b]$. Hence $[f^*D + E] \geq [f^*D] + [E]$. Thus $f_*\mathcal{O}_Y([f^*D + E]) = \mathcal{O}_X(D)$.

Fact 2. $[f^*D + E] - (K_Y + f_*^{-1}[\Delta] + \{-f^*D - E\})$ is nef and log big on $(Y, f_*^{-1}[\Delta] + \{-f^*D - E\})$. □

Proposition 2. (Kawamata, cf. the proofs of Lemma 3 of [Ka] and Theorem 3-1-1 of [KMM]) Assume that $(X, \Delta)$ is wklt. Let $L$ be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{N}$. If $\text{Bs}|mL| \cap [\Delta] = \emptyset$ for every $m \gg 0$, then $\text{Bs}|mL| = \emptyset$ for every $m \gg 0$. 

2. Proof of Main Theorem

We prove the main theorem by induction on \( \dim X \).

**Step 0.** Let \( f : Y \to X \) be a log resolution of \( (X, \Delta) \) such that the following conditions are satisfied:

1. \( Y \) is projective (from the assumption that \( X \) is projective and the definition of the notion ofwklt),
2. \( \text{Exc}(f) \) consists of divisors,
3. \( K_Y + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E \),
4. \( E \) and \( F \) are \( f \)-exceptional effective \( \mathbb{Q} \)-divisors such that \( \text{Supp}(E) \) and \( \text{Supp}(F) \) do not have common irreducible components,
5. \( |F| = 0 \).

Note that, for any member \( G \in \text{Strata}(f_*^{-1}[\Delta]) \), \( \text{Exc}(f) \) does not include \( G \) (from an argument in p. 99 of [Sh], cf. the proof of Proposition of [Fu]).

**Step 1.** We shall show that we may assume that \( X \) is \( \mathbb{Q} \)-factorial.

Apply the relative log minimal model program to \( f : (Y, f_*^{-1}\Delta + F) \to X \).

We end up with a \( \mathbb{Q} \)-factorial weakly Kawamata log terminal variety (over \( X \)) \( \mu : (Z, \mu_*^{-1}\Delta + (F)_Z) \to X \) such that \( K_Z + \mu_*^{-1}\Delta + (F)_Z \) is \( \mu \)-nef.

Because \( K_Z + \mu_*^{-1}\Delta + (F)_Z = \mu^*(K_X + \Delta) + (E)_Z \), \((E)_Z \) is \( \mu \)-nef. Thus \( (E)_Z \) is a \( \mu \)-exceptional \( \mu \)-nef effective divisor. Applying 1.1 of [Sh] to \(- (E)_Z \), we obtain the fact that \((E)_Z = 0 \). Hence \( \mu^*(K_X + \Delta) = K_Z + \mu_*^{-1}\Delta + (F)_Z \).

We put \( M := aL - (K_X + \Delta) \). Here we prove that \( \mu^*M \) is nef and log big on \((Z, \mu_*^{-1}\Delta + (F)_Z) \). Let \( G_0 \) be a divisor of \( \mathbb{C}(X) \) whose discrepancy with respect to \((Z, \mu_*^{-1}\Delta + (F)_Z) \) is \(-1 \). Let \( W \) be a non-singular projective variety and \( f_1 : W \to Y \) and \( \mu_1 : W \to Z \) birational morphisms, such that \( G_0 \) is a non-singular prime divisor on \( W \) and that \( f f_1 = \mu \mu_1 \). Then \( f_1(G_0) \in \text{Strata}(f_*^{-1}[\Delta]) \) (cf. the proof of Claim of [Fu]). So \( \text{Exc}(f) \) does not include \( f_1(G_0) \).

Put \( B := f(\text{Exc}(f)) \). By Zariski’s Main Theorem, the set-theoretical inverse image \( f^{-1}(B) = \text{Exc}(f) \). We note that \( Y \setminus \text{Exc}(f) \cong Z \setminus \mu^{-1}(B) \).

