Mixed Hodge Structures of Siegel Modular Varieties and Siegel Eisenstein Series

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Abstract. In this paper we study the mixed Hodge structure on the middle degree cohomology of the Siegel modular variety of level $n$. We attach some global automorphic forms to its highest weight quotient space and also show a vanishing of the next weight quotient. As an appendix, we also consider the universal family of abelian varieties over the moduli space and treat its middle degree mixed Hodge structure similar to the above case.

1. Introduction

The purpose of this paper is to give a description of some graded quotients associated with the weight filtration on the mixed Hodge structure defined for the middle degree cohomology group of Siegel modular variety. Here is a more precise statement of the main result.

Let $H_g$ be the Siegel upper half space of degree $g$. The Siegel modular group $\Gamma_g = Sp(g, \mathbb{Z})$ acts on it properly discontinuously as usual by

$$Z \mapsto \gamma(Z) = (AZ + B)(CZ + D)^{-1}, \text{ for } Z \in H_g, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g.$$ 

Moreover the principal congruence subgroup of level $n \geq 3$ of $\Gamma_g : \Gamma_g(n) = \ker(\Gamma_g \to Sp(g, \mathbb{Z}/n\mathbb{Z}))$, acts freely. Then the quotient space $V_g(n) = \Gamma_g(n) \backslash H_g$ becomes to be a smooth (open) algebraic variety over $\mathbb{C}$ of dimension $N = \frac{1}{2}g(g + 1)$, and is known to be quasi-projective because there exists a projective minimal compactification: Baily-Borel-Satake compactification ([3], [20]).

The cohomology group $H^i(V_g(n), \mathbb{Q})$ has the mixed Hodge structure by Deligne [5, 6]. Let $\{W_k\}$ be the weight filtration. Then we have the following main result.

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Main theorem. Let $V_g(n)^*$ be the minimal compactification of a Siegel modular variety, and $N = \frac{1}{2}g(g + 1)$ be its dimension. Then for $g \geq 2$, we have that

(i) $\dim G_{2N}^W H^N(V_g(n),\mathbb{C}) =$ the number of 0-dimensional cusps in $V_g(n)^*$.  
(ii) $G_{2N-1}^W H^N(V_g(n),\mathbb{C}) = \{0\}$.

The main ingredients of the proof are Poincaré residue map of the mixed Hodge structures, toroidal compactification of Siegel modular variety and some regularity results on the Eisenstein series proved by Shimura [22].

Let us explain the contents of this paper. We review the mixed Hodge structure of the cohomology groups of a non-singular algebraic variety in §2. We explain about the edge component of mixed Hodge structure on $H^i(V,\mathbb{C})$ in §3. The Poincaré residue map is also explained in this section. After brief review on the toroidal compactification of a Siegel modular variety in §4, we rewrite in §5 a Fourier-Jacobi expansion of Siegel modular forms of weight $g+1$ in the local coordinates of a smooth compactification of $V_g(n)$. There we describe the image of a modular form by the Poincaré residue map in terms of the constant term of its Fourier expansion. In §6, we give two examples of the smooth compactification in case of $g = 2$, 3 as in Namikawa [16] and Nakamura [15], and explain about degenerate boundary coordinates. In §7, we review the holomorphic Siegel Eisenstein series. We restate the main result in §8, and give a proof for it. There we use theorems of Shimura [22] for the holomorphy of the Eisenstein series of low weight. In the last section §9, as an appendix, we remark some results about the weight filtrations in the case of the universal family of principally polarized abelian varieties which is similar with the case of Siegel modular varieties.

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2. Mixed Hodge structures

We recall briefly the theory of mixed Hodge structures for a smooth algebraic variety defined by P. Deligne. The references are [5], [6].

Given a smooth quasi-projective algebraic variety $V$ over $\mathbb{C}$ of dimension $N$, it can be imbedded into a smooth projective variety $\bar{V}$ as a Zariski open subset, and $D = \bar{V} - V$ is a finite union of smooth irreducible divisors
\{D_i\}_{i \in I} \text{ which have at most simply normal crossings (theorem of Nagata and the resolution of singularities by Hironaka). Choose such an embedding: } j = V \hookrightarrow \tilde{V}. \text{ Then } \{H^i(V, \mathbb{C})\}_{i \in \mathbb{Z}} \text{ make the mixed Hodge structures defined as follows.}

**Definition 2.1.** A mixed Hodge structure (abbreviated M.H.S.) is triple data: \(\{H_\mathbb{Z}, W_\bullet, F_\bullet\}\);

(i) \(H_\mathbb{Z}\) is a \(\mathbb{Z}\)-module of finite type,
(ii) A finite increasing filtration \(W_\bullet\) on \(H_\mathbb{Q} = H_\mathbb{Z} \otimes \mathbb{Q}\),
(iii) A finite decreasing filtration \(F_\bullet\) on \(H_\mathbb{C} = H_\mathbb{Z} \otimes \mathbb{C}\);

with following requirement. Denote also \(W_\bullet\) the naturally induced filtration on \(H_\mathbb{C}\) and define \(F_p(\text{Gr}_{W_k} H_\mathbb{C})\) to be the image of \(F_p H_\mathbb{C} \cap W_k H_\mathbb{C} \rightarrow Gr_k W_k H_\mathbb{C}\), then for all \(k \in \mathbb{Z}\), \(\{Gr_{W_k} H_\mathbb{Q}, F_\bullet\}\) is the pure \(\mathbb{Q}\)-Hodge structure of weight \(k\). \(W_\bullet\) and \(F_\bullet\) are called as weight and Hodge filtration respectively.

Write \(H^{p,q} = Gr_p^W Gr_q^W Gr_k^W H_\mathbb{C}\), \((F^\bullet\) is the complex conjugation to \(F_\bullet\)). Then the definition means that

(i) \(H^{p,q} = 0\), if \(p + q \neq k\), and
(ii) \(Gr_k^W H_\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}\) : direct sum decomposition, \(H^{p,q} = \overline{H}^{q,p}\).

(When these properties are satisfied, \(F\) and \(F^\bullet\) are said to be \(k\)-opposite to each other.)

For a smooth (open) algebraic variety \(V\), the mixed Hodge structures on \(H^*(V, \mathbb{C})\) are obtained in the following manner. First for the holomorphic de Rham complex \(\Omega^\bullet\), there are isomorphisms (Grothendieck [10]):

\[H^*(V, \mathbb{C}) \simeq H^*(V, \Omega^\bullet_V) \simeq H^*(\tilde{V}, j_* \Omega^\bullet_{\tilde{V}})\].

We take the subcomplex \(\Omega^\bullet_{\tilde{V}}(logD)\) of \(j_* \Omega^\bullet_{\tilde{V}}\), the logarithmic de Rham differential complex as follows. The sheaf \(\Omega^1_{\tilde{V}}(logD)\) is the locally free \(\mathcal{O}_{\tilde{V}}\)-module generated by sections \(\frac{dz_i}{z_i}\) (\(1 \leq i \leq l\)), \(dz_j\) (\(l + 1 \leq j \leq N\)) at a point where \(D\) is defined locally by \(\{z_1 \cdots z_l = 0\}\) with a local coordinates \(\{z_i\}_{1 \leq i \leq N}\) of \(\tilde{V}\). Setting \(\Omega^k_{\tilde{V}}(logD) = \wedge^k \Omega^1_{\tilde{V}}(logD)\), we obtain the
complex $\Omega^\bullet_V(logD)$, and the natural inclusion $\Omega^\bullet_V(logD) \hookrightarrow j_*\Omega^\bullet_V$ is a quasi-isomorphism of complexes (Deligne [5] (3.1.8)). Therefore we get isomorphisms:

$$H^\bullet(V,\mathbb{C}) \cong H^\bullet(\tilde{V},j_*\Omega^\bullet_V) \cong H^\bullet(\tilde{V},\Omega^\bullet_V(logD)).$$

Thus we can identify the singular cohomology of $V$ with the hypercohomology of the logarithmic differential complex on $\tilde{V}$ (Deligne [5], (3.1.5)).

With this identification the Hodge and weight filtrations on $H^\bullet(V,\mathbb{C})$ are induced by the following filtrations on the complex. For any complex $C^\bullet$ in abelian category, define a subcomplex $\sigma_{\geq p}$ as

$$\left\{ \begin{array}{cl} 0 & (p < 0) \\
\Omega^k_V(logD) & (p > k) \\
\Omega^k_V \otimes \Omega^p_V(logD) & (0 \leq p \leq k). 
\end{array} \right.$$ 

For each $p$, one has inclusions $F^p\Omega^\bullet_V(logD) \hookrightarrow \Omega^\bullet_V(logD)$, $W^p\Omega^\bullet_V(logD) \hookrightarrow \Omega^\bullet_V(logD)$ and these induce maps between hypercohomologies. Define the Hodge and weight filtration on $H^\bullet(V,\mathbb{C}) = H^\bullet(\tilde{V},\Omega^\bullet_V(logD))$ as follows.

\begin{equation}
F^pH^i(V,\mathbb{C}) = \text{image of } H^i(\tilde{V},F^p\Omega^\bullet_V(logD)) \rightarrow H^i(\tilde{V},\Omega^\bullet_V(logD))
\end{equation}

\begin{equation}
W^pH^i(V,\mathbb{C}) = \text{image of } H^i(\tilde{V},W^p\Omega^\bullet_V(logD)) \rightarrow H^i(\tilde{V},\Omega^\bullet_V(logD))
\end{equation}

Observe that though the weight filtration is constructed over $\mathbb{C}$ in the above, it can be defined over $\mathbb{Q}$ (Deligne [5], (3.2.4)). Write $D = \bigcup_{i \in I} D_i = \tilde{V} - V$ with a finite (ordered) index set $I$. Each $D_i$ is a smooth irreducible divisor which is also projective, for $\tilde{V}$ is projective. We fix one orientation for each $D_i$. Consider the disjoint union of all $m$-fold intersections of $\{D_i\}$,

$$D^{[m]} = \bigcup_{\{i_1 < \cdots < i_m\} \subset I} D_{i_1} \cap \cdots \cap D_{i_m} : \text{disjoint union}.$$
It is a complex manifold of dimension of $N - m$. We set $D^{[0]} = \tilde{V}$. We consider the Poincaré residue maps $\text{Res}_{[m]}$:

$$\text{Res}_{[m]} : W_m \Omega^\bullet_{\tilde{V}}(\log D) \to i_{m*} \Omega^\bullet_{D^{[m]}}[-m],$$

where $i_m : D^{[m]} \to \tilde{V}$ are natural maps and, for any complex $C^\bullet$, define $C^\bullet[m]$ as $(C^\bullet[m])^i = C^{i+m}$. With local coordinates on $\tilde{V}$ it is given by

$$\omega \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}} \mapsto \omega|_{D_{i_1} \cap \cdots \cap D_{i_m}},$$

for holomorphic differential forms $\omega$ on $\tilde{V}$. Here the order of components $D_{i_j} = \{z_{i_j} = 0\}$ is taken to be increasing. Moreover it should be noted that we must consider the contribution of orientation to the target complex (Deligne [5], (3.1.4), (3.1.5)), but we omit the explicit suitable notation. $\text{Res}_{[m]}$ becomes a morphism of complexes. It is surjective, trivial on $W_{m-1} \Omega^\bullet_{\tilde{V}}(\log D)$, and induces an isomorphism of complexes:

$$\text{Res}_{[m]} : Gr_m W \Omega^\bullet_{\tilde{V}}(\log D) \simeq i_{m*} \Omega^\bullet_{D^{[m]}}[-m],$$

