On a Zelevinsky Theorem and the Schur Indices of the Finite Unitary Groups

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Abstract. Let $G$ be the finite unitary group $U_n(F_q)$ over a finite field $F_q$ of characteristic $p$. Let $U$ be a Sylow $p$-subgroup of $G$. We prove that, for any irreducible character $\chi$ of $G$ that is contained in a certain class, there is a linear character $\lambda$ of $U$ such that $(\lambda^G, \chi)_G = 1$. As an application, we shall determine the local Schur indices of an irreducible character of $G$ which belongs to such class.

1. Introduction

Let $F_q$ be a finite field with $q$ elements of characteristic $p$. In [4] I. M. Gel’fand and M. I. Graev proved:

Theorem A (Gel’fand-Graev [4, Theorems 1, 2]). Let $H$ be the special linear group $SL_n(F_q)$ over $F_q$, and let $U$ be the upper-triangular maximal unipotent subgroup of $H$. Then

(i) For any irreducible character $\chi$ of $H$, there is a linear character $\lambda$ of $U$ such that $(\lambda^H, \chi)_H \neq 0$.

(ii) If $\lambda$ is a linear character of $U$ in “general position”, then $\lambda^H$ is multiplicity-free.

It is well known that the assertion (ii) of Theorem A holds for any finite group of Lie type (T. Yokonuma [22], R. Steinberg [21, Theorem 49]; cf. R. W. Carter [1, Theorem 8.1.3]). But the assertion (i) of Theorem A does not hold generally for a finite group of Lie type (e.g. for $U_n(F_q)$, $Sp_{2n}(F_q)$, etc.).

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In [23] A. V. Zelevinsky proved:

**Theorem B** (Zelevinsky [23, 12.5]). *Let* $H$ *be the general linear group* $GL_n(F_q)$ *and let* $U$ *be the upper-triangular maximal unipotent subgroup of* $H$. *Then, for any irreducible character* $\chi$ *of* $H$, *there is a linear character* $\lambda$ *of* $U$ *such that* $(\lambda^H, \chi)_H = 1$.

As an application, Zelevinsky proved:

**Theorem C** (Zelevinsky [23, 12.6], A. A. Kljačko [10]; cf. [15] for $p \neq 2$). *The Schur index* $m_Q(\chi)$ *of any irreducible character* $\chi$ *of* $GL_n(F_q)$ *with respect to* $Q$ *is equal to one.

The purpose of this paper is to show that Zelevinsky’s Theorem B holds for a certain class of irreducible characters of $U_n(F_q)$ (Theorem 1), and, as an application, we show that, for any irreducible character $\chi$ of $U_n(F_q)$ contained in such class, we can determine the local Schur indices of $\chi$ in principle (Theorem 4).

As to the Schur indices of the irreducible characters of $G = U_n(F_q)$, it is known that $m_Q(\chi) \leq 2$ for any irreducible character $\chi$ of $G$ (R. Gow [5]) and that, for any irreducible character $\chi$ of $G$, we have $m_Q(\chi) = 1$ for any prime number $l \neq p$ ([16]; for $p = 2$, we use some properties of the generalized Gelfand-Graev characters of $G$ [9]). For $n \leq 5$, all the local Schur indices of every irreducible character of $G$ are completely determined ([17, 6]). Our result here is a certain contribution to the complete determination of the local Schur indices of all the irreducible characters of $G$. (In another paper [19], we give some sufficient conditions subject for that $m_Q(\chi) = 1$.)

As to the use of Kawanaka’s generalized Gelfand-Graev characters of a finite group of Lie type for the study of the rationality-properties of the irreducible character of such a group, we refer [18].

2. The unipotent values

2.1. Partitions

Let $m$ be a positive integer. Let $\mathcal{P}_m$ be the set of all partitions of $m$. If $\mu$ is a partition of $m$, then we write $|\mu| = m$. We denote by 0 the unique partition of the number 0. $\mathcal{P}_m$ has the lexicographical ordering.
If \( \mu = (m_1, \ldots, m_s) \) is a partition of \( m > 0 \) and \( \mu' = (m'_1, \ldots, m'_s) \) is a partition of \( m' > 0 \), then we denote by \( \mu + \mu' \) the partition \( (m_1 + m'_1, \ldots, m_s + m'_s) \) of \( m + m' \). If \( \mu = (m_1, m_2, \ldots, m_s) \) is a partition of \( m \) such that \( m_1 \geq m_2 \geq \cdots \geq m_s \geq 0 \) and \( \mu' = (m'_1, m'_2, \ldots, m'_s) \) is a partition of \( m' \) such that \( m'_1 \geq m'_2 \geq \cdots \geq m'_s \geq 0 \), then we denote by \( \mu \cdot \mu' \) the partition \( (m_1 + m'_1, m_2 + m'_2, \ldots, m_s + m'_s) \) of \( m + m' \). If \( d, v \) are positive integers and if \( \pi = (p_1, p_2, \ldots, p_s) \) is a partition of \( v \), then we denote by \( d \cdot \pi \) the partition \( (dp_1, dp_2, \ldots, dp_s) \) of \( dv \). If \( \mu \) is a partition of \( m \), then \( \bar{\mu} \) will denote the conjugate partition of \( \mu \).

Let \( S_m \) denote the symmetric group of order \( m! \). Then, as is well known, the conjugacy classes of \( S_m \) and the irreducible characters of \( S_m \) can be naturally parametrized by the partitions of \( m \). For \( \lambda, \rho \in \mathcal{P}_m \), let \( \chi^\lambda \) or \( \chi^\lambda(\rho) \) denote the value of the irreducible character \( \chi^\lambda \) of \( S_m \) corresponding to \( \lambda \) at the class of \( S_m \) corresponding to \( \rho \). It is well known that \( \chi^{(m)} = 1_{S_m} \), \( \chi^\lambda = \text{sgn} \) and \( \chi^\lambda = \text{sgn} \cdot \chi^\lambda \) and it is easy to see by induction on \( v \) that \( \text{sgn}(d \cdot \pi) = (-1)^{(d-1)r} \text{sgn}(\pi) \), \( \pi \in \mathcal{P}_v \).

