J. Math. Sci. Univ. Tokyo
4 (1997), 341–349.

*Spectrum of Functions in Orlicz Spaces**

By Ha Huy Bang

Abstract. Some geometrical properties of the spectrum of functions in Orlicz spaces are given first in this paper.

The study of properties of functions in the connection with the support of their Fourier transform has been considered by S.N. Bernstein, R.E.A.C. Paley, N. Wiener, L. Schwartz, L. Hörmander, S.M. Nikol'skii, V.S. Vladimirov, O.V. Besov, L.D. Kudrjavtsev, V.P. Il'in, A.F. Timan, N.I. Akhiezer, N.K. Bari, P.I. Lizorkin, B.I. Burenkov, V.N. Temlyakov, H. Triebel, E.Görlich, R.J. Nessel, G. Wilmes, M. Morimoto, C. Watari, Y. Okuyama, C. Markett, P. Nevai, G. Freud, G. Björck, C. Roumieu, R.W. Braun, R. Meise, B.A. Taylor, M. Reed, B. Simon, S. Saitoh, Ha Huy Bang, and many other mathematicians (see, for example, [2 - 4, 10, 14, 16, 18] and their references). To this study, in particular, belong the inequalities of Bernstein, Nikol'skii, Bohr, and the Paley - Wiener - Schwartz theorem.

Let $f \in S'$. The spectrum of f is by definition the support of its Fourier transform $\hat{f}$ (see, [11, 17]). Denote $\mathrm{sp}(f) = \mathrm{supp}\hat{f}$. Then the geometry of $\mathrm{sp}(f)$, in general, can have enough arbitrary character. In this paper we give some geometrical properties of the spectrum of functions in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ (a subset of S'). This study is necessary for us to characterize behaviour of the sequence of norms of derivatives $\|D^\alpha f\|_\Phi, \alpha \geq 0$ in the connection with the spectrum $\mathrm{sp}(f)$ (for L_p - norms it is given in [7]) and completely describe functions in Sobolev - Orlicz spaces of infinite order in the sense of their spectrum. Note that Sobolev - Orlicz spaces of infinite

1991 *Mathematics Subject Classification.* Primary 46F99; Secondary 46E30.

Key words: Geometry of spectrum, Orlicz spaces.

*Supported by the National Basic Research Program in Natural Science and by the NCNST "Applied Mathematics".

order arise in the study of nonlinear differential equations of infinite order (see the definition in [5, 9] and their references).

Let $\Phi(t) : [0, +\infty) \to [0, +\infty]$ be an arbitrary Young function [1, 12 - 13, 15], i.e., $\Phi(0) = 0, \Phi(t) \geq 0, \Phi(t) \not\equiv 0$ and $\Phi(t)$ is convex. Denote by

$$\overline{\Phi}(t) = \sup_{s\geq 0} \big\{ts - \Phi(s)\big\}$$

the Young function conjugate to $\Phi(t)$ and by $L_\Phi(\mathbb{R}^n)$ the space of measurable functions $u(x)$ such that

$$|< u, v >| = | \int u(x)v(x)dx | < \infty$$

for all $v(x)$ with $\rho(v, \overline{\Phi}) < \infty$, where

$$\rho(v, \overline{\Phi}) = \int \overline{\Phi}(| v(x) |)dx.$$

Then $L_\Phi(\mathbb{R}^n) \subset \mathcal{S}'$ and $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the Orlicz norm

$$\| u \|_\Phi = \sup_{\rho(v,\overline{\Phi})\leq 1} | \int u(x)v(x)dx |,$$

which is equivalent to the Luxemburg norm

$$||f||_{(\Phi)} = \inf\{\lambda > 0 : \int \Phi(|f(x)|/\lambda)dx \leq 1\} < \infty.$$

Let $u \in L_\Phi(\mathbb{R}^n), h \in L_1(\mathbb{R}^n)$ and $v \in L_{\overline{\Phi}}(\mathbb{R}^n)$. Then $\|u * h\|_\Phi \leq \|u\|_\Phi \|h\|_1$ and

$$\int |u(x)v(x)|dx \leq \|u\|_\Phi \|v\|_{(\overline{\Phi})}.$$

Recall that $||\cdot||_\Phi = ||\cdot||_p$ when $1 \leq p < \infty$ and $\Phi(t) = t^p$; and $||\cdot||_\Phi = ||\cdot||_\infty$ when $\Phi(t) = 0$ for $0 \leq t \leq 1$ and $\Phi(t) = \infty$ for $t > 1$.