(Deligne [7], [5]). Hence there is an isomorphism of hypercohomologies:

$$\text{(1.3) } \text{Res}_{[m]} : H^i(\tilde{V}, Gr_m W \Omega^\bullet_{\tilde{V}}(\log D)) \simeq H^i(\tilde{V}, i_{m*} \Omega^\bullet_{D^{[m]}}[-m]) \simeq H^{i-m}(D^{[m]}, C)(-m).$$

The last term defines a pure Hodge structure of weight $i + m$ by classical Hodge theory. $(-m)$ means the $(-m)$-th Tate twist.) Then it follows that a spectral sequence for hypercohomology of filtered complex

$$F E^{p,i-p}_{1} = H^i(\tilde{V}, Gr^{p}_{F} Gr_{m} W \Omega^\bullet_{\tilde{V}}(\log D)) \Rightarrow H^i(\tilde{V}, Gr_{m} W \Omega^\bullet_{\tilde{V}}(\log D))$$

degenerates at $E_1$-term because, by the above residue map, this is identified with de Rham-Hodge spectral sequence for $H^*(D^{[m]}, C)$. And the induced filtration on the right hand term becomes to be $i+m$-opposite to its complex conjugation (Deligne [5], (3.2.6), (3.2.7)). On the other hand, the left hand side of the isomorphism (1.3) gives an $E_1$-term of a spectral sequence

$$W E^{i-m,m+i}_{1} = H^i(\tilde{V}, Gr_{m} W \Omega^\bullet_{\tilde{V}}(\log D)) \Rightarrow H^i(\tilde{V}, \Omega^\bullet_{\tilde{V}}(\log D)) \simeq H^i(V, C).$$
It is shown that each of its differentials $d_1$ (the connecting homomorphisms of a long exact sequence of hypercohomologies induced from a short exact sequence $0 \to Gr_{m-1}^W \to W_m/W_{m-2} \to Gr_m^W \to 0$) is strictly compatible with the filtration on $WE_{1-m,i+m}$ induced above ([5], (3.2.8)). Then it is proved that a unique filtration is defined on $WE_2$-term from the one on $WE_1$. (There are different types of filtrations on $WE_2$ which canonically induced from the filtration $F^\bullet$ on the complex $\Omega_V^\bullet(logD)$, but now these are coincident with each other.) Also we get $d_j = 0$, $j \geq 2$, hence above spectral sequence $WE_r$ degenerates at $WE_2$-term ([5], (3.2.9), (3.2.10)). Thus we obtain a filtration on $Gr^{W+i}H^i(V, C) = WE_2$ which is $i+m$-opposite to its complex conjugation, which is proved to coincide with the filtration defined by (1.1), (1.2). Therefore $\{H^\bullet(V, C), WE^\bullet, F^\bullet\}$ defines a M.H.S. ([5], (3.2.5)). Together with [5, (3.2.13)] and [6, (7.2.8)] we have the following.

**Theorem 2.2.** (Deligne)

(i) A spectral sequence $F^1_{p,q} = H^q(V, Gr_p^W \Omega^\bullet_V(logD)) \Rightarrow H^{p+q}(V, C)$ degenerates at $F^1$-terms.

(ii) A spectral sequence $H^i(\tilde{V}, Gr_p^W \Omega^k_V(logD)) \Rightarrow H^i(\tilde{V}, \Omega^k_V(logD))$ degenerates at $E_2$-terms.

(iii) There is an isomorphism of spectral sequences

$$Gr^1_p E_r(\mathcal{R}\Gamma(\Omega^\bullet_V(logD)), W) \simeq E_r(\mathcal{R}\Gamma(Gr^1_p \Omega^\bullet_V(logD)), W),$$

here $\mathcal{R}\Gamma(K^\bullet)$ is a filtered complex with filtration $W_\bullet$ which is derived from an acyclic bi-filtered resolution $K^I \bullet$ of a bi-filtered complex $K^\bullet$.

2. Residue map on edge parts

In this section we define a certain homomorphism from subspaces of a weight quotient space of $H^i(V)$ to the cohomology groups $H^j(D^{[m]})$, which is induced from the Poincaré residue map. This construction is necessary in the proof of the main result (§7).

We begin with isomorphisms:

$$Gr^W_{m+1}H^i(V, C)$$

$$= H(H^{i-1}(\tilde{V}, Gr^W_{m+1} \Omega^\bullet_V(logD)) \xrightarrow{d_1} H^i(\tilde{V}, Gr^W_m \Omega^\bullet_V(logD)) \xrightarrow{d_1} H^{i+1}(\tilde{V}, Gr^W_{m-1} \Omega^\bullet_V(logD)))$$
\[
\begin{align*}
= \text{H}(H^{i-m-2}(D^{[m+1]}, C)(-m-1)) & \xrightarrow{d_1} \text{H}(H^{i-m}(D^{[m]}, C)(-m)) \\
& \xrightarrow{d_1} \text{H}(H^{i-m+2}(D^{[m-1]}, C)(-m+1)).
\end{align*}
\]

Here \(H(* \to * \to *)\) means the cohomology of the 3-terms complexes. The first isomorphism comes from \(E_2\)-terms of the spectral sequence in (ii) of Theorem (1.2). The second one is obtained on passing to the targets of the Poincaré residue map \(\text{Res}_{[m]}\) in §2.

By the first isomorphism, \(F^i Gr^{W}_{m+i} H^i(V, C)\) is regarded as a subquotient space of \(H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D))\).

**Lemma 3.1.** Let \(\{F^*\}\) be the Hodge filtration on \(Gr^W_{m+i} H^i(V, C)\), then \(F^i Gr^W_{m+i} H^i(V, C)\) injects into the \(H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D))\).

**Proof.** By Theorem (1.2), we have
\[
Gr^i_F Gr^W_{m+i} H^i(\tilde{V}, \Omega^i_V(\log D)) \cong Gr^i_F \text{E} \text{G}(\Omega^i_V(\log D)), W)
\]
\[
\cong \text{E} \text{E}(Gr^i_F R_{\text{G}}(\Omega^i_V(\log D)), W) \cong \text{E} \text{E}(Gr^i_F R_{\text{G}}(\Omega^i_V(\log D)), W),
\]
last of which equals the cohomology of the next complex;
\[
\cong \text{H}(H^{i-1}(\tilde{V}, Gr^W_{m+1} \Omega^{i-1}_V(\log D)[i])
\]
\[
\xrightarrow{d_1} H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D)[i]) \xrightarrow{d_1} H^{i+1}(\tilde{V}, Gr^W_{m-1} \Omega^{i+1}_V(\log D)[i]).
\]

Since the first term of above 3-term complex is zero space, we know that the space \(Gr^i_F Gr^W_{m+i} H^i(\tilde{V}, \Omega^i_V(\log D))\) can be seen as a subspace of \(H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D))\). Also we have that
\[
H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D)) = Gr^i_F H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D)).
\]

On the other hand, we have \(F^{i+1} H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D)) = \{0\}\). In fact, since the Poincaré residue map \(\text{Res}_{[m]}\) is compatible with the Hodge filtration on \(H^i(V, C)\), we get
\[
F^j H^i(\tilde{V}, Gr^W_m \Omega^i_V(\log D)) \simeq F^{j-i} H^{i-m}(D^{[m]}, C)(-m).
\]
Here the Hodge types of \(H^{i-m}(D^{[m]}, C)(-m)\) is only \(\{p+m, q+m\}\), where \(p + q = i - m\), and \(p, q \geq 0\). Therefore for \(j > i\), considering above Hodge
type, we obtain that $F^{j-m}H^{i-m}(D^{[m]}, C)(-m) = \{0\}$. Hence combining this with above, the lemma follows. □

By this lemma, we can consider a restriction of the Poincaré residue map on this edge subspace and this restriction map gives a isomorphism of $F^i Gr^W_{m+i} H^i(V, C)$ into its image. By an abuse of notation, we denote this homomorphism

$$Res_{[m]} : F^i Gr^W_{m+i} H^i(V, C) \rightarrow H^{i-m}(D^{[m]}, C)(-m).$$

by the same symbol as the Poincaré residue map on the complexes. Moreover the following lemma shows that the domain of above map is a subquotient of the space of global sections of $\Omega^i_{V}(log D)$.

**Lemma 3.2.**

$$Gr^i_F Gr^W_{m+i} H^i(V, C) \simeq Gr^W_{m+i} H^0(\tilde{V}, \Omega^i_{\tilde{V}}(log D)).$$

**Proof.** This results from Theorem (1.2). □

4. **Toroidal compactification of $V_g(n)$**

In this section, we recall the construction of toroidal compactification of a quotient variety of Hermitian symmetric space. Main references are, for example, Ash, Mumford, Rapoport, Tai [1], Namikawa [16], [17], Nakamura [15].

4.1. **Minimal compactification**

First we recall the Baily-Borel-Satake minimal compactification. The Siegel upper half space $H_g$ is analytically isomorphic to $\mathcal{D}_g = \{ \tau = t\tau \in M_g(\mathbb{C}) \; \text{s.t.} \; 1 - \tau \bar{\tau} > 0 \}$ by $H_g \ni z \mapsto (z - \sqrt{-1} \mathbf{1}_g)(z + \sqrt{-1} \mathbf{1}_g)^{-1}$. $\mathcal{D}_g$ is a bounded symmetric domain in $\mathbb{C}^{1/2g(g+1)}$. We set $\mathcal{D}'_g = \{ \tau = t\tau \in M_g(\mathbb{C}) \; \text{s.t.} \; 1 - \tau \bar{\tau} \geq 0 \} \supset \mathcal{D}_g$, which is a union of $\mathcal{D}_g$ and its boundary components. We take the only rational boundaries defined over $\mathbb{Q}$. Then we set $\mathcal{D}'_g^* = \mathcal{D}_g \cup \{ \text{rational boundary components of } \mathcal{D}_g \}$. According to this, we also define for the Siegel upper half space

$$H^*_g = H_g \cup \{ \text{rational boundary components of } H_g \}.$$
The action of \( Sp(g, \mathbb{Q}) \) on \( H_g \) extends to \( H_g^* \) and we make the quotient \( V_g(n)^* = \Gamma_g(n) \backslash H_g^* \). Then \( V_g(n)^* \) becomes a compact Hausdorff space by defining a suitable topology on \( H_g^* \) (called Satake topology). This is the minimal compactification of \( V_g(n) \), which is a projective variety with singularities on \( V_g(n)^* - V_g(n) \) (Satake [20]).