### 2.2. The irreducible characters of \( U_n(F_q) \)

Let \( G = U_n(F_q) \). Then, as to the character theory, by thanks to the truth of Ennola conjecture ([3]; R. Hotta and T. A. Springer [8], G. Lusztig and B. Srinivasan [13], G. Lusztig, D. Kazhdan, N. Kawanaka [9]), we can use V. Ennola’s formulation in [3].

Let \( s \) be a positive integer. Then a set \( g = \{k, k(-q), k(-q)^2, \ldots, k(-q)^{s-1}\} \) of integers will be called an \( s \)-simplex with the roots \( k(-q)^i \), \( 0 \leq i \leq s-1 \), if the \( k(-q)^i \) are all distinct modulo \( q - (-1)^s \); we write \( d(g) = s \). Let \( \mathcal{Y} \) be the set of all \( s \)-simplexes for \( s \geq 1 \). Put \( \mathcal{P} = \bigcup_{m \geq 0} \mathcal{P}_m(\mathcal{P}_0 = \{0\}) \). Let \( X \) be the set of functions \( \nu: \mathcal{Y} \to \mathcal{P} \) such that

\[
\sum_{g \in \mathcal{Y}} |\nu(g)| d(g) = n.
\]

For \( \nu \in X \), set (formally)

\[
\chi_\nu = (\cdots g^{\nu(g)} \cdots) = (g_1^{\nu_1} \cdots g_N^{\nu_N}),
\]

where \( g_1, \ldots, g_N \) are all the \( g \in \mathcal{Y} \) such that \( \nu(g) \neq 0 \) and, for \( 1 \leq i \leq N \), \( \nu_i = \nu(g_i) \). Then the \( \chi_\nu, \nu \in X \), parametrize the irreducible characters of \( G \).
For $\nu \in X$, we identify $\chi_\nu$ with the irreducible character of $G$ corresponding to it.

Let $Q_\rho^\lambda(q)$ be the Green polynomial of $GL_n(F_q)$ ([7]). For $\pi = (1^{r_1}2^{r_2}3^{r_3}\cdots)$ $\in \mathcal{P}_\nu$, put $z_\pi = 1^{r_1}r_1!2^{r_2}2!3^{r_3}3!\cdots$. If $n_1,\ldots,n_N$ are positive integers, then we put $\mathcal{P}(n_1,\ldots,n_N) = \mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_N}$.

**Proposition 1.** Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be any irreducible character of $G = U_n(F_q)$. For $1 \leq i \leq N$, put $d_i = d(g_i)$ and $v_i = |\nu_i|$. Let $\lambda$ be a partition of $n$, and let $u_\lambda$ be any unipotent element of $G$ of type $\lambda$. Then we have:

$$\chi(u_\lambda) = \eta(\chi) \sum_{(\pi_1,\ldots,\pi_N) \in \mathcal{P}(v_1,\ldots,v_N)} \frac{1}{z_{\pi_1} \cdots z_{\pi_N}} \chi_{\pi_1}^{\nu_1} \cdots \chi_{\pi_N}^{\nu_N} \times Q_{d_1,\pi_1+\cdots+d_N,\pi_N}^\lambda(-q),$$

where $\eta(\chi) = \pm 1$ such that $\chi(u_\lambda) > 0$ if $\lambda = (1^n)$.

**Remark.** We remark here about the relation between Ennola’s parametrization of the irreducible characters of $G = U_n(F_q)$ and G. Lusztig’s parametrization ([11, 12]; also see [1, pp. 391–2]). Let $G = GL_n(F_q)$, where $F_q$ is an algebraic closure of $F_q$, and let $F': G \to G$ be the endomorphism of $G$ given by $F'([g_{ij}]) = t[q_{ij}]^{-1}$ for $[g_{ij}] \in G$. Then $F'$ is the Frobenius map relative to some $F_q$-structure on $G$, and the group $G(F_q) = G^{F'}$ of $F$-fixed points of $G$ is isomorphic to $G$. The dual group $G^\#$ of $G$ is isomorphic to $G$. Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be an irreducible character of $G$. Then $\chi$ is a unipotent character of $G$ if and only if $N = 1, d(g_1) = 1$ and 0 is the root of $g_1$. $\chi$ is a semisimple character of $G$ (i.e. $p \nmid \chi(1)$) if and only if, $1 \leq i \leq N$, $\nu_i = (v_i)$ ($v_i = |\nu_i|$). And $\chi$ is a regular character of $G$ (i.e. an irreducible component of the Gelfand-Graev character $\Gamma_G$ of $G$) if and only if, for $1 \leq i \leq N$, $\nu_i = (1^{v_i})$. Generally, the dual class $(g_1^{1^{v_1}} \cdots g_N^{1^{v_N}})$ determines the unique semisimple conjugacy class $(s)$ of $G = G^\#(F_q)$ (see [3, pp. 6–7]). The partitions $\nu_1,\ldots,\nu_N$ determine a unique unipotent character $\rho$ of $H(s) = (Z_G^\#(s))^\#(F_q)$ ($Z_G^\#(s)$ is the centralizer of $s$ in $G^\#$). We see easily that $\chi(1) = \chi_s(1)\rho(1)$, where $\chi_s$ is the semisimple character $(g_1^{v_1} \cdots g_N^{v_N})$. This may be regarded as the “Jordan decomposition” of $\chi$. Thus we can regard the mapping $(s, \rho) \to \chi$ as Lusztig’s parametrization mapping for the irreducible characters of $G$ (cf. [11]).
3. Linear characters of $U$

3.1. We say that a partition $\mu$ of $n$ is involutive if the parts of $\mu$ are arranged so that $\mu = (n_1, n_2, \ldots, n_s, n_{s+1}, n_s, \ldots, n_2, n_1)$ (possibly $n_{s+1} = 0$). For example, if $n = 4$, then $(4), (2^2), (21^2)$ and $(1^4)$ are the involutive partitions of 4 and (31) is not involutive.