LEMMA 1. *Let $f \in L_\Phi(\mathbb{R}^n)$ and $\mathrm{sp}(f)$ be bounded. Then $f(x)$ is bounded.*

PROOF. Without loss of generality we may assume that $\int \Phi(|f(x)|)dx < \infty$. Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n), \hat{\psi} = 1$ in some neighbourhood of sp(f) and M_1, M_2 be positive numbers such that $\overline{\Phi}(\|\psi\|_\infty/M_1) < \infty$ and $\|\psi\|_\infty \le M_2$. Then the Young inequality and the property $\overline{\Phi}(\lambda t) \le \lambda\overline{\Phi}(t)$ for $0 \le \lambda \le 1, t \ge 0$ yield

$$\begin{aligned} |f(x)|/M_1M_2 &\le \int \Phi(|f(y)|)dy + \int \overline{\Phi}(|\psi(x-y)|/M_1M_2)dy \\ &\le \int \Phi(|f(y)|)dy + \overline{\Phi}(\|\psi\|_\infty/M_1) \int |\psi(y)|/M_2 dy < \infty. \end{aligned}$$

The proof is complete. □

THEOREM 1. *Let $\Phi(t) > 0$ for $t > 0$, $f \in L_\Phi(\mathbb{R}^n), f(x) \not\equiv 0$ and $\xi^0 \in$ sp(f) be an arbitrary point. Then the restriction of $\hat{f}$ on any neighbourhood of ξ^0 cannot concentrate on any finite number of hyperplanes.*

PROOF. We choose a function $\hat{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\hat{\varphi} = 1$ in some neighbourhood of ξ^0. Then $F^{-1}\hat{\varphi}\hat{f} = \varphi * f \in L_\Phi(\mathbb{R}^n)$. Therefore, it is enough to prove our theorem for functions with bounded spectrum.

Put $\hat{h}(\xi) = \hat{f}(\xi - \xi^0)$. Then $h(x) = e^{i\xi^0 x} f(x)$ belongs to $L_\Phi(\mathbb{R}^n)$ and has bounded spectrum. So we can assume that $\xi^0 = 0$.

We prove by contradiction: Assume that there exist a neighbourhood $U \ni 0$ and hyperplanes $H_1, \ldots, H_m$ such that the restriction of $\hat{h}(\xi)$ on U concentrates on $H_1, \ldots, H_m$. Without loss of generality we may assume that $0 \in H_j, j = 1, \ldots, m$. Then H_j can be defined by the equation

$$a_{j1}\xi_1 + \cdots + a_{jn}\xi_n = 0,$$

where $(a_{j1}, \ldots, a_{jn})$ is a unit vector in $\mathbb{R}^n$.

We put for each $j = 1, \ldots, m$

$$G_j = \mathbb{R}^n \backslash (\bigcup_{i \ne j} H_i)$$

Then G_j is open. For any $\psi(\xi) \in C_0^\infty(G_j)$, the distribution $\psi(\xi)\hat{h}(\xi)$ concentrates on the hyperplane H_j. We introduce the transformation

$$x = (x_1, \cdots, x_n) \quad \rightleftarrows \quad (y_1, \cdots, y_n) = y,$$

where $y_1, \cdots, y_n$ are the coordinates of x in the new rectangular system of coordinates, which is chosen such a way that the hyperplane

$$a_{j1}x_1 + \cdots + a_{jn}x_n = 0$$

will be transformed into the hyperplane $y_j = 0$. The coordinate transformation

$$x_k = \sum_{s=1}^{n} \alpha_{k,s} y_s, \quad k = 1, \cdots, n$$

is defined by a real orthogonal matrix $A = (\alpha_{k,s})$ and $|\det A| = 1$.