4.2. Rational boundary components

We fix some symbols to denote rational boundary components of \( H_g \).

Symbols:
- \( \{ F_\alpha \} \); \( \Gamma_g(n) \)-equivalence classes of rational boundary components of \( H_g \).
- \( F_\alpha \simeq H_{g_0} \) \((0 \leq g_0 \leq g)\).
- \( P_\alpha = \{ g \in Sp(g, \mathbb{R}) ; gF_\alpha = F_\alpha \} \); maximal \( \mathbb{Q} \)-parabolic subgroups associated to \( F_\alpha \).
- \( W_\alpha \subset P_\alpha \); a unipotent radical.
- \( U_\alpha \subset W_\alpha \); the center of \( W_\alpha \), \( \simeq \{ b \in M_{g_1}(\mathbb{R}) ; t^b = b \} = Q_{g_1} \), \( g_1 = g - g_0 \).
- \( \Omega_\alpha \); a self dual open cone, \( \simeq \{ b \in M_{g_1}(\mathbb{R}) ; t^b, b > 0 \} = Q_{g_1}^+ \).

Among \( \{ F_\alpha \} \), the standard boundaries \( F_{\text{st}}^{g_0} \) can be chosen for each \( g_0 \).

The maximal \( \mathbb{Q} \)-parabolic subgroup \( P_\text{st}_{g_0} \) associated to this standard boundary is given as follows:

\[
P_{g_0} = \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & t_u^{-1} \end{pmatrix} \in Sp(g, \mathbb{R}) \right\}, \quad \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(g_0, \mathbb{R}), \quad u \in GL(g_1, \mathbb{R})
\]

\[
W_{g_0} = \left\{ \begin{pmatrix} 1_{g_0} & 0 & n \\ t_m & 1_{g_1} & t_n & b \\ 0 & 0 & 1_{g_0} & -n \\ 0 & 0 & 0 & 1_{g_1} \end{pmatrix} \in P_{g_0} \right\}, \quad t^nm + b = t^mn + t^b
\]

\[
U_{g_0} = \left\{ \begin{pmatrix} 1_{g_0} & 0 & 0 \\ 1_{g_1} & 0 & b \\ 0 & 0 & 1_{g_0} & 0 \\ 0 & 0 & 0 & 1_{g_1} \end{pmatrix} \in W_{g_0} \right\}, \quad t^b = b = Q_{g_1}.
\]

All rational boundaries \( F_\alpha \) are transformed into one \( F_{\text{st}}^{g_0} \simeq H_{g_0} \) by the action of \( Sp(g, \mathbb{Z}) = \Gamma_g \); \( F_\alpha = \gamma_\alpha F_{\text{st}}^{g_0} \), \( \exists \gamma_\alpha \in \Gamma_g \). Under this situation we first construct a partial compactification in direction of a rational boundary component \( F_\alpha \).
4.3. Partial compactification

Fix $F = F_{\ast}^{st}$, a standard rational boundary and we consider specially the partial compactification for it. Let $P_{g_0} \mathbb{Z} = P_{\mathbb{Z}} = P_{g_0} \cap \Gamma_g$, $U_{g_0} \mathbb{Z} = U_{\mathbb{Z}} = U_{g_0} \cap \Gamma_g$. $P_{g_0}(n) = P(n) = P_{g_0} \cap \Gamma_g(n)$, $U_{g_0}(n) = U(n) = U_{g_0} \cap \Gamma_g(n)$, $U_{\mathbb{C}} = U_{\mathbb{Z}} \otimes \mathbb{C}$. We make a next map:

$$e : H_g \rightarrow H_{g_0} \times V_{g_0} \times (U_{\mathbb{Z}} \setminus U_{\mathbb{C}}), \quad Z = \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) \mapsto (z_1, z_2, e\left( \frac{z_3}{n} \right)),$$

here $e(z) = \exp(2\pi \sqrt{-1}z) = \exp(2\pi \sqrt{-1}z_{k,l})$, $V_{g_0} \mathbb{Z}$ is the space of $g_0 \times g_1$-matrices with coefficients in $\mathbb{C}$. $T_{g_1} = U_{\mathbb{Z}} \setminus U_{\mathbb{C}}$ is a complex torus of dimension $\frac{1}{2}g_1(g_1 + 1)$, $\simeq (\mathbb{C}^*)^{g_1(g_1 + 1)}$. It can be seen that the above map $e$ factors through $U(n) \setminus H_g$. Then, the image $T_{g_0,g_1}^e$ of $e$ is an open subset of $T_{g_0} = H_{g_0} \times V_{g_0} \times T_{g_1}$, and $U(n) \setminus H_g$ is isomorphic to this image. We identify them, thus consider $T_{g_0,g_1}$ in $T_{g_0,g_1}$.

For the third factor of complex torus $T_{g_1}$, there exists a toroidal embedding as following. We remark that $U_{\mathbb{Z}} \simeq Q_{g_1} \mathbb{Z} \simeq \text{Hom}_{\text{alg-grp}}(G_m, T_{g_1}) \simeq \pi_1(T_{g_1})$, where $Q_{g_1,\mathbb{Z}}$ is the $\mathbb{Z}$-lattice of symmetric integral matrices in $Q_{g_1}$. Take $\hat{Q}_{g_1}$ to be a dual real vector space of $Q_{g_1}$, and denote by $\langle , , \rangle : \hat{Q}_{g_1} \times Q_{g_1} \rightarrow \mathbb{R}$ the natural pairing. Then the dual lattice $M$ of $Q_{g_1,\mathbb{Z}}$ is defined by

$$M = \{ \hat{y} \in \hat{Q}_{g_1} : \langle \hat{y}, y \rangle \in \mathbb{Z} \text{ for } \forall y \in Q_{g_1} \}.$$

Let $Q_{g_1}^+$ be the set of positive definite real quadratic forms in $Q_{g_1}$. By $\overline{Q}_{g_1}^+$ we denote the rational closure in the space of nonnegative real quadratic forms which is, by definition, the convex hull of the set of nonnegative integral quadratic forms. The group $GL(g_1,\mathbb{Z})$ operates on $Q_{g_1}^+$ as $y \mapsto uy^t u$ for $u \in GL(g_1,\mathbb{Z})$, and the action preserves $Q_{g_1}^+$ and $\overline{Q}_{g_1}^+$. Every element of $\overline{Q}_{g_1}^+$ can be transformed by a unimodular integral matrix $u$ to $uy^t u = \left( \begin{array}{cc} 0 & 0 \\ 0 & y' \end{array} \right)$; $y' > 0$ (Namikawa [16]).

On $\overline{Q}_{g_1}^+ \simeq \overline{Q}_{g_0}^+$, we consider a $GL(g_1,\mathbb{Z})$-admissible cone decomposition $\Sigma_{g_1} = \{ \sigma \}$ which satisfies following properties:

1. each $\sigma \in \Sigma_{g_1}$ is a rational convex cone, namely, generated by finite number of semipositive integral quadratic forms,
2. $\sigma \in \Sigma_{g_1}, \tau < \sigma$ ($\tau$ is a face of $\sigma$) $\Rightarrow \tau \in \Sigma_{g_1}, \sigma, \tau \in \Sigma_{g_1} \Rightarrow \sigma \cap \tau \in \Sigma_{g_1}$,
3. the decomposition is invariant under the action of $GL(g_1,\mathbb{Z})$. 


(4) there are only a finite number of classes of \(\sigma\)'s modulo \(GL(g_1, \mathbb{Z})\),

\[
\bigcup_{\sigma \in \Sigma_{g_1}} \sigma = \overline{\mathcal{Q}}_{g_1}^+.
\]

Later we glue all partial compactifications into one \(\overline{V}_g(n)\). For this purpose, we have to assume that the family of cone decompositions \(\{\Sigma_{g_1}^{F_\alpha}\}\), each of which is associated to a rational boundary \(F_\alpha\), satisfies next compatibility conditions:

(6) if \(F_\alpha = \gamma F_\beta\) with \(\gamma \in \Gamma_g(n)\), then \(\Sigma_{g_1}^{F_\alpha} = \gamma \Sigma_{g_1}^{F_\beta}\), via the natural isomorphism \(\gamma : \Omega_\alpha \to \Omega_\beta\).

(7) if \(g'_1 < g_1\), for natural embedding \(\overline{\mathcal{Q}}_{g_1}^+ \to \overline{\mathcal{Q}}_{g'_1}^+ : y' \mapsto \left(\begin{array}{cc}0 & 0 \\ 0 & y'\end{array}\right)\), the restriction of \(\overline{\mathcal{Q}}_{g_1}^+\) to \(\overline{\mathcal{Q}}_{g'_1}^+\) is the cone decomposition \(\Sigma_{g_1}^{F_\beta}\).

A family of cone decompositions for each rational boundaries satisfying from (1) to (7) is called \(\Gamma_g(n)\)-admissible collection. For given admissible \(\Sigma_{g_1}\) for \(\overline{\mathcal{Q}}_{g_1}^+\), we can construct an affine torus embeddings \(\{T_\sigma\}\) of \(T_{g_1}\) to every \(\sigma \in \Sigma_{g_1}\). Set the dual cone of \(\sigma\) to be \(\hat{\sigma} = \{\hat{y} \in \hat{\mathcal{Q}}_{g_1}^+; \langle \hat{y}, y \rangle \geq 0 \text{ for } \forall y \in \sigma\}\).

\(T_{g_1} = Spec \ C[M] = Spec \ C[z_{ij}, z_{ij}^{-1}; 1 \leq i \leq j \leq g_1]\). Then we get an embedding

\[
T_{g_1} \hookrightarrow T_\sigma = Spec \ C[z^A; A \in \hat{\sigma} \cap M],
\]

where \(z^A = \prod_{1 \leq i \leq j \leq g_1} z_{ij}^{A_{ij}}\), and \(\hat{\sigma} \cap M\) is a sub-semigroup of \(M\).

From the property (1), \(T_\sigma\) becomes an algebraic scheme. And from (2) \(\{T_\sigma\}_{\sigma \in \Sigma_{g_1}}\) can be glued with each other (i.e. for \(\sigma' \prec \sigma'\), use natural open embedding \(T_{\sigma'} \subset T_\sigma\)). Then we get a torus embedding \(T_{g_1}\);

\[
T_{g_1} \hookrightarrow T_{g_1} = \bigcup_{\sigma \in \Sigma_{g_1}} T_\sigma \text{ (gluing)}.
\]

The scheme \(T_{g_1}\) is not necessary of finite type, but locally of finite type. Here the natural action of \(T_{g_1}\) on its image in \(T_{g_1}\) by the product of torus extends to all over \(T_{g_1}\). Each of the \(T_{g_1}\)-orbits in \(T_{g_1}\) is in one-to-one correspondence to cones \(\sigma \in \Sigma_{g_1}\) (Namikawa [16, Theorem(4.6)], [17, Prop.(6.12)]):

\[
\Sigma_{g_1} \ni \sigma \leftrightarrow \mathcal{O}(\sigma) \in \{T_{g_1}\text{-orbits } \subset T_{g_1}\}.
\]

\[
\mathcal{O}(\sigma) = \{ \lim_{t \to \infty} t^n z; z \in T_{g_1}\}.
\]

Here \(\eta \in \sigma^n \cap \overline{\mathcal{Q}}_{g_1}^+, t^n z = (t^{nij} z_{ij})_{i,j}\), and \(\eta\) is considered as an element of one parameter subgroup \(\mathcal{Q}_{g_1}^+\). In this correspondence, one get also \(\sigma' \prec
Therefore $T_{\{0\}} = T_{g_1}$ is the $T_{g_1}$-orbit corresponding to $\{0\} \in \Sigma_{g_1}$, and
$T_{g_1} \hookrightarrow T_g$ defines a Zariski open subset.