Let $G = GL_n(\mathbb{F}_q)$, and let $F: G \to G$ be the endomorphism of $G$ given by $F([g_{ij}]) = w_0^t [g_{ij}^q]^{-1} w_0$, where $w_0 = \begin{bmatrix} 0 & 1 \\ \vdots & \ddots & 1 \\ 1 & \cdots & 0 \end{bmatrix}$. Then $G^F \simeq U_n(\mathbb{F}_q)$.

Let $U$ be the upper triangular maximal unipotent subgroup of $G$. Then $U$ is $F$-stable and $U = U^F$ is a Sylow $p$-subgroup of $G$. For $1 \leq k \leq n - 1$, set $U_k = \{u = [u_{ij}] \in U \mid u_{i,i+1} = 0 \text{ for } i \neq k \text{ and } u_{ij} = 0 \text{ if } j - i \geq 2\}$. Then, for $1 \leq k \leq n - 1$, we have $F(U_k) = U_{n-k}$, so $F$ acts on $\Delta = \{1, 2, \ldots, n-1\}$ by $F(U_k) = U_{F(k)}$. Let $I$ be the set of orbits of $F$ on $\Delta$. Let $U_i$ be the derived group of $U$. Then $U/U_i = \prod_{k \in I} U_k$. For $i \in I$, set $U_i = \prod_{k \in I} U_k$. Then we have $U^F/U_i^F = (U/U_i)^F = \prod_{i \in I} U_i^F$. For $i \in I$, we have $U_i^F \simeq \mathbb{F}_{q^2}$ or $\mathbb{F}_q$.

Let $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$ be any involutive partition of $n$, and put

$$L_\mu = \begin{cases} \begin{bmatrix} A_1 & \cdots & 0 \\ & \ddots & \vdots \\ & & A_s \\ B_{s+1} & & \\ 0 & \ddots & \vdots \\ & & A'_s \\ & & & \ddots \end{bmatrix} & | A_i, A'_i \in GL_{n_i}(\mathbb{F}_q), \end{cases}$$

$1 \leq i \leq s, A_{s+1} \in GL_{n_{s+1}}(\mathbb{F}_q)$

(A_{s+1} does not occur in the above expression if $n_{s+1} = 0$). Put $P_\mu = L_\mu U$. Then $P_\mu$ is an $F$-stable parabolic subgroup of $G$ and $L_\mu$ is an $F$-stable Levi subgroup of $P_\mu$. We put $P_\mu = P^F_\mu$ and $L_\mu = L^F_\mu$. 
Let $\phi$ be a linear character of $U$. Then $\phi$ can be regarded as a character of $U/U. (U. = U.F )$. We say that $\phi$ is of type $\mu$ if, for $i \in I$, $\phi$ is non-trivial on $U_i = U_i^F$ if $U_i \subset L_\mu$ and trivial on $U_i$ if $U_i \not\subset L_\mu$. Conversely, it will be clear that if $\phi$ is any linear character of $U$, then there is uniquely determined involutive partition $\mu$ of $n$ such that $\phi$ is of type $\mu$.

Let $\phi$ be any linear character of $U$ of type $\mu$. Let $\Gamma_\mu$ be the Gelfand-Graev character of $L_\mu$. We have

$$\phi^G = \text{Ind}_{P_\mu}(\Gamma_\mu),$$

where we regard $\Gamma_\mu$ as a character of $P_\mu$ through the natural map $P_\mu \to P_\mu/V_\mu = L_\mu$ ($V_\mu$ is the unipotent radical of $P_\mu$).

4. Induced characters of $G$

4.1.

Let $G = GL_n(\bar{F}_q)$ and let $F': G \to G$ be the endomorphism of $G$ given by $F'([g_{ij}]) = t^{[g_{ij}^q]}$. Then $F'$ is the Frobenius map of $G$ corresponding to some $F_q$-rational structure on $G$. We have $G^{F'} \cong U_n(F_q)$.

Let $T_0$ be the diagonal maximal torus of $G$. Then $T_0$ is $F'$-stable. Let $W = W_G = N_G(T_0)/T_0$, where $N_G(T_0)$ is the normalizer of $T_0$ in $G$. Then $F'$ acts on $W$ trivially. $W$ can be naturally identified with the symmetric group $S_n$. The $G^{F'}$-conjugacy classes of $F$-stable maximal tori of $G$ can be parametrized by the conjugacy classes of $W = S_n$, and the latter can be parametrized by the partitions of $n$. For $\rho \in P_n$, let $T_\rho$ denote one of the $F'$-stable maximal tori of $G$ corresponding to $\rho$.

Let $\rho$ be a partition of $n$, and suppose that $T_\rho = yT_0y^{-1}$, $y \in G$. Put $w = y^{-1}F'(y) \mod T_0 \in W$. Then $y$ induces an identification: $(F' \text{ on } T_\rho) = (\text{ad } w \circ F' \text{ on } T_0)$, so we have:

$$|T_\rho^{F'}| = |T_\rho^{\text{ad } w \circ F'}| = |c_\rho(-q)|,$$

where if $\rho = (r_1^1r_2^2r_3^3 \ldots)$, then $c_\rho(q) = (q - 1)^{r_1}(q^2 - 1)^{r_2}(q^3 - 1)^{r_3} \ldots$.