Put $g(y) = F^{-1}\psi * h(x)$. Then $\|g(y)\|_\Phi = \|F^{-1}\psi * h(x)\|_\Phi$, supp$\hat{y}$ is compact and, clearly, the Fourier transform of $g(y)$ will concentrate on the hyperplane $\xi_j = 0$ (see formula (7.1.17) [10] or the proof of Theorem 2 [8]). Therefore, taking account of a remark on Theorem 2.3.5 mentioned in Example 5.1.2 [10], we get

$$g(y) = \sum_{\ell=0}^{N} g_\ell(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)(-iy_j)^\ell \,, \tag{1}$$

where N is the order of the distribution $\hat{h}(\xi)$ ($N < \infty$ because supp$\hat{h}$ is compact) and $\hat{g}_\ell(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n), 0 \le \ell \le N$, are distributions with compact support.

Because of Lemma 1, equality (1) is possible only if $N = 0$. So the function $g(y)$ does not depend on y_j.

Further, by the definition we get

$$\int \Phi(|g(y)|/\lambda)dy < \infty \tag{2}$$

for some $\lambda > 0$. Then

$$\Phi(|g(y)|/2\lambda) \equiv 0. \tag{3}$$

Actually, assume the contrary that $\Phi(|g(y^0)|/2\lambda) > 0$ for some point y^0. Because of the fact that $\Phi(t)$ never decreases and the continuity of $g(y) = F^{-1}(\psi\hat{h})(x)$, there is a number $\delta > 0$ such that $\Phi(|g(y^0)|/2\lambda) \ge \delta$ in some neighbourhood of y^0, which contradicts (2) because $g(y)$ does not depend on y_j.

Combining (3) and the assumption that $\Phi(t) > 0, t > 0$, we get $g(y) \equiv 0$. It means $\psi(\xi)\hat{h}(\xi) \equiv 0$. Since $\psi(\xi) \in C_0^\infty(G_j)$ is arbitrarily chosen, we get $\hat{h}(\xi) \equiv 0$ on the hyperplane H_j. So $\hat{h}(\xi)$ must concentrate on the planes $H_i \cap H_j, i, j = 1, \dots, m, i \neq j$.

We put for $i, j = 1, \dots, m, i \neq j$

$$G_{ij} = \mathbb{R}^n \backslash \cup \{H_k \cap H_\ell : (k, \ell) \neq (i, j), k \neq \ell\}.$$

Then G_{ij} is open. For any $\psi(\xi) \in C_0^\infty(G_{ij})$, the distribution $\psi(\xi)\hat{h}(\xi)$ concentrates on the plane $H_i \cap H_j$.

Introducing a suitable transformation of coordinates, by an argument analogous to the previous one, we obtain $\psi(\xi)\hat{h}(\xi) \equiv 0$. Hence, since $\psi \in C_0^\infty(G_{ij})$ is arbitrarily chosen, we get that $\hat{h}(\xi)$ must concentrate on $H_i \cap H_j \cap H_\ell, i, j, \ell = 1, \dots, m, i \neq j \neq \ell$.

Repeating the above arguments $(m-3)$ times more, we obtain that the distribution $\hat{h}(\xi)$ concentrates on $\bigcap_{i=1}^{m} H_i$ and then, by the same way, we get $\hat{h}(\xi) \equiv 0$, which contradicts $h(x) \not\equiv 0$. The proof is complete. □

Corollary 1. *Let $\Phi(t) > 0, t > 0, f \in L_\Phi(\mathbb{R}^n)$ and $f(x) \not\equiv 0$. Then for any $\xi^0 \in \mathrm{sp}(f)$ there exists a sequence $\{\xi^m\} \subset \mathrm{sp}(f)$ such that $\xi_j^m \neq \xi_j^0, j = 1, \dots, n$ and $\xi^m \to \xi^0$.*