Then we set $T_{g_0g_1} \hookrightarrow X_{g,g_1} = H_g \times V_{g_0g_1} \times T_{g_1}$, and define the partial compactification $(U(n) \setminus H_g)_{\Sigma_{g_1}}$ of $T_{g_0g_1}^o \subset T_{g_0g_1}$ in the direction of $F^s_{g_0}$ as

$$(U(n) \setminus H_g)_{\Sigma_{g_1}} = \text{the interior of the closure of } T_{g_0g_1}^o \text{ in } X_{g,g_1}.$$ 
(Namikawa [16, Prop.(6.3)].) For this we have the following proposition
(Namikawa [16, Prop.(6.6), (6.9)]):

**Proposition 4.1.** $\overline{P(n)} = P(n)/U(n)$ acts properly discontinuously on $(U(n) \setminus H_g)_{\Sigma_{g_1}}$. Moreover, if level $n \geq 3$, then this action is without fixed points.

Therefore the quotient $\overline{P(n)} \setminus (U(n) \setminus H_g)_{\Sigma_{g_1}}$ has a structure of normal analytic space. (We remark that $P(n)(\hookrightarrow GL(g_1, \mathbb{Z}))$ is considered as a subgroup of finite index.) On the other hand, let

$$O_{\Sigma_{g_0}} = \bigcup_{\sigma \cap Q^+_g \neq \phi, \sigma \in \Sigma_{g_1}} H_{g_0} \times V_{g_0g_1} \times O(\sigma) \subset X_{g,g_1},$$
then $O_{\Sigma_{g_0}} \subset (U(n) \setminus H_g)_{\Sigma_{g_1}}$ and one can see that the quotient $\overline{P(n)} \setminus O_{\Sigma_{g_0}}$ is a closed subset in the above normal analytic space. By the reduction theory of Siegel, we have the following:

For any point $p \in O_{\Sigma_{g_0}}$, there is a neighborhood $Y$ of $p$ in $(U(n) \setminus H_g)_{\Sigma_{g_1}}$ such that if $z_1 = M \cdot z_2$, $\exists M \in \Gamma_g(n)$ for $z_1, z_2 \in e^{-1}(Y) \cap H_{g_1}$ then $M \in P_{g_0}(n)$.

This means, near $\overline{P(n)} \setminus O_{\Sigma_{g_0}}$, the variety $V_g(n) = \Gamma_g(n) \setminus H_g$ is locally isomorphic to $\overline{P(n)} \setminus (U(n) \setminus H_g)_{\Sigma_{g_1}}$. Therefore we can glue these in a neighbourhood of each boundary point, and put an analytic structure on these neighbourhoods from the one on $\overline{P(n)} \setminus (U(n) \setminus H_g)_{\Sigma_{g_1}}$. The above procedure (partial compactification, and gluing pieces nearby boundary orbits) for all $\Gamma_g(n)$-equivalent classes of rational boundaries, is canonically compatible by the properties of $\Gamma_g(n)$-admissible family of cone decompositions. Finally we obtain a toroidal compactification $\bar{V}_g(n)$ of $V_g(n)$, with underlying set

$$\bar{V}_g(n) = \bigcup_{0 \leq g_0 \leq g} \bigcup_{\text{mod } \Gamma_g(n)} \overline{P_{g_0}(n)} \setminus O_{\Sigma_{g_0}}, \quad (\overline{P_{g_0}} \setminus O_g = V_g(n)).$$
It is shown that $\tilde{V}_g(n)$ is a compact normal space, and by choosing certain suitable cone decompositions, it becomes a smooth and projective variety over $\mathbb{C}$. Moreover $\tilde{V}_g(n) - V_g(n) = D = \bigcup D_i$ is a finite union of smooth irreducible divisors with simply normal crossing.

4.4. The Map from $\tilde{V}_g(n)$ to $V_g(n)^*$

As a set, $V_g(n)^*$ is a disjoint union of modular varieties of lower dimension $= V_g(n) \amalg V^{(g-1)} \amalg \cdots \amalg V^{(0)}$. Here $V^{(i)} = \coprod_{\Gamma_g(n)} \Gamma'_i \setminus F_i$ is the $i$-th rational boundary component ($F_i \simeq H_i$, $0 \leq i \leq g - 1$). Moreover, $V_g(n)^*$ is a projective over $\mathbb{C}$. We have a holomorphic map from $\tilde{V}_g(n)$ to $V_g(n)$,

$$\pi : \tilde{V}_g(n) \to V_g(n)^*.$$ 

The restriction of $\pi$ over a rational boundary component $\Gamma'_i \setminus F_i$ comes from the naturally extended map:

$$(U(n) \setminus H_g)_{\Sigma g_1} \overset{\text{mod} \, \Gamma_g(n)}{\to} V_g(n)^*,$$

and the inverse image of a rational boundary is given by

$$p_{F_{g_0}}^{-1}(\Gamma'_g \setminus F_g) = \mathcal{O}_{\Sigma g_0}$$

(Namikawa [16], §6). Then $\pi$ gives an identity on the open stratum $V_g(n)$ and

$$\pi^{-1}(V^{(g_0)}) = \coprod_{\{F_{g_0}\} \text{ mod } \Gamma_g(n)} \overline{P_{g_0}(n)} \setminus \mathcal{O}_{\Sigma g_0}.$$ 

Later we consider the intersections of the boundary divisors $D_i$ of $\tilde{V}_g(n)$ inside $\pi^{-1}(V^{(g_0)})$. A description of these intersections in local coordinates can be given from the local structure of

$$\mathcal{O}_{\Sigma g_0} = \bigcup H_{g_0} \times V_{g_0 g_1} \times \mathcal{O}(\sigma)$$

or more precisely from the local structure of

$$\bigcup_{\sigma \cap \mathcal{Q}_{g_1}^+ \neq \emptyset, \sigma \in \Sigma g_1} \mathcal{O}(\sigma).$$

Details are discussed in §6.
5. Poincaré residue maps on the space of holomorphic Siegel modular forms

From now on we set \( V = V_g(n) \), and \( N = \dim V = \frac{1}{2} g(g + 1) \). The space of holomorphic Siegel modular forms of weight \( g + 1 \) is identified with a subspace of \( H^N(V, \mathbb{C}) \). By use of the Poincaré residue map, we study the weight filtrations on this subspace of \( H^N(V, \mathbb{C}) \). In §3, it is shown that the Poincaré residue map defined for the edge part of \( Gr^W_{m+N}H^N(V, \mathbb{C}) \), can be transferred to the space \( \Gamma(\tilde{V}, \Omega^N_V(\log D)) \) (cf. Lemma (3.1) and (3.2)). For \( \omega \in \Gamma(\tilde{V}, \Omega^N_V(\log D)) \), denote the pull-back of \( \omega \) to \( H^g \) by \( \omega_0 = f(z_1, \cdots, z_N)dz_1 \wedge \cdots \wedge dz_N \) in the coordinates of \( H^g \).

**Lemma 5.1.** If \( g \geq 2 \), \( \Gamma(\tilde{V}, \Omega^N_V(\log D)) \cong M_{g+1}(\Gamma_g(n)) \). Here \( M_{g+1}(\Gamma_g(n)) \) is the space of holomorphic Siegel modular forms on \( H_g \) of weight \( g + 1 \) for \( \Gamma_g(n) \),

\[
M_{g+1}(\Gamma_g(n)) = \{ f : H_g \to \mathbb{C}; \text{ holomorphic} \mid f(\gamma(Z)) = \det(CZ + D)^{g+1}f(Z) \text{ for } \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n) \}.
\]

**Proof.** This is a standard fact (e.g. Chai-Faltings [4, Chap. V]). But for our purpose, we here review its proof.

We put \( \omega_s = \bigwedge_{1 \leq i \leq j \leq g} dz_{ij} \) where \( Z = (z_{i,j})_{i, j=1,\ldots,g} \in H_g \). For \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n) \), it is well known that \( \gamma^*\omega_s = \det(CZ + D)^{-g+1}\omega_s \), see for example Maass [13, §3, p.23]. From the \( \Gamma_g(n) \)-invariance of \( \omega_0 = f(Z)\omega_s \), \( f(z) \) is a holomorphic modular form of weight \( g + 1 \). Thus the inclusion \( \Gamma(\tilde{V}, \Omega^N_V(\log D)) \subset M_{g+1}(\Gamma_g(n)) \) is shown. The converse inclusion is proved as followings.

We fix a \( g_0, 0 \leq g_0 \leq g - 1 \), and denote by \( \{ F_0, F_1, \cdots, F_r \} \) all of the \( \Gamma_g(n) \)-equivalence classes of \( g_0 \)-th rational boundaries \( (\simeq H_{g_0}) \) of \( H_g \). We take \( F_0 \) as the standard \( g_0 \)-th rational boundary \( (= F_{g_0}^{st}) \) of \( H_g \), and write corresponding maximal \( \mathbb{Q} \)-parabolic subgroup as \( P_0 \subset Sp(g, \mathbb{R}) \). Each
$F_l$ transformed into $F_0$ by some $\gamma_l \in \Gamma_g = Sp(g, Z)$, and we fix $\gamma_l \in \Gamma_g$ such that $F_l = \gamma_l F_0$ for $l = 1, \ldots, r$. Then Fourier-Jacobi expansion of $f \in M_{g+1}(\Gamma_g(n))$ at $F_l$ is

$$j(\gamma_l^{-1}, Z)^{g+1} f(Z) = a_0(z_1^l, z_2^l; F_l) + \sum_{\{T\}} a_T(z_1^l, z_2^l; F_l) e\left(\frac{tr T z_3^l}{n}\right),$$

where

$$\gamma_l^{-1}(Z) = \begin{pmatrix} z_1^l & z_2^l & z_3^l \\ z_2^l & z_3^l & \end{pmatrix},$$

with $z_1^l \in M_{g_0}(C)$, $z_2^l \in M_{g_1}(C)$, $g_1 = g - g_0$, and

$$j(\gamma, Z) = \det(CZ + D) \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $\{T\}$ runs all the set of nonzero non-negative, half-integral symmetric matrices of degree $g_0$. Recall that for $g \geq 2$, the non-negativity of $T$ is a consequence of the Koecher principle.