In the following, if $S$ is a maximal torus of a connected reductive group $M$, then we write $W_M(S) = N_M(S)/S$. 
Let $\rho$ be a partition of $n$, and let $s_\rho$ be an element of $S_n$ contained in the class of $S_n$ corresponding to $\rho$. Then $W_G(T_\rho)^{F'}$ is isomorphic to $W_G(T_0)^{\text{ad } w_0 F'} = Z_{S_n}(s_\rho)$, so we have

$$|W_G(T_\rho)^{F'}| = |Z_{S_n}(s_\rho)| = z_\rho.$$

Let $F: G \to G$ be as in §3. Then $F$ acts on $W$ by $\text{ad } w_0$, and the $G^F$-conjugacy classes of $F$-stable maximal tori of $G$ can be parametrized by the $F$-conjugacy classes in $S_n$ ([2, p. 107]). For $w \in W$, let $T(w)$ denote one of the $F$-stable maximal tori of $G$ corresponding to $w$.

Let $w \in W$, and suppose that $T(w) = zT_0z^{-1}$ with $z^{-1}F(z) \mod T_0 = w$. Then $\text{ad } z$ induces an identification: $(F \text{ on } T(w)) = (\text{ad } w \circ F \text{ on } T_0)$, so we have

$$|T(w)^F| = |T_0^{\text{ad } w_0 F'}| = |T_0^{\text{ad } w_0 \circ F'}| = |T_{\rho(ww_0)}^F|,$$

where $\rho(ww_0)$ is the partition of $n$ corresponding to the class of $W = S_n$ containing $ww_0$.

In the following, if $M$ is an $F$-stable (resp. $F'$-stable) reductive subgroup of $G$, then we denote by $\sigma(M)$ (resp. by $\sigma'(M)$) the $F_q$-rank of $M$ with respect to the $F_q$-rational structure on $M$ determined by $F$ (resp. by $F'$). Then it is easy to see that, for $w \in W$, $\sigma(T(w)) = \sigma'(T_{\rho(ww_0)})$.

Let $\rho$ be any partition of $n$. Then it is easy to see that $\sigma'(T_\rho)$ is equal to the number of even parts of $\rho$. So we have

$$(-1)^{\sigma(T_\rho)} = \text{sgn}(\rho).$$

We have $\sigma(G) = \sigma(T_0) = \lfloor n/2 \rfloor$, the integral part of $n/2$, so we have

$$(-1)^{\sigma(G) - \sigma(T(w))} = (-1)^{\lfloor n/2 \rfloor} \text{sgn}(\rho(ww_0)), \quad w \in W.$$

4.2. Green function

Let $M$ be a connected, reductive algebraic group, defined over $F_q$, and let $F'' : M \to M$ be the corresponding Frobenius endomorphism. If $S$ is an $F''$-stable maximal torus of $M$ and $\theta$ is a character of $S^{F''}$, then we denote by $R_M^S(\theta)$ the Deligne-Lusztig virtual character of $M^{F''}$, and by $Q_{S,M}$ the corresponding Green function. We shall often consider $Q_{S,M}$ as a function on all $M^{F''}$ by putting $Q_{S,M}(x) = 0$ whenever $x$ is not unipotent.
Now assume that $M = G$ with $F'' = F$ or $F'$. Let $x$ be an element of $G$ such that $x^{-1}F(x) = w_0$. Then $\text{ad} \ x$ induces a bijection from $G^{F'}$ onto $G^F$, and we have $Q_{T_{\rho(w_0)}, G}(g) = Q_{T(w), G}(\text{ad} \ x(g)), g \in G^{F'}$. Let $\lambda$ be a partition of $n$, and let $u_\lambda$ (resp. $u'_\lambda$) denote a unipotent element of $G^F$ (resp. of $G^{F'}$) of type $\lambda$. Then, by the result of Hotta-Springer-Kawanaka ([8], [9]), we have

$$Q_{T(ww_0), G}(u_\lambda) = Q_{T(w), G}(u'_\lambda), \quad w \in W.$$

### 4.3. The Gelfand-Graev character

Let $\Gamma_G$ be the Gelfand-Graev character of $G^{F'}$ (let $\phi'$ be the linear character of $U' = (\text{ad} \ x)^{-1}(U)$ corresponding (via the bijection $\text{ad} \ x: G^{F'} \to G^F$ in 4.2) to a linear character $\phi$ of $U$ of type $(n)$; then $\Gamma_G = \text{Ind}_{U'}^{G^{F'}}(\phi')$). Then, by Theorem 10.7 of [2], we have

$$\Gamma_G = \sum_{(T, \theta) \mod G^{F'}} (-1)^{\sigma(G) - \sigma(T)} \frac{(R_T^G(\theta), R_T^G(\theta))_{G^{F'}}}{R_T^G(\theta)} G^{F'},$$

where the sum is taken over all $G^{F'}$-conjugacy classes of pairs $(T, \theta)$ of $F'$-stable maximal tori $T$ of $G$ and characters $\theta$ of $T^{F'}$. Let $\rho$ be any partition of $n$. Then, by using [2, Theorem 6.8], we see that

$$\sum_{\theta \mod W_G(T_\rho)^{F'}} \frac{1}{(R_{T_\rho}(\theta), R_{T_\rho}(\theta))_{G^{F'}}} = \frac{|T_\rho^{F'}|}{z_\rho},$$

Thus we get

$$\Gamma_G = (-1)^{[n/2]} \sum_{\rho \in P_n} \text{sgn}(\rho) \frac{|T_\rho^{F'}|}{z_\rho} Q_{T_\rho, G}.$$