Because \( \mu_1(G_0) \cap (Z \setminus \mu^{-1}(B)) \cong f_1(G_0) \cap (Y \setminus \text{Exc}(f)) \neq \emptyset \), \( \mu_1(G_0) \) and \( f_1(G_0) \) are birationally equivalent.
Thus $\mu_1(G_0)$ and $f(f_1(G_0))$ are birationally equivalent. Hence $\mu^* M$ is nef and big on $\mu_1(G_0)$. Therefore $\mu^* M$ is nef and log big on $(Z, \mu_*^{-1}\Delta + (F)_Z)$.

Thus we may assume that $X$ is $\mathbb{Q}$-factorial.

**Step 2.** From 3.8 of [Sh], every irreducible component of $|\Delta|$ is normal. Let $S$ be an irreducible component of $|\Delta|$.

We put $S_0 := f^{-1}_* S$, $\text{Diff}(0) := (f | S_0)_* (f^*(K_S + S) | S_0 - (K_Y + S_0) | S_0)$ and $\text{Diff}(\Delta - S) := \text{Diff}(0) + (\Delta - S) | S$. We note that $\text{Diff}(0) \geq 0$ (the Subadjunction Lemma, 3.2.2 of [Sh], cf. 5-1-9 of [KMM]). Then $K_{S_0} + (f_*^{-1}\Delta - S_0) | S_0 + F | S_0 = f^*(K_S + \text{Diff}(\Delta - S)) + E | S_0$.

Because $(f | S_0)_* ((f_*^{-1}\Delta - S_0) | S_0 + F | S_0 - E | S_0) = \text{Diff}(\Delta - S)$ and $\text{Diff}(\Delta - S) = \text{Diff}(0) + (\Delta - S) | S \geq 0$, $[\text{Diff}(\Delta - S)]$ is reduced. Here $M | S$ is nef and big. And, for any member $\Gamma$ of $\text{Strata}((f_*^{-1}[\Delta] - S_0) | S_0)$, $M | f(\Gamma)$ is nef and big. Hence $(S, \text{Diff}(\Delta - S))$ is log terminal and $M | S$ is nef and log big on $(S, \text{Diff}(\Delta - S))$.

We note that $\text{Exc}(f | S_0)$ may not consist of divisors. But, because $\text{Exc}(f) \cap S_0$ includes $\text{Exc}(f | S_0)$, $\text{Exc}(f | S_0)$ does not include any member of $\text{Strata}((f_*^{-1}[\Delta] - S_0) | S_0)$. Hence, by a finite composition of blowing ups with centers included in $\text{Exc}(f | S_0)$, we get a log resolution of $(S, \text{Diff}(\Delta - S))$ such that the exceptional locus consists of divisors with discrepancies $>-1$ (because, for every divisor $\nu \in \mathbb{C}(S)$ whose discrepancy with respect to $(S, \text{Diff}(\Delta - S))$ is $-1$, center $S_0(\nu) \in \text{Strata}((f_*^{-1}[\Delta] - S_0) | S_0)$). Therefore $(S, \text{Diff}(\Delta - S))$ is wklt (Lemma 4 of [Ka], [Sz]).

Thus $|mL | S |$ is base point free for $m \gg 0$, by induction hypothesis.

**Step 3.** We consider the exact sequence:

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$ 

Tensoring with $\mathcal{O}_X(mL)$ for $m \geq a$, we have an exact sequence

$$0 \to \mathcal{O}_X(mL - S) \to \mathcal{O}_X(mL) \to \mathcal{O}_S(mL) \to 0$$

(Since the sheaf $\mathcal{O}_X(mL)$ is invertible, $\mathcal{O}_X(mL) \otimes \mathcal{O}_X(-S) = \mathcal{O}_X(mL - S)$).

Because $mL - S - (K_X + \Delta - S)$ is nef and log big on $(X, \Delta - S)$, $H^1(X, \mathcal{O}_X(mL - S)) = 0$ from Proposition 1. Hence the linear system
$|mL|$ on $X$ cuts out a complete linear system $|mL|_S$ on $S$. Therefore $\text{Bs}|mL| \cap |\Delta| = \emptyset$ for $m \gg 0$.

Thus Proposition 2 implies the assertion. □

Remark. The proof above implies that the main theorem holds also under the conditions that $\dim X = 4$ and that $X$ is $\mathbb{Q}$-factorial.

More generally our proof shows the following implication:
If the log minimal model program works in all dimensions, then the main theorem holds without the condition that $\dim X \leq 3$.

References


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