We make a toroidal compactification $\tilde{V}_g(n)$ of $V_g(n)$ by taking a $\Gamma_g(n)$-admissible family of cone decompositions. We consider the map $\pi : \tilde{V}_g(n) \to V_g(n)^\ast$. And for $V^{(l)} \subset V_g(n)^\ast$, let $m = m(l)$ be the greatest integer such that $\pi^{-1}(V^{(l)}) \cap i_m(D^{[m]}) \neq \emptyset$. (Here we concern ourselves with the maximally degenerate boundary in $\pi^{-1}(V^{(l)})$.) Then local coordinates system at a point of $\pi^{-1}(V^{(l)}) \cap i_m(D^{[m]})$ is written as $\{(z_i), (u_j), (q_k = e(w_k/n))\}$. Here $z_i$, $1 \leq i \leq \frac{1}{2} g_0(g_0 + 1)$ (resp. $u_j$, $1 \leq j \leq g_0 g_1$) run those upper triangle coefficients of $z_1^l$ (resp. $z_2^l$). For $1 \leq k \leq \frac{1}{2} g_1 g_1 + 1 = d$, $w_k$ is a linear combination of uppertriangle coefficients of $z_3^l$. Now we can rewrite above Fourier-Jacobi expansion in this coordinates as follows.

$$j(\gamma_l^{-1}, Z) f(Z) = a_0((z_i), (u_j); F_l) + \sum_{\{T\}} a_T((z_i), (u_j); F_l) q_1^{t_1} \cdots q_d^{t_d}.$$

Note that for nonnegativity of $T$, all $t_n \geq 0, n = 1 \cdots d$, and $(t_1, \ldots, t_d) \neq (0, \ldots, 0)$.

**Remark.** For example, if $g_1 = 2$, writing

$$z_3^l = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix},$$
then
\[
\begin{align*}
\mathbf{e} & \left(t \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix} \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \right) \\
& = \mathbf{e}(t_1(\tau_1 + \tau_2) + t_3(\tau_2 + \tau_3) - (t_1 + t_3 - 2t_2)\tau_2) \\
& = q_1^{t_1}q_2^{t_1 + t_3 - 2t_2}q_3^{t_3}.
\end{align*}
\]

Now for the semi-positivity of \( T \), we have that \( t_1, t_3 \geq 0 \) and \( t_1 + t_3 - 2t_2 = (1, -1)T \begin{pmatrix} 1 \\ -1 \end{pmatrix} \geq 0 \).

On the other hand, since we have that
\[
\wedge d\omega^l_1, \wedge d\omega^l_{2,j} \wedge d\omega^l_{3,k} = \text{const.} \times \wedge dz_i \wedge du_j \wedge dq_k \prod_{k=1}^d q_k,
\]
the form \( \omega \) is described with the above local coordinates as
\[
\omega = c_l \left\{ (a_0((z_i), (u_j); F_l) \\
+ \sum_{T} a_T((z_i), (u_j); F_l)q_1^{t_1} \cdots q_d^{t_d} \right\} \wedge dz_i \wedge du_j \wedge \frac{dq_k}{\prod_{k=1}^d q_k},
\]
in which \( c_l \neq 0 \) is a constant which depends on \( \gamma_l \). (Ash, Mumford, Rapoport, Tai [1] chap.4) If we take \( g_0 = 0 \) then it is shown that \( \omega_0 = f(Z)\omega_s \) defines a meromorphic differential form at most with poles of order one on rational boundaries. This settles the proof of Lemma (5.1). \( \square \)

Moreover by the definition of \( \text{Res}_{[m]} \), we also conclude the following lemma.

**Lemma 5.2.** With local coordinates at one point of \( \pi^{-1}V(l) \cap i_m(D^{[m]}) \), the image of \( \omega \) by residue map: \( \text{Res}_{[m]} \omega \in F^{l-m}H^l(D^{[m]}, \mathbb{C})(-m) \), is described as
\[
\text{Res}_{[m]} \omega = c_l \cdot a_0((z_i), (u_j); F_l) \wedge dz_i \wedge du_j.
\]
We rewrite above statement by Siegel’s \( \Phi \)-operator (cf. Maass [13, §13]).

For \( f \in M_k(\Gamma_g(n)) \), Siegel’s \( \Phi \)-operator is defined as

\[
\Phi f(Z_1) = \lim_{t \to \infty} f \left( \begin{array}{cc} Z_1 & 0 \\ 0 & \sqrt{-1} t \end{array} \right), \quad Z_1 \in H_{g-1}.
\]

This is well defined and defines a holomorphic Siegel modular form of weight \( k \) for \( \Gamma_{g-1}(n) \) on \( H_{g-1} \).

In Fourier expansion \( f(Z) = \sum_{\{T\}} a(T)e(\text{tr}(TZ)/n) \),

\[
\Phi f(Z_1) = \sum_{\{T_1\}} a(T_1)e(\text{tr}(T_1Z_1)/n),
\]

\[
a(T_1) = a \left( \begin{array}{cc} T_1 & 0 \\ 0 & 0 \end{array} \right); \quad T_1 \text{ is of rank } \leq g-1.
\]

Iterating this, then

\[
\Phi^j f(Z_j) = \sum_{\{T_j\}} a \left( \begin{array}{cc} T_j & 0 \\ 0 & 0 \end{array} \right)e(\text{tr}(T_jZ_j)/n), \quad Z_j \in H_{g-j}.
\]

Then each cusp component of \( \text{Res}_{\omega} \) is written as \((0\text{-th term of } \Phi^j(f)) \wedge dz_i \wedge du_j\) (cf. Chai-Faltings [4, Chap. V, Prop. 1.6]). Hence to show our main results we need to find a modular form of weight \( g+1 \) which remains non zero under the action of Siegel \( \Phi \)-operator at one specified cusp, but vanishes at all other cusps. This is obtained by an Eisenstein series in §7.

6. **Structure of degenerate coordinates over an Satake rational boundary**

Let \( \pi : \tilde{V}_g(n) \to V_g(n)^* \) is the natural morphism defined in §§4.4, and \( D^{[m]} \) as in §1. In order to know explicitly the local defining equations of \( D^{[m]} \), in this section we investigate the structure of \( \pi^{-1}(V^{(g_0)}) \cap i_m(D^{[m]}) \), where \( V^{(g_0)}(\subset V_g(n)^*) \) is a union of \( g_0 \)-th rational boundary components. Now \( \tilde{V}_g(n) \) can be taken to be a smooth projective variety. Indeed, as in Igusa [11], there is a \( \Gamma_g(n) \)-admissible family of cone decomposition: the central cone decompositions. When we make a toroidal compactification associated with these cone decompositions, it is a normalized blowing-up of Satake compactification at some ideals defining boundary components. This
is non-singular projective variety over $\mathbb{C}$ if $g \leq 3$ (in this case it is the same as the Delony-Voronoi compactification by Namikawa [16]). If $g \geq 4$, we take a suitable subdivision of the central cone decompositions and we can get a smooth projective compactification (Namikawa [17, (7.20), (7.26)]).

As in $\S$ 5, define the torus coordinates $\{q_k\}_{k=1,\ldots,d}$, $d = \frac{1}{2}g_1(g_1 + 1)$, $g_1 = g - g_0$, for the cone decompositions of the space $\{z_3 = t z_3 \in M_{g_1}(\mathbb{C})\}$.

Then within $\pi^{-1}(V(g_0))$, the intersections of divisors $i_m(D)(m)$ are defined as common zeros of $m$ coordinates among $\{q_k\}_{k=1,\ldots,d}$. Remark that inside $\pi^{-1}(V(g_0))$ the coordinates $\{q_k\}$ are the only those which can determine the boundary components. Hence over the $g_0$-th rational boundaries the number of degenerated coordinates can be at most $d = \frac{1}{2}g_1(g_1 + 1)$ in the toroidal compactification. We get the following:

**Lemma 6.1.** $\pi^{-1}(V(g_0)) \cap i_m(D)(m)$ is non-empty for only $m \leq \frac{1}{2}g_1(g_1 + 1)$, $g_1 = g - g_0$. Especially the locus $i_N(D)(N)$, $N = \frac{1}{2}g(g + 1)$ intersects with only $\pi^{-1}(V(0))$.

We have two examples which appear in Namikawa [16] and Nakamura [15].

**Example 1.** $g = 2$.

$V_2(n)^* = V_2(n) \sqcup V_1 \sqcup V_0$. Each cone in the Delony-Voronoi (abbreviated D-V) decomposition $\Sigma_2$ of $\overline{Q}_2^{+}$ is transformed into one of the followings by the action of $GL(2,\mathbb{Z})$.

\[
\sigma_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad \sigma_1 = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} ; \lambda \geq 0 \right\},
\]

\[
\sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} ; \lambda_1, \lambda_2 \geq 0 \right\},
\]

\[
\sigma_3 = \left\{ \begin{pmatrix} \lambda_1 + \lambda_2 & -\lambda_2 \\ -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix} ; \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}.
\]

For $\overline{Q}_{0,\mathbb{Z}}^{+}: \overline{Q}_{1,\mathbb{Z}}^{+} \hookrightarrow \overline{Q}_2^{+}$, we can restrict $\Sigma_2$ on $\overline{Q}_{0,\mathbb{Z}}^{+}: \overline{Q}_{1,\mathbb{Z}}^{+}$ then get $\Sigma_0 = \{\sigma_0\}$, $\Sigma_1 = \{\sigma_0, \sigma_1\}$. We have a bilinear form on $\overline{Q}_2^{+}: \overline{Q}_2 \times \overline{Q}_2 \to \mathbb{R}$ : $(y, y') \mapsto tr(yy')$. Now $\sigma_3 \cap \overline{Q}_2^{+}$ is generated by

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\]
hence dual bases in $\hat{\sigma}_3 \cap M$ are

$$
\begin{pmatrix}
1 & 1/2 \\
1/2 & 0
\end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\
1/2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/2 \\
-1/2 & 0 \end{pmatrix}.
$$

We take coordinates as

$$
q_1 = e\left(\frac{z_1 + z_2}{n}\right), q_2 = e\left(\frac{-z_2}{n}\right), q_3 = e\left(\frac{z_2 + z_3}{n}\right),
$$

and construct affine torus embeddings:

$$
T_{\sigma_3} = \text{Spec } \mathbb{C}[q_1, q_2, q_3] = \{t = (t_1, t_2, t_3)\}
\supset T_{\sigma_2} = \{t \in T_{\sigma_3}; t_2 \neq 0\}
\supset T_{\sigma_1} = \{t \in T_{\sigma_3}; t_1 \neq 0, t_2 \neq 0\}
\supset T_{\sigma_0} = \{t \in T_{\sigma_3}; t_1 \neq 0, t_2 \neq 0, t_3 \neq 0\} \simeq T_2.
$$

Moreover the orbit for each $\{\sigma_i\}$ is written as

$$
\mathcal{O}(\sigma_0) = T_2,
\mathcal{O}(\sigma_1) = \{(t_1, t_2, 0); t_1 \neq 0, t_2 \neq 0\},
\mathcal{O}(\sigma_2) = \{(0, t_2, 0); t_2 \neq 0\},
\mathcal{O}(\sigma_3) = \{(0, 0, 0)\}.
$$

We have for instance $\sigma_2 \prec \sigma_3 \Leftrightarrow \mathcal{O}(\sigma_3) \subset \overline{\mathcal{O}(\sigma_2)}$ etc.