(1) $\Gamma_G = (-1)^{[n/2]} \sum_{\rho \in P_n} \text{sgn}(\rho) \frac{|T_\rho^{F'}|}{z_\rho} Q_{T_\rho, G}.$
4.4. Degenerate linear characters

Let $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$ be any involutive partition of $n$, and let $\phi$ be any linear character of $U = U^F$ of type $\mu$. We assume that $\mu \neq (n)$. Let $W_\mu = W_{L_\mu}(T_0)$ (a subgroup of $W = W_G(T_0)$). Then we have

$$W_\mu = S_\mu = S_{n_1} \times \cdots \times S_{n_s} \times S_{n_{s+1}} \times S_{n_s} \times \cdots \times S_{n_1}.$$ 

By [2, Theorem 10.7, Proposition 8.2], we have

$$\phi^{G^F} = \sum_{T \mod L_\mu \atop (T \subset L_\mu)} (-1)^{\sigma(G) - \sigma(T)} \frac{|T^F|}{|W_{L_\mu}(T)^F|} Q_{T,G},$$

where the sum is taken over all $L_\mu$-conjugacy classes of $F$-stable maximal tori $T$ of $L_\mu$.

$F$ acts on $W_{L_\mu}(T_0) = S_\mu$ by ad $w_0$. The $L_\mu$-conjugacy classes of $F$-stable maximal tori of $L_\mu$ can be parametrized by the $F$-conjugacy classes of $S_\mu$. $S_\mu$ acts on $S_\mu w_0$ by conjugations. We see that, for $w_1, w_2 \in S_\mu, w_1$ is $F$-conjugate to $w_2$ in $S_\mu$ if and only if $w_1 w_0$ is $S_\mu$-conjugate to $w_2 w_0$ in $S_\mu w_0$.

Let $w$ be an element of $S_\mu$, and suppose that $T(w) = y T_0 y^{-1}, y \in L_\mu (y^{-1} F(y) \mod T_0 = w)$. Then ad $y$ induces an identification: $(F$ on $W_{L_\mu}(T(w))) = (\text{ad } w \circ F$ on $W_{L_\mu}(T_0))$. Therefore we have:

$$|W_{L_\mu}(T(w))^F| = |W^{\text{ad } w \circ F'}|$$
$$= |W^{\text{ad } w_0 \circ F'}|$$
$$= |Z_{W_\mu}(w w_0)| \quad (F' = \text{id. on } W_\mu)$$
$$= \frac{|W_\mu|}{|K_{W_\mu w_0}(w w_0)|},$$

where $K_{W_\mu w_0}(w w_0)$ is the $W_\mu$-orbit of $w w_0$ in $W_\mu w_0$ under the conjugate action of $W_\mu$.

Therefore we have

$$\phi^{G^F} = \sum_{w w_0 \mod W \atop (w \in W_\mu)} (-1)^{\sigma(G) - \sigma(T(w))} \frac{|T(w)^F|}{|W_{L_\mu}(T(w))^F|} Q_{T(w),G},$$
so, if $\phi'$ is the linear character of $U' = (\text{ad } x)^{-1}(U)$ corresponding to the linear character $\phi$ of $U$, we have

$$
\phi' G^{F'} = \sum_{w w_0 \mod W_\mu \atop (w \in W_\mu)} (-1)^{\sigma'(G) - \sigma'(T_{\rho(w w_0)})} |T_{\rho(w w_0)}^{F'}| \times \frac{|K_{w_\mu w_0}(w w_0)|}{|W_\mu|} Q_{T_{\rho(w w_0)}, G}
$$

$$
= \sum_{w' \mod W \atop (w' \in W)} \sum_{w w_0 \mod W_\mu \atop (w \in W_\mu)} (-1)^{\sigma'(G) - \sigma'(T_{\rho(w')})} \times |T_{\rho(w')}^{F'}| \frac{|K_{w_\mu w_0}(w w_0)|}{|W_\mu|} \bigg\{ Q_{T_{\rho(w')}, G} \bigg\}
$$

(“$\sim$” means conjugate in $W$)

$$
= \sum_{\rho \in \mathcal{P}_n} (-1)^{[n/2]} \text{sgn}(\rho) |T_{\rho}^{F'}| \frac{|K_{S_n}(s_\rho) \cap S_\mu w_0|}{|S_\mu|} Q_{T_{\rho}, G},
$$

where, for $\rho \in \mathcal{P}_n$, $s_\rho$ is an element of $S_n$ contained in the class of $S_n$ corresponding to $\rho$ and $K_{S_n}(s_\rho)$ denotes the class of $s_\rho$ in $S_n$.

Let us express the $|K_{S_n}(s_\rho) \cap S_\mu w_0|/|S_\mu|$ in terms of characters of $S_n$. Let $H = \langle S_\mu, w_0 \rangle$. Then $(H : S_\mu) = 2$ (note that $w_0 \notin S_\mu$ and $w_0 S_\mu w_0 = S_\mu$). Let $\xi$ be the linear character of $H$ defined by

$$
\xi(x) = \begin{cases} 
1 & \text{if } x \in S_\mu, \\
-1 & \text{if } x \in H - S_\mu.
\end{cases}
$$

Let

$$
\chi = 1_H - \xi.
$$

Then we have

$$
\chi_{S_n}(s_\rho) = |K_{S_n}(s_\rho) \cap S_\mu w_0|/|S_\mu| |Z_{S_n}(s_\rho)|.
$$