Corollary 2. *Assume the hypotheses of Corollary 1. Then for any $\xi^0 \in \mathrm{sp}(f)$ there exists a sequence $\{\xi^m\} \subset \mathrm{sp}(f)$ such that $\xi_j^m \neq 0, j = 1, \dots, n$ and $\xi^m \to \xi^0$.*

Corollary 3. *Assume the hypotheses of Corollary 1. Then*

$$\mathrm{span}(\mathrm{sp}(f)) = \mathrm{span}(\mathrm{sp}(f) - \xi^0) = \mathbb{R}^n$$

for any $\xi^0 \in \mathrm{sp}(f)$.

Corollary 4. *Clearly, $\mathrm{sp}(D^\alpha f) \subset \mathrm{sp}(f)$. Further, if the hypotheses of Corollary 1 is satisfied, then $\mathrm{sp}(D^\alpha f) = \mathrm{sp}(f)$.*

Remark 1. In all above conclusions, the assumption $\Phi(t) > 0, t > 0$ cannot be dropped because, in the contrary case, $L_\Phi(\mathbb{R}^n)$ contains all constant functions.

Remark 2. In contrast with hyperplanes, $\hat{f}(\xi)$ may concentrate on surfaces. Actually, let $n = 3$ and $f(x) = \frac{\sin|x|}{|x|}$. Then $\mathrm{sp}(f) = \{\xi : |\xi| = 1\}$ (see [6]) and, clearly, $f(x) \in L_p(\mathbb{R}^n)$ for any $p > 3$.

Theorem 2. *Let $\Phi(t)$ be an arbitrary Young function, $f \in L_\Phi(\mathbb{R}^n)$ and $\alpha \geq 0$ be a multiindex. Then $\sup_{\mathrm{sp}(f)} |\xi^\alpha| = 0$ if and only if $D^\alpha f(x) \equiv 0$, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, D_j = -i\partial/\partial x_j$.*

Proof. We have to prove only the "only if" part. Without loss of generality we may assume that $\alpha_j \neq 0, j = 1, \ldots, k$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$ $(1 \leq k \leq n)$. Then the distribution $\hat{f}(\xi)$ concentrates on the hyperplanes $\xi_j = 0, j \in \{1, \ldots, k\} = I$. For the sake of convenience we assume that $\alpha_1 = \cdots = \alpha_k = 1$.

We shall begin with showing that if $\xi^\alpha \psi(\xi) \hat{f}(\xi)$ concentrates on the plane $\xi_{i_1} = \cdots = \xi_{i_\ell} = 0$ for some $i_1, \ldots, i_\ell \in I$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, then $D^\alpha F^{-1}\psi * f(x) \equiv 0$. Actually, because $\xi^\alpha \psi(\xi) \hat{f}(\xi)$ concentrates on the plane $\xi_1 = \ldots = \xi_\ell = 0$ (we assume that $i_j = j, j = 1, \ldots, \ell$ for simplicity of notation), we have

$$F^{-1}\big(\xi^\alpha \psi(\xi) \hat{f}(\xi)\big)(x) = \sum_{|\beta| \leq N} g_\beta(x'')(-ix')^\beta \,, \tag{4}$$

where N is the order of the distribution $\psi(\xi)\hat{f}(\xi), x' = (x_1, \ldots, x_\ell), x = (x', x''), \beta \in \mathbb{Z}_+^\ell$ and $\hat{g}_\beta(\xi_{\ell+1}, \ldots, \xi_n), |\beta| \leq N$ are distributions with compact support.