Now we consider the structure of the D-V compactification over a Satake rational boundary.

(1) $\pi^{-1}(V_2(n)) = V_2(n)$.

(2) $\pi^{-1}(V^{(1)}) (g_0 = g_1 = 1)$,

$$
\mathcal{O}_{\Sigma_1} = \bigcup_{\sigma \cap Q^+ \neq \emptyset, \sigma \in \Sigma_1} H_1 \times \mathbb{C} \times \mathcal{O}(\sigma)
= H_1 \times \mathbb{C} \times \mathcal{O}(\sigma_1) = H \times \mathbb{C} \times \{t_3 = 0\}.
$$

Therefore, in $\pi^{-1}(V^{(1)}) = \bigcup \overline{P_1(n)} \setminus \mathcal{O}_{\Sigma_1}$, the number of degenerating coordinates is $\leq 1$. 


(3) \( \pi^{-1}(V^{(0)}) \) \((g_0 = 0, g_1 = 2)\),

\[
O_{\Sigma_2} = \bigcup_{\sigma \in \Sigma_2} O(\sigma) = \bigcup_{\{\sigma_2\}} O(\sigma_2) \cup \bigcup_{\{\sigma_3\}} O(\sigma_3)
\]

\[
= \bigcup \{t_1 = t_3 = 0\} \cup \bigcup \{t_1 = t_2 = t_3 = 0\}.
\]

Here, \( \{\sigma_2\} \) and \( \{\sigma_3\} \) denote the set of \( GL(2, \mathbb{Z}) \)-transformations of \( \sigma_2 \) and \( \sigma_3 \) respectively. Hence, in \( \pi^{-1}(V^{(0)}) = \bigcup P_0(n) \setminus O \), the number of degenerating coordinates is \( \leq 3 \). Also \( D^{[2]} \) is a disjoint union of \( \mathbb{P}^1 \)'s whose image in \( \tilde{V}_g(n) \) intersect with each other at \( i_3(D^{[3]}) = \{ \text{points} \} \).

**Example 2.** \( g = 3 \).

\( V_3(n)^* = V_3(n) \Pi V^{(2)} \Pi V^{(1)} \Pi V^{(0)} \). Each cone of the D-V decomposition of \( \Sigma_3 \) is transformed into one of the followings by the action of \( GL(3, \mathbb{Z}) \).

\[
\sigma_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \sigma_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix} ; \lambda \geq 0 \right\},
\]

\[
\sigma_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} ; \lambda_1, \lambda_2 \geq 0 \right\},
\]

\[
\sigma_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix} ; \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\},
\]

\[
\sigma_4 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} ; \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\},
\]

\[
\sigma_5 = \left\{ \begin{pmatrix} \lambda_4 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix} ; \lambda_1, \lambda_2 \geq 0 \right\},
\]

\[
\sigma_6 = \left\{ \begin{pmatrix} \lambda_1 + \lambda_4 & -\lambda_1 & 0 \\ -\lambda_1 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix} ; \lambda_1, \lambda_2 \geq 0 \right\},
\]
\[ \sigma_7 = \left\{ \begin{pmatrix} \lambda_4 + \lambda_5 & -\lambda_5 & 0 \\ -\lambda_5 & \lambda_1 + \lambda_2 + \lambda_5 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix}; \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}, \]

\[ \sigma_8 = \left\{ \begin{pmatrix} \lambda_4 + \lambda_5 + \lambda_6 & -\lambda_5 & -\lambda_6 \\ -\lambda_5 & \lambda_1 + \lambda_2 + \lambda_5 & -\lambda_2 \\ -\lambda_6 & -\lambda_2 & \lambda_2 + \lambda_3 + \lambda_6 \end{pmatrix}; \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}. \]

As in the case of \( g = 2 \), we consider subdecomposition \( \Sigma_0 = \{\sigma_0\}, \Sigma_1 = \{\sigma_0, \sigma_1\}, \Sigma_2 = \{GL(2, \mathbb{Z})\text{-transformations of} \sigma_0, \sigma_1, \sigma_2\} \).

For each generator of \( \sigma_8 \cap \mathbb{Q}_3^+ \) over \( \mathbb{Z} \),

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},
\]

the dual bases as in case \( g = 2 \) are

\[
\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix},
\]

\[
\begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}.
\]

Then we set as coordinates:

\[ q_1 = e^{\left(\frac{z_1 + z_2 + z_3}{n}\right)}, q_2 = e^{\left(\frac{-z_2}{n}\right)}, q_3 = e^{\left(\frac{-z_3}{n}\right)}, q_4 = e^{\left(\frac{z_2 + z_4 + z_5}{n}\right)}, q_5 = e^{\left(\frac{-z_5}{n}\right)}, q_6 = e^{\left(\frac{z_3 + z_5 + z_6}{n}\right)}. \]

Now we get an affine torus embedding:

\[ T_{\sigma_8} = Spec \mathbb{C}[q_1, q_2, q_3, q_4, q_5, q_6] \simeq \{t = (t_1, t_2, t_3, t_4, t_5, t_6)\}. \]
Also the orbits associated with cones are

\[ \mathcal{O}(\sigma_0) = T_3 \]
\[ \mathcal{O}(\sigma_1) = \{(t_1, t_2, t_3, t_4, t_5, 0) ; t_i \neq 0, \ i = 1, 2, 3, 4, 5\} \]
\[ \mathcal{O}(\sigma_2) = \{(t_1, t_2, t_3, 0, t_5, 0) ; t_i \neq 0, \ i = 1, 2, 3, 5\} \]
\[ \mathcal{O}(\sigma_3) = \{(t_1, t_2, t_3, 0, 0, 0) ; t_i \neq 0, \ i = 1, 2, 3\} \]
\[ \mathcal{O}(\sigma_4) = \{(0, t_2, t_3, 0, t_5, 0) ; t_i \neq 0, \ i = 2, 3, 5\} \]
\[ \mathcal{O}(\sigma_5) = \{(0, t_2, t_3, 0, 0, 0) ; t_2 \neq 0, \ t_3 \neq 0\} \]
\[ \mathcal{O}(\sigma_6) = \{(0, 0, t_3, t_4, 0, 0) ; t_3 \neq 0, \ t_4 \neq 0\} \]
\[ \mathcal{O}(\sigma_7) = \{(0, 0, t_3, 0, 0, 0) ; t_3 \neq 0\} \]
\[ \mathcal{O}(\sigma_8) = \{(0, 0, 0, 0, 0, 0)\}. \]

(1) \( \pi^{-1}(V_3(n)) = V_3(n) \).
(2) \( \pi^{-1}(V^{(2)}) \ (g_0 = 2, \ g_1 = 1) \),

\[ \mathcal{O}_{\Sigma_1} = \bigcup_{\sigma \cap Q_1^+, \sigma \in \Sigma_1} H \times V_{2,1} \times \mathcal{O}(\sigma) \]
\[ = H_2 \times V_{2,1} \times \mathcal{O}(\sigma_1) \]
\[ = H_2 \times V_{2,1} \times \{t_6 = 0\}. \]

Hence in \( \pi^{-1}(V^{(2)}) = \bigcup \overline{P_2(n)} \smallsetminus \mathcal{O}_{\Sigma_1} \), the number of degenerating coordinates is \( \leq 1 \).

(3) \( \pi^{-1}(V^{(1)}) \ (g_0 = 1, \ g_2 = 2) \),

\[ \mathcal{O}_{\Sigma_2} = \bigcup_{\sigma \cap Q_2^+, \sigma \in \Sigma_2} H_1 \times V_{1,2} \times \mathcal{O}(\sigma) \]
\[ = \bigcup_{\{\sigma_2\}} H_1 \times V_{1,2} \times \mathcal{O}(\sigma_2) \cup \bigcup_{\{\sigma_3\}} H_1 \times V_{1,2} \times \mathcal{O}(\sigma_3) \]
\[ = \bigcup_{\{\sigma_2\}} H_1 \times V_{1,2} \times \{t_4 = t_6 = 0\} \cup \bigcup_{\{\sigma_3\}} H_1 \times V_{1,2} \times \{t_4 = t_5 = t_6 = 0\}. \]

Hence in \( \pi^{-1}(V^{(1)}) = \bigcup \overline{P_1(n)} \smallsetminus \mathcal{O}_{\Sigma_2} \), the number of degenerating coordinates is \( \leq 3 \).

(4) \( \pi^{-1}(V^{(0)}) \ (g_0 = 0, \ g_1 = 3) \),

\[ \mathcal{O}_{\Sigma_3} = \bigcup_{\sigma \cap Q_3^+, \sigma \in \Sigma_3} \mathcal{O}(\sigma) \]
\[
\bigcup_{\{\sigma_4\}} \mathcal{O}(\sigma_4) \cup \bigcup_{\{\sigma_5\}} \mathcal{O}(\sigma_5) \cup \bigcup_{\{\sigma_6\}} \mathcal{O}(\sigma_6) \cup \bigcup_{\{\sigma_7\}} \mathcal{O}(\sigma_7) \cup \bigcup_{\{\sigma_8\}} \mathcal{O}(\sigma_8) = \bigcup \{t_1 = t_4 = t_6 = 0\} \cup \bigcup \{t_1 = t_4 = t_5 = t_6 = 0\}
\]
\[
\bigcup \{t_1 = t_2 = t_5 = t_6 = 0\} \cup \bigcup \{t_1 = t_2 = t_4 = t_5 = t_6 = 0\}
\]
\[
\bigcup \{t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0\}.
\]
Hence in \(\pi^{-1}(V(0)) = \overline{P_0(n) \setminus \Sigma_3}\), the number of degenerating coordinates is \(\leq 6\).

**Remark 6.2.** We can obtain that \(D^{[N-1]}\) is a union of \(P_1^1\). Indeed the components of \(D^{[N-1]}\) correspond to those \(N-1\) dimensional cones in \(\Sigma_g\). By the construction of the partial compactification each components of \(D^{[N-1]}\) contains an affine line \(A^1\) as Zariski dense subset. Then we obtain our assertion.

### 7. Eisenstein series

In this section we review the some basic facts on holomorphic Eisenstein series. These are used in the proof of main results in \(\S 8\) and \(\S 9\).

#### 7.1. Siegel Eisenstein series of higher weights

Let \(\{e_0, \cdots, e_r\}\) be the set of all 0-dimensional cusps in \(V_g(n)^*\), and choose \(e_0\) to be the standard one. Fix an element \(\gamma_i \in \Gamma_g\) such that \(e_i = \gamma_i e_0\). Then we have only to construct holomorphic Siegel modular forms of weight \(g + 1\) for \(\Gamma_g(n)\) on \(H_g\) such that the constant term of its Fourier expansion at some \(e_i\) does not vanish. Equivalently we construct holomorphic Siegel modular forms of weight \(g + 1\) such that \(\Phi^g_i f = a_0(e_i) \neq 0\), here \(\Phi_i\) is the Siegel operator at \(e_i\).