It is well known that one has:

$$
1_{S_\mu}^{S_n} = \chi^\mu + \sum_{\nu > \mu} k^{\mu \nu} \chi^\nu,
$$
where the $k^\nu$ are certain non-negative integers. As $1^{S_n}_{S_\mu} = 1^S_n + \xi^S_n$, we see that the irreducible components of $1^S_n$ and $\xi^S_n$ are contained in $1^{S_n}_{S_\mu}$; we have

$$
\chi^S_n = 1^S_n - \xi^S_n = \epsilon_\mu \chi^\mu + \sum_{\nu > \mu} c^\nu_\mu \chi^\nu,
$$

where $\epsilon_\mu = 1$ or $-1$ according as $\chi^\mu$ is contained in $1^S_n$ or $\xi^S_n$ respectively and the $c^\nu_\mu$ are some integers. Thus we have:

\begin{equation}
\phi^{G,F'} = \sum_{\rho \in P_n} (-1)^{[n/2]} \text{sgn}(\rho) \left( \epsilon_\mu \chi^\mu + \sum_{\nu > \mu} c^\nu_\mu \chi^\nu \right) \frac{|T^F_\rho|}{z_\rho} Q_{T_\rho,G}.
\end{equation}

5. Inner products

5.1. Some preliminaries

Let $m$ be a positive integer, and let $x_1, \ldots, x_m$ be $m$ different variables over $\mathbb{Q}$. For a partition $\lambda = (l_1, \ldots, l_m)$ of $m$ with $l_1 \geq \cdots \geq l_m \geq 0$, set

$$
s_\lambda(x_1, \ldots, x_m) = \frac{\det[x_{i+m-j}]_{1 \leq i,j \leq m}}{\det[x_{i-j}]_{1 \leq i,j \leq m}},
$$

which we call the $S$-function in the variables $x_1, \ldots, x_m$ corresponding to $\lambda$ (see Macdonald [14, p. 24]).

Let $m_1, \ldots, m_k$ be positive integers such that $m_1 + \cdots + m_k = m$, and, for $1 \leq i \leq k$, let $\lambda_i$ be a partition of $m_i$. Let $x_1, \ldots, x_{m_1}; y_1, \ldots, y_{m_2}; \ldots, z_1, \ldots, z_{m_k}$ be independent variables. Suppose that

$$
s_{\lambda_1}(x_1, \ldots, x_{m_1})s_{\lambda_2}(y_1, \ldots, y_{m_2}) \cdots s_{\lambda_k}(z_1, \ldots, z_{m_k})
= \sum_{\lambda \in P_m} c_{\lambda_1\lambda_2\ldots\lambda_k}^\lambda s_{\lambda}(x_1, \ldots, x_{m_1}; y_1, \ldots, y_{m_2}; \ldots; z_1, \ldots, z_{m_k}),
$$

where $c_{\lambda_1\lambda_2\ldots\lambda_k}^\lambda$'s are some non-negative integers. Then we have ([14, I, (7.3))):

$$
\text{Ind}_{S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}}^{S_m} (\chi^{\lambda_1} \times \chi^{\lambda_2} \times \cdots \times \chi^{\lambda_k}) = \sum_{\lambda \in P_m} c_{\lambda_1\lambda_2\ldots\lambda_k}^\lambda \chi^\lambda.
$$
Lemma 1 (see, e.g., [15, (2.4)]). If $\lambda > \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_k$ or $\lambda < \lambda_1 + \lambda_2 + \cdots + \lambda_k$, then we have $c^\lambda_{\lambda_1 \lambda_2 \cdots \lambda_k} = 0$, and, if $\lambda = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_k$ or $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, then we have $c^\lambda_{\lambda_1 \lambda_2 \cdots \lambda_k} = 1$.

By the Frobenius reciprocity law, we get:

$$\chi^\lambda |_{S_{m_1} \times \cdots \times S_{m_k}} = \sum_{(\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_k}} c^\lambda_{\lambda_1 \cdots \lambda_k} \chi^\lambda_1 \times \cdots \times \chi^\lambda_k.$$

5.2. Let $G = G^{F'} \simeq U_n(F_q)$. Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be any irreducible character of $G$. For $1 \leq i \leq N$, put $d_i = d(g_i)$, $v_i = |\nu_i|$. For a partition $\rho$ of $n$, we put $Q_{\rho,G} = Q_{T_\rho,G}$. Then, by Proposition 1, by the formula (1) and by the orthogonality relations for the Green functions of $G$, we have:

$$(\Gamma_G, \chi)_G = (-1)^{[n/2] + \sum_{i=1}^N (d_i - 1)v_i} \eta(\chi) \prod_{i=1}^N (\chi^{\nu_i}, 1_{S_{v_i}})_{S_{v_i}}$$

$$= \begin{cases} 1 & \text{if } \nu_i = (1^{v_i}) \text{ for } 1 \leq i \leq N, \\ 0 & \text{if } \nu_i \neq (1^{v_i}) \text{ for some } i. \end{cases}$$

This is a known result (see the remark in §2.2).

Next, suppose that $\phi$ is any linear character of $U = U^F$, of type $\mu$, and suppose that $\mu \neq (n)$. Let $x : G \rightarrow G^F$ be an isomorphism as before $(x^{-1}F(x) = w_0)$, and let $\phi'$ be the linear character of $U' = (ad x)^{-1}(U)$ corresponding to $\phi$ via $ad x$. Then, by Proposition 1 and the formula (2), we get:

$$(\phi'^G, \chi)_G = (-1)^{[n/2]} \sum_{\rho \in \mathcal{P}_n} \sgn(\rho) \left( \epsilon_\mu \chi_\rho + \sum_{\nu > \mu} c^\nu_\mu \chi_\rho \frac{|c_\rho(-q)|}{z_\rho} \right) \times \eta(\chi)$$