Further, we choose $\omega \in C_0^\infty(\mathbb{R}^n)$ such that $\omega(\xi) = 1$ in some neighbourhood of $\mathrm{supp}\psi$. Then by virtue of Lemma 1, we obtain

$$||F^{-1}\big(\xi^\alpha \psi(\xi) \hat{f}(\xi)\big)||_\infty = ||F^{-1}\big(\xi^\alpha \psi(\xi) \omega(\xi) \hat{f}(\xi)\big)||_\infty$$

$$\leq ||F^{-1}\big(\xi^\alpha \psi(\xi)\big)||_1 ||F^{-1}(\omega \hat{f})||_\infty < \infty.$$

Therefore, since (4) we get

$$F^{-1}\big(\xi^\alpha\psi(\xi)\hat f(\xi)\big)(x) = D^\alpha F^{-1}\psi * f(x) = g_0(x'').$$

Put $\gamma_1 = 0, \gamma_2 = \cdots = \gamma_k = 1, \gamma_{k+1} = \cdots = \gamma_n = 0$. Then

$$D_1 D^\gamma F^{-1}\psi * f(x) = g_0(x'').$$

Hence,

$$D^\gamma F^{-1}\psi * f(x) = ix_1 g_0(x'') + h(x_2, \ldots, x_n).$$

Therefore, taking account of $D^\gamma F^{-1}\psi * f \in L_\infty$, we obtain $g_0(x'') \equiv 0$, i.e.,

$$D^\alpha F^{-1}\psi * f(x) \equiv 0.$$

Next we prove that $\xi^\alpha \hat f(\xi)$ concentrates on the plane $\xi_1 = \cdots = \xi_k = 0$. Actually, we put for any $j \in I$

$$G_j = \big\{\xi \in \mathbb{R}^n : \xi_i \neq 0, i \in I\backslash\{j\}\big\}.$$

Then G_j is open. For each $\varphi \in C_0^\infty(G_j)$ we choose a function $\psi(\xi) \in C_0^\infty(G_j)$ such that $\psi = 1$ in some neighbourhood of suppφ. Then $\psi(\xi)\hat f(\xi)$ concentrates on the plane $\xi_j = 0$ and by the fact proved above, we get

$$< \xi^\alpha \hat f(\xi), \varphi(\xi) > = < \xi^\alpha\psi(\xi)\hat f(\xi), \varphi(\xi) >$$

$$= < D^\alpha F^{-1}\psi * f, \hat\varphi > = 0.$$

So we have proved that $\xi^\alpha \hat f(\xi)$ must concentrate on the planes $\xi_i = \xi_j = 0, i, j \in I$.

We put for $i, j \in I, i \neq j$

$$G_{ij} = \{\xi \epsilon \mathbb{R}^n : \xi_\ell \neq 0, \ell \in I\backslash\{i, j\}\}.$$

Then G_{ij} is open. Arguing as in the case G_j, we get

$$< \xi^\alpha \hat f(\xi), \varphi(\xi) > = 0, \quad \forall\varphi \in C_0^\infty(G_{ij}).$$

So $\xi^\alpha \hat{f}(\xi)$ must concentrate on the planes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0, i_1, i_2, i_3 \in I, i_1 \neq i_2 \neq i_3$.

Repeating the above arguments $(k-3)$ times more, we obtain that the distribution $\xi^\alpha \hat{f}(\xi)$ concentrates on the plane $\xi_1 = \cdots = \xi_k = 0$.

Finally, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we choose $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi = 1$ in some neighbourhood of suppφ. Then

$$\begin{aligned} < D^\alpha f, \hat{\varphi} > &= < \xi^\alpha \hat{f}(\xi), \varphi(\xi) > = < \xi^\alpha \psi(\xi) \hat{f}(\xi), \varphi(\xi) > \\ &= < D^\alpha F^{-1}\psi * f, \hat{\varphi} > = < 0, \hat{\varphi} > = 0. \end{aligned}$$

Therefore, it follows from the density of $C_0^\infty(\mathbb{R}^n)$ in $\mathcal{S}$ that $< D^\alpha f, \hat{\varphi} > = 0$ for all $\varphi \in \mathcal{S}$. Therefore, $D^\alpha f(x) \equiv 0$. The proof is complete. □

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(Received April 8, 1996)

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