We consider Siegel Eisenstein series of weight \(k\) for \(\Gamma_g(n)\) on \(H_g\) associated with each \(e_i\). It is defined for \(k > g + 1\), \(Z \in H_g\) as

\[
E_{e_i}(Z; k) = \sum_{\sigma \in \Gamma_g(n) \setminus P_i \Gamma_g(n)} j(\gamma_i^{-1} \sigma, Z)^{-k},
\]

where \(j(\sigma, Z) = \det(CZ + D), \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n), P_i = \gamma_i P_0 \gamma_i^{-1},\) and \(P_0\) is the stabilizer of \(e_0\) in \(Sp(g, \mathbb{R})\), that is the standard maximal
Q-parabolic subgroup:

\[
P_0 = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(g, \mathbb{R}), \ A, D \in GL(g, \mathbb{R}) \right\}.
\]

**Proposition 7.1.** (i) For \( k > g + 1 \), the above infinite series is absolutely convergent, and it defines a holomorphic modular form of weight \( k \) for \( \Gamma_g(n) \). We call this series Siegel Eisenstein series of weight \( k \).

(ii) The constant term of the Fourier expansion of the series at the cusp \( e_i \) of \( E_{e_i}(Z; k) \) does not vanish, and the constant terms at the other cusps \( e_j \) not \( \Gamma_g(n) \)-equivalent to \( e_i \) are equal to zero.

**Proof.** Statements for the absolutely convergence and the constant term of Fourier expansion at \( e_i \) are well known (Maass [13, §14]). We prove the last statement. Now we define a relation for elements of \( \Gamma_g \).

For \( N_1, N_2 \in \Gamma_g, \ N_1 \sim \Gamma_g(n) \Leftrightarrow \exists M \in \Gamma_g(n) \) such that \( N = N_1^{-1}MN_2 \in P_0 \cap \Gamma_g \).

Then we claim that:

For \( \gamma_i, \gamma_j \in \Gamma_g \) which are not equivalent to each other and \( \gamma_ie_0 = e_i, \gamma_je_0 = e_j \), the constant term of the Fourier expansion at \( e_j \) of \( E_{e_i}(Z, k) \) is equal to zero. That is,

\[
\Phi_g(\sum_{\sigma \in \Gamma_g(n) \cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-k}|\gamma_j) = 0.
\]

**Proof of the claim.**

\[
\Phi_g(\sum_{\sigma \in \Gamma_g(n) \cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-k}|\gamma_j) = \Phi_g(j(\gamma_j, Z)^{-k}\sum_{\sigma \in \Gamma_g(n)} j(\gamma_i^{-1}\sigma, \gamma_j(Z))^{-k})
\]

\[
= \lim_{\lambda \to \infty} \sum_{\sigma \in \Gamma_g(n)} j(\gamma_i^{-1}\sigma, \sqrt{-\overline{1}\lambda}1_g)^{-k} = \sum_{\lambda \to \infty} \lim_{\lambda \to \infty} j(\gamma_i^{-1}\sigma, \sqrt{-\overline{1}\lambda}1_g)^{-k}.
\]

We put \( \gamma_i^{-1}\sigma\gamma_j = N_\sigma = \begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix} \in \Gamma_g \) (\( N_\sigma \) depends on \( \sigma \) under fixed \( \gamma_i, \gamma_j \)). Then \( j(\gamma_i^{-1}\sigma\gamma_j, \sqrt{-\overline{1}\lambda}1_g) = \det(\sqrt{-\overline{1}\lambda}C_\sigma + D_\sigma) \) is a polynomial in \( \lambda \) (including the case of degree 0). And if the degree of this polynomial is
greater than 0, then we have \( \lim_{\lambda \to \infty} |\det(\sqrt{-1}\lambda C_\sigma + D_\sigma)| \to \infty \) as \( \lambda \to \infty \).

Hence in the above infinite series the term associated to this \( N_\sigma \) is equal to zero. Therefore we want to know in which case \( \det(\sqrt{-1}\lambda C_\sigma + D_\sigma) \) is a constant independent of \( \lambda \).

Because \( N_\sigma \in \Gamma_g \), we have \( C_\sigma^t D_\sigma = D_\sigma^t C_\sigma \) for \( C_\sigma, D_\sigma \) (It means that \( C_\sigma \) and \( D_\sigma \) make a symmetric pair). Suppose \( \operatorname{rank} C_\sigma = r \), then we can take \( U_1, U_2 \in GL(g, \mathbb{Z}) \) such that \( U_1 C_\sigma = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \) where \( C_1 \) is a \( r \times r \) matrix and its determinant is not zero. Write formally as \( U_1 D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} U_2^{-1} \), then we get \( U_1 C_\sigma^t (U_1 D_\sigma) = U_1 C_\sigma^t D_\sigma^t U_1 = U_1 D_\sigma^t (U_1 C_\sigma). \) This means that \( U_1 C_\sigma \) and \( U_1 D_\sigma \) is also a symmetric pair.

In particular, since \( \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} tD_1 & tD_2 \\ tD_3 & tD_4 \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} tC_1 & 0 \\ 0 & 0 \end{pmatrix} \), we get \( C_1^t D_1 = D_1^t C_1 \) and \( D_3 = 0 \). Under this consideration, we investigate \( \det(\sqrt{-1}\lambda C_\sigma + D_\sigma) \) in the followings. Our consideration is separated into three cases.

(Case 1) \( \operatorname{rank} C_\sigma = g \).

In this case, since \( \det(\sqrt{-1}\lambda C_\sigma + D_\sigma) = \det C_\sigma \det(\sqrt{-1}\lambda \mathbf{1}_g + C_\sigma^{-1} D_\sigma) \)
and \( \det(\sqrt{-1}\lambda \mathbf{1}_g + C_\sigma^{-1} D_\sigma) \) has non-zero degree \( g \)-term, we obtain that \( |\det(\sqrt{-1}\lambda C_\sigma + D_\sigma)| \to \infty \) as \( \lambda \to \infty \).

(Case 2) \( 0 < \operatorname{rank} C_\sigma = k \leq g - 1 \).

First we have
\[
0 \neq \det((\sqrt{-1}\lambda C_\sigma + D_\sigma) = \det U_1^{-1} \det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} tU_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} U_2^{-1}).
\]

where \( C_1 \in M_k(\mathbb{Z}) \) is of rank \( k \), \( D_2 \in M_{g-k}(\mathbb{Z}) \), and \( \det D_4 \neq 0 \). Moreover we have that
\[
\det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} tU_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} U_2^{-1})
= \det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} tU_2 U_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}) \det U_2^{-1}. \quad (*)
\]
Here we consider the matrix $U_2 U_2 = \begin{pmatrix} V & * \\ * & * \end{pmatrix}$, with $tV = V \in M_k(\mathbb{Z})$, for $U_2 \in GL(g, \mathbb{Z})$. Then the $k \times k$ symmetric matrix $V$ is of full rank. Indeed if $V$ is not invertible then take a nonzero vector $v \in \text{Ker} V$, and make $w = (v, 0) \in \mathbb{R}^g$. Then $tw^t U_2 U_2 w = |U_2 w|^2 = 0$, for $w \neq 0, \in \mathbb{R}^g$, which contradicts to the condition that $\det U_2 \neq 0$. Therefore we obtain that

$$\det(\sqrt{-1} \lambda \begin{pmatrix} C_1 V & * \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}) \times (\text{nonzero constant}) = \det(\sqrt{-1} \lambda 1_k + *) \times (\text{nonzero constant}).$$

(We remark that $\det C_1 V \neq 0$ in the last equality.)

Thus, since $\det(\sqrt{-1} \lambda 1_k + *)$ has a nontrivial term of degree $k$ of $\lambda$, the term $|j(N_\sigma, \sqrt{-1} \lambda 1_g)^{-k}|$ associated with this $N_\sigma$ vanishes, when $\lambda \to \infty$.

*(Case 3)* rank $C_\sigma = 0$.

This means $\gamma_i^{-1} \sigma \gamma_j \in P_0 \cap \Gamma_g$. Since $\sigma \in \Gamma_g(n)$, it contradicts to that $\gamma_i$ and $\gamma_j$ is not equivalent to each other.

These prove the above claim. □

**Proof of the Proposition 7.1.** Now we take two cusps $e_i = \gamma_i e_0$ and $e_j = \gamma_j e_0$ which are not $\Gamma_g(n)$-equivalent with each other. Suppose $\gamma_i \sim \gamma_j$, thus there exists an element $M \in \Gamma_g(n)$ such that $N = \gamma_i^{-1} M \gamma_j \in P_0 \cap \Gamma_g$. Then we have the following equality for those maximal $\mathbb{Q}$-parabolic subgroup corresponding to each of $e_i, e_j$

$$P_j = \gamma_j P_0 \gamma_j^{-1} = M^{-1} \gamma_i N P_0 N^{-1} \gamma_i^{-1} M = M^{-1} P_i M,$$

which contradicts to the choice of $e_i$ and $e_j$. Therefore, together with above claim, the proposition is proved. □

**7.2. Eisenstein series of low weight**

As far as $Gr_{2n}^W H^N(V_g(n), \mathbb{C})$ is concerned, what we want to obtain is a holomorphic modular form of weight $g + 1$. But for $k = g + 1$, above infinite series which defines $E_{e_i}(Z, k)$ does not converge. Therefore some modification is needed for above series to make sense. For this purpose we
consider a series $E_{e_i}(Z, s; g+1)$ as below. This infinite series of $Z \in H_g$, $s \in \mathbb{C}$ is defined as

$$E_{e_i}(Z, s; g+1) = \sum_{\sigma \in \Gamma_{g}(n) \cap \mathbb{P}_i \backslash \Gamma_{g}(n)} j(\gamma_i^{-1}\sigma, Z)^{-(g+1)}|j(\gamma_i^{-1}\sigma, Z)|^{-s}.$$ 

This series absolutely converges for $\text{Re } s > 0$, and is known to be meromorphically continued to all $s$-plane (after some modification [22, p.426], then by a theory of Langlands [12], Arthur [2]). Moreover, we recall some theorems of Shimura ([22, Theorem (7.1)]).

\textbf{Theorem 7.2.} (Shimura)

(i) If $g \geq 2$, $E_{e_i}(Z, s; g+1)$ is holomorphic in $s$ at $s = 0$, and moreover $E_{e_i}(Z, s; g+1)|_{s=0}$ defines a holomorphic function of $Z \in H_g$.

(ii) The same result as in (ii) of Proposition (7.1) is valid for the constant terms of the Fourier expansions of $E_{e_i}(Z, s; g+1)|_{s=0}$ at 0-dimensional cusps.