$$\times \sum_{\pi_1, \ldots, \pi_N \in \mathcal{P}_{v_1} \times \cdots \times \mathcal{P}_{v_N}} \frac{1}{z_{\pi_1} \cdots z_{\pi_N}} \chi^{\nu_1}_{\pi_1} \cdots \chi^{\nu_N}_{\pi_N} (Q_{\rho,G}, Q_{\sigma,G})_G.$$
By the orthogonality relations for the Green functions, we see that the latter expression of the above equality is equal to

\[ (-1)^{[n/2]} \eta(\chi) \sum_{\rho=d_1 \cdot \pi_1 + \cdots + d_N \cdot \pi_N} \text{sgn}(\rho) \left( \epsilon_\mu \chi_\rho^\mu + \sum_{\nu > \mu} c^\nu_\mu \chi_\rho^\nu \right) \]

\[ \times \frac{1}{z_{\pi_1} \cdots z_{\pi_N}} \chi_1^\nu_1 \cdots \chi_N^\nu_N \]

\[ = (-1)^{[n/2] + \sum_{i=1}^N (d_i - 1) v_i} \eta(\chi) \sum_{\rho=d_1 \cdot \pi_1 + \cdots + d_N \cdot \pi_N} \left( \epsilon_\mu \chi_\rho^\mu + \sum_{\nu > \mu} c^\nu_\mu \chi_\rho^\nu \right) \]

\[ \times \frac{1}{z_{\pi_1} \cdots z_{\pi_N}} \chi_1^\nu_1 \cdots \chi_N^\nu_N . \]

Put \( \eta(\chi)' = (-1)^{[n/2] + \sum_{i=1}^N (d_i - 1) v_i} \eta(\chi) \). For \( 1 \leq i \leq N \), put \( n_i = d_i v_i \). Then, by a remark in 5.1, we see that the last expression in the above equality is equal to:

\[ \eta(\chi)' \sum_{\pi_1, \ldots, \pi_N} \sum_{(\xi_1, \ldots, \xi_N) \in P_{n_1} \times \cdots \times P_{n_N}} \left( \epsilon_\mu c^\mu_\xi_1 \cdots \xi_N + \sum_{\nu > \mu} c^\nu_\mu c^\nu_{\xi_1} \cdots \xi_N \right) \]

\[ \times \chi_{d_1 \cdot \pi_1}^\xi_1 \cdots \chi_{d_N \cdot \pi_N}^\xi_N \frac{1}{z_{\pi_1} \cdots z_{\pi_N}} \chi_{\bar{\nu}_1} \cdots \chi_{\bar{\nu}_N} \]

\[ = \eta(\chi)' \left\{ \epsilon_\mu \sum_{\xi_1, \ldots, \xi_N} c^\mu_{\xi_1} \cdots \xi_N \prod_{i=1}^N \left( \sum_{\pi_i \in P_{v_i}} \frac{1}{z_{\pi_i}} \chi_{\bar{\nu}_i \pi_i \xi_i} \right) \right. \]

\[ + \sum_{\nu > \mu} c^\nu_\mu \sum_{\xi_1, \ldots, \xi_N} c^\nu_{\xi_1} \cdots \xi_N \prod_{i=1}^N \left( \sum_{\pi_i \in P_{v_i}} \frac{1}{z_{\pi_i}} \chi_{\bar{\nu}_i \pi_i \xi_i} \right) \left. \right\} . \]

**Lemma 2** ([15, (2.8)]). Let \( d, v \) be positive integers. Then one has

\[ \sum_{\pi \in P_v} \frac{1}{z_{\pi}} \chi_{\xi}^\nu \chi_{d \cdot \pi} = \begin{cases} 1 & \text{if } \xi = d \cdot \nu, \\ 0 & \text{id } \xi > d \cdot \nu. \end{cases} \]
Assume that $\mu = (d_1 \cdot \tilde{\nu}_1) \cdots (d_N \cdot \tilde{\nu}_N)$. Then, by Lemmas 1, 2, we see easily that the last expression in the above equality is equal to $\eta(\chi)'\epsilon_\mu$. But, as $(\phi^G, \chi)_G \geq 0$, we must have $(\phi^G, \chi)_G = 1$.

Thus we get

**THEOREM 1.** Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be an irreducible character of $G = U_n(F_q)$. Suppose that $\mu = (d(g_1) \cdot \tilde{\nu}_1) \cdots (d(g_N) \cdot \tilde{\nu}_N)$ is an involutive partition of $n$. Let $\phi$ be a linear character of a Sylow $p$-subgroup $U'$ of $G$ of “type $\mu$”. Then we have $(\phi^G, \chi)_G = 1$.

6. **The Schur index**

6.1.

Let $G = U_n(F_q)$. Then the following two results are known:

**THEOREM 2 (R. Gow [5]).** The Schur index $m_Q(\chi)$ of any irreducible character $\chi$ of $G$ with respect to $Q$ is at most two.

**THEOREM 3 (cf. [16] for $p \neq 2$).** Let $\chi$ be any irreducible character of $G$. Then, for any prime number $l \neq p$, we have $m_{Q_l}(\chi) = 1$.

In [16] Theorem 2 is proved for $p \neq 2$. We give here a proof of this theorem which is valid for all $p$. Let $\chi$ be any irreducible character of $G$. Then, by a result of Kawanaka [9], there is a generalized Gelfand-Graev character $\gamma_\mu$ of $G$ such that $(\gamma_\mu, \chi)_G = 1 ([9, (3.2.18), (3.3.24)(i)])$. $\gamma_\mu$ is of $Q$-valued ([9, (3.2.14)]) and is supported by a set of unipotent elements of $G$ (this is clear from the construction of $\gamma_\mu$). Then, by [20, Theorem 34 in p. 145, Proposition 33 in p. 106], we see that, for any prime number $l \neq p$, $\gamma_\mu$ is realizable in $Q_l$. Thus we have $m_{Q_l}(\chi) = 1$.

6.2.

Let us review some results in [17, §3]. Let $G$ be as above, and let $U$ be a Sylow $p$-subgroup of $G$. Let $U$. be the derived group of $U$. If $p = 2$, then $U/U$. is an elementary abelian 2-group, so that any linear character of $U$ is realizable in $Q$.

Assume that $p \neq 2$. Let $\zeta_p$ be a fixed primitive $p$-th root of unity, and let $\alpha$ be a certain generator of $\text{Gal}(Q(\zeta_p)/Q)$. 

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First, assume that $n$ is odd. Then there is an element $t$ in $N_G(U)$, of order $p-1$, such that $\phi^t = \phi^\alpha$ for any linear character $\phi$ of $U$, where $\phi^\alpha$ is the linear character of $U$ defined by $\phi^t(u) = \phi(tu^{-1})$, $u \in U$. Put $M = U(t)$. Then we see that, if $\phi$ is any non-principal linear character of $U$, $\phi^M$ is an irreducible character of $M$ which is realizable in $Q$. Therefore, for any linear character $\phi$ of $U$, $\phi^G$ is realizable in $Q$.

Next, assume that $n$ is even ($p \neq 2$). We use the notation in 3.1. Let $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$ be any involutive partition of $n$, and let $\phi$ be any linear character of $U$ of type $\mu$. Then we have $\phi^G = \text{Ind}_{P_\mu}^G(\Gamma_\mu)$. We have

$$L_\mu \simeq \prod_{i=1}^s GL_{n_i}(F_{q^2}) \times U_{n_{s+1}}(F_q).$$

Therefore, if $n_{s+1} = 0$, then, by Gow’s theorem ([5]), $\Gamma_\mu$ is realizable in $Q$, so $\phi^G$ is realizable in $Q$.

Assume that $n_{s+1} \neq 0$. Then, as $n$ is even, $n_{s+1}$ is even. There is an element $t'$ in $N_G(U)$, of order $(p-1)(q+1)$, such that $\phi^{t'} = \phi^\alpha$ and $c = t'^{p-1}$ is a generator of the centre $Z$ of $G$. Put $M' = U\langle t' \rangle$. For $0 \leq j \leq q$, let $\phi_j$ be the extension of $\phi$ to $U\langle c \rangle$ given by $\phi_j(c) = \zeta_{q+1}^j$, where $\zeta_{q+1}$ is a previously fixed primitive $(q+1)$-th root of unity. For $0 \leq j \leq q$, let $\nu_j = \phi_j^{M'}$. Then we see that the $\nu_j$ are irreducible characters of $M'$ and $\phi^{M'} = \nu_0 + \cdots + \nu_q$. For $0 \leq j \leq q$, let $k_j = Q(\nu_j)$, the field generated over $Q$ by the values of $\nu_j$. Then we have $k_j = Q(\zeta_{q+1}^j)$, $0 \leq j \leq q$. For $0 \leq j \leq q$, let $A_j$ be the simple component of the group algebra $k_j[M']$ of $M'$ over $k_j$ associated with $\nu_j$. Then, for $0 \leq j \leq q$, if $j \neq (q+1)/2$, $A_j$ splits in $k_j$, and if $j = (q+1)/2$, $k_j$ has non-zero Hasse invariants ($\equiv \frac{1}{2} \pmod{1}$) only at the places $\infty$, $p$ of $k_j = Q$.

We have:

**Theorem 4.** Let $\chi = (g_1^{F_1} \cdots g_N^{F_N})$ be an irreducible character of $G = U_n(F_q)$. Let $\mu = (d(g_1) \cdot \nu_1) \cdots (d(g_N) \cdot \nu_N)$. Assume that $\mu$ is an involutive partition of $n$, and suppose that $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$. Then:

(i) If $p = 2$, or $n$ is odd, or $n_{s+1} = 0$, then we have $m_Q(\chi) = 1$.

(ii) Assume that $p \neq 2$, $n$ is even, and $n_{s+1} \neq 0$.

Recall that $c$ is a generator of the centre of $G$. Then, if $\chi(c) \neq -\chi(1)$, we have $m_Q(\chi) = 1$. Assume that $\chi(c) = -\chi(1)$. Then we have $m_R(\chi) = 2$ or 1 according as $\chi$ is real or not respectively, and we have $m_{Q_R}(\chi) = 2$ or
$1$ according as $[\mathbb{Q}_p(\chi) : \mathbb{Q}_p]$ is odd or even respectively ($\mathbb{Q}_p(\chi)$ is the field generated over $\mathbb{Q}_p$ by the values of $\chi$).

**Proof.** We use the notation in 3.1. Let $\phi$ be any linear character of $U$ of type $\mu$. Then, by Theorem 1, we have $(\phi^G, \chi)_G = 1$. Then, as we have observed above, if $p = 2$, or $n$ is odd, or $n_{s+1} = 0$, $\phi^G$ is realizable in $\mathbb{Q}$, so we have $m_{\mathbb{Q}}(\chi) = 1$. Assume therefore that $p \neq 2$, $n$ is even, and $n_{s+1} \neq 0$.

We have $\chi(c) = \zeta_{q+1}^j \chi(1)$ for some $j$, $0 \leq j \leq q$. Then, by Schur’s lemma, we must have $(\chi, \nu_j)_{M'} = 1$. If $j \neq (q+1)/2$, then $\nu_j$ is realizable in $k_j$ and $\mathbb{Q}(\chi) \supset k_j$, so we have $m_{\mathbb{Q}}(\chi) = m_{k_j}(\chi) = 1$. Suppose that $j = (q+1)/2$. Then we have $m_{\mathbb{R}}(\nu_j) = m_{\mathbb{Q}_p}(\nu_j) = 2$. Therefore the last assertion follows from properties of Hasse invariants. □

**References**


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