Shimura proved this theorem by explicit calculations of the Fourier coefficients of some related Eisenstein series at some cusp (image of an intertwining operator) and showing its analytic continuation. He also get some other statement on Eisenstein series (see remarks below). We remark, in the case of $g = 1$, Eisenstein series of weight 2 of above type (without non-trivial character) does not define a holomorphic function of $Z \in H_g$ at $s = 0$. (This is, in other terms, the residue theorem for a Riemann surface.) The holomorphy of above Eisenstein series at $s = 0$ is a property of Siegel ($g \geq 2$) modular forms.

8. Weight filtration of $H^N(V, \mathbb{C})$

Finally we state some results about the graded quotients $Gr^W_\bullet H^N(V_g(n), \mathbb{C})$. $N = 1/2g(g+1)$ is the dimension of $V_g(n)$. For some related other results, in particular in the case of $g = 2$, see also Oda-Schwermer [19]. By the definition of $W_\bullet$ on $\Omega^*_V(logD)$, we obtain that $W_m H^N(V_g(n), \mathbb{C}) = 0$ for $m \leq N$, and $W_{2N} H^N(V_g(n), \mathbb{C}) = H^N(V_g(n), \mathbb{C})$. By Deligne [5] Cor(3.2.17), we also know that $W_N H^N(V_g(n), \mathbb{C})$ is the image of the natural map: $H^N(V_g(n), \mathbb{C}) \rightarrow H^N(V_g(n), \mathbb{C})$.

\textbf{Theorem 8.1.} Assume that $g \geq 2$. 

(i) The dimension of the space $\text{Gr}^W_{2N}H^N(V_g(n), \mathbb{C})$ is equal to the number of 0-dimensional rational cusps in $V_g(n)^*$. The corresponding classes to this space are constructed by the global automorphic Siegel Eisenstein series of weight $g + 1$ for $\Gamma_g(n)$ on $H_g$.

(ii) $\text{Gr}^W_{2N-1}H^N(V_g(n), \mathbb{C}) = \{0\}$.

**Proof.** First we prove (i). We consider induced residue map:

$$\text{Res}_{[N]} : \mathbb{H}^N(\tilde{V}_g(n), \text{Gr}^W_N \Omega^*_V(logD)) \simeq \mathbb{H}^0(D^{[N]}, \mathbb{C})(-N).$$

In the right hand side of above isomorphism, $D^{[N]} = \{\text{points}\}$ is a finite union of points. Hence,

$$\mathbb{H}^0(D^{[N]}, \mathbb{C})(-N) = \prod_{p \in D^{[N]}} \mathbb{C}(-N)$$

is a direct sum of $\mathbb{C}(-N)$. Therefore the Hodge type of this space is only of $(N, N)$, and the whole space is itself an edge component. Then from Lemma (3.1) we can consider a restriction of $\text{Res}_{[N]}$ to this subspace:

$$\text{Res}_{[N]} : \text{Gr}^W_{2N}H^N(V_g(n), \mathbb{C}) \to \mathbb{H}^0(D^{[N]}, \mathbb{C})(-N),$$

which defines an isomorphism into its image. Also as in §3, we consider the map $\text{Res}_{[n]}$ on $\Gamma(\tilde{V}_g(n), \Omega^N_V(logD))$. By §6, $i_N(D^{[N]})$ must be all contained within the fiber of $\pi$ over 0-dimensional cusps of $V_g(n)^*$.

Now we apply Theorem (7.2) to $E_{e_i}(Z, s = 0; g + 1)$. Then for $\tilde{e} \in i(D^{[n]})$ we obtain that

$$\tilde{e}-\text{component of } \text{Res}_{[N]} E_{e_i}(Z, 0; g + 1) = \begin{cases} a_0(E_{e_i}(Z, 0; g + 1); e_i) \neq 0, & \text{if } \pi(\tilde{e}) = e_i \\ 0, & \text{if } \pi(\tilde{e}) = e_j \neq e_i \end{cases}$$

For $f \in M_{g+1}(\Gamma_g(n))$, we have shown that $\text{Res}_{[N]} f \in \prod_{\tilde{e} \in i(D^{[N]})} \mathbb{C}(-N)$ is determined by the constant terms of Fourier expansions at $\pi(\tilde{e})$'s. On the other hand, in the above consideration we have attached to each of cusps in $V_g(n)^*$ one Eisenstein series with nonzero residue. Thus we get (i) of the theorem.
We want to prove (ii). Recall that $D^{[N]}$ is a union of boundary components of codimension $N-1$ of a toric variety of dimension $N = \frac{1}{2}g(g+1)$, and is a union of $\mathbb{P}^1$’s (see Remark 6.2. For $g = 2, 3$, see Igusa [11], Namikawa [16]). Then in the residue map:

$$Res_{[N-1]} : H^N(\tilde{V}_g(n), Gr_{N-1}^W \Omega^*_V(logD)) \simeq H^1(D^{[N-1]}, \mathbb{C})(-N + 1),$$

we have that the target space:

$$H^1(D^{[N-1]}, \mathbb{C})(-N + 1) \simeq \prod H^1(\mathbb{P}^1, \mathbb{C})(-N + 1) = 0.$$ 

Therefore we obtain (ii).

**Remarks.**

(1) For lower weight quotient it might be need to consult with properties of something like Klingen Eisenstein series as nearly holomorphic forms studied by Shimura.

(2) We know that Fourier coefficients of above Siegel Eisenstein series are all in $\mathbb{Q}^{ab}$ by the results of Shimura [22, Theorem (7.1)]. This arithmetic structure of Fourier coefficients should be compatible with the rational structure of the de Rham realization of the mixed Hodge structure.

(3) The following is suggested by T. Oda. Start with the exact sequence of relative cohomology:

$$\cdots \to H^N(V_g(n), \mathbb{Q}) \to H^N(\partial V_g(n), \mathbb{Q}) \to H^{N+1}_c(V_g(n), \mathbb{Q}) \to \cdots,$$

where $\partial V_g(n)$ is the Borel-Serre compactification of the Siegel modular variety, $\partial V_g(n)$ its boundary, and $H^*_c(\star)$ means the cohomology with compact support. Then we can consider the above sequence as an exact sequence of mixed Hodge structure thanks to the theory of mixed Hodge structure on the cohomology of links (A.H.Durfee and M.Saito [8]). Then we have that the term $H^N(V_g(n), \mathbb{Q})$ has weight $N, \cdots, 2N$. On the other hand $H^{N+1}_c(V_g(n), \mathbb{Q})$ has weight $2, \cdots, N+1$. (This is a dual of $H^{N-1}(V_g(n), \mathbb{Q})$ with weight $N-1, \cdots, 2N-2$.) Then the term $H^N(\partial V_g(n), \mathbb{Q})$ has possible weight $2, \cdots, 2N$. Hence, for the weight compatibility of above exact sequence, we conclude that $H^N(V_g(n), \mathbb{Q})$ maps surjectively to those spaces with weights $N, \cdots, 2N$ derived from $H^N(\partial V_g(n), \mathbb{Q})$. 
9. An appendix: On a mixed Hodge structure of universal family over $V_2(n)$

Let $A \xrightarrow{f} V_2(n)$ be the universal family of principal polarized abelian varieties of dimension 2 with level $n$ structure. The $f$ is a proper smooth morphism and a fiber over $Z \bmod \Gamma_2(n)$ is the complex torus $\mathbb{C}^2/(1, Z)(\mathbb{Z})^4$. Denote by $e$ the canonical 0-section of $f$. The family $A$ is an open quasi-projective variety over $\mathbb{C}$ of dimension 5. We realize a compactification $\tilde{A}$ of $A$ as a smooth irreducible divisor in $\tilde{V}_3(n) - V_3(n)$ (Namikawa [16]). In terms of §6 example (2), its local defining equation in $\tilde{V}_3(n)$ is given by $\{t_6 = 0\}$. It is a smooth compactification of $A$. We can write $\tilde{A} - A = \bigcup_i Y_i$ with each $Y_i$ a smooth irreducible divisor of $\tilde{A}$. The structure of $Y^{[m]}$ is following.

$m = 0 \quad Y^{[0]} = \tilde{P}_2(n) \setminus H_2 \times V_{2,1} \times \mathcal{O}(\sigma_1) = \tilde{P}_2(n) \setminus H_2 \times V_{2,1}$.

$m = 1 \quad Y^{[1]} = \bigcup \tilde{P}_1(n) \setminus H_1 \times V_{1,2} \times \mathcal{O}(\sigma_2).
\quad \mathcal{O}(\sigma_2) = \{t_4 = t_6 = 0\} \subset \mathbb{C}^3 = \{t_4, t_5, t_6\}.$

$m = 2 \quad Y^{[2]} = \bigcup \tilde{P}_1(n) \setminus H_1 \times V_{1,2} \times \mathcal{O}(\sigma_3) \cup \bigcup \tilde{P}_0(n) \setminus \mathcal{O}(\sigma_4).
\quad \mathcal{O}(\sigma_3) = \{t_4 = t_5 = t_6 = 0\} \subset \mathbb{C}^3 = \{(t_4, t_5, t_6)\},
\quad \mathcal{O}(\sigma_4) = \{t_1 = t_4 = t_6 = 0\} \subset \mathbb{C}^6 = \{(t_1, t_2, t_3, t_4, t_5, t_6)\}.$

$m = 3 \quad Y^{[3]} = \bigcup \tilde{P}_0(n) \setminus \mathcal{O}(\sigma_5) \cup \bigcup \tilde{P}_0(n) \setminus \mathcal{O}(\sigma_6),
\quad \mathcal{O}(\sigma_5) = \{t_1 = t_4 = t_5 = t_6 = 0\} \subset \mathbb{C}^6,
\quad \mathcal{O}(\sigma_6) = \{t_1 = t_2 = t_5 = t_6 = 0\} \subset \mathbb{C}^6.$

$m = 4 \quad Y^{[4]} = \bigcup \tilde{P}_0(n) \setminus \mathcal{O}(\sigma_7).
\quad \mathcal{O}(\sigma_7) = \{t_1 = t_2 = t_4 = t_5 = t_6 = 0\} \subset \mathbb{C}^6.$

$m = 5 \quad Y^{[5]} = \bigcup \tilde{P}_0(n) \setminus \mathcal{O}(\sigma_8),
\quad \mathcal{O}(\sigma_8) = \{t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0\}.$
The $Y^{[1]}$ above is a fiber space over $V_1(n)$ whose fiber is an extension by $\mathbb{P}^1$’s of 2-copies of an elliptic curve $E \cong \mathbb{C}^2/(1,z)(\mathbb{Z})^2, z \mod \Gamma_1(n) \in V_1(n)$.

**Theorem 9.1.** In the above case, we have that
(i) the dimension of the space of $Gr_{W_{10}}^1 H^5(A, C)$ is equal to the number of 0-dimensional cusps in $V_2(n)^*$. And its cohomology classes are constructed by using the global Siegel Eisenstein series of degree two and weight four.
(ii) $Gr_{W_9}^1 H^5(A, C) = \{0\}$.

**Proof.** (i). Since $H^0(Y^{[5]}, C)(-5)$ has only edge components, we can restrict residue map on $Gr_{W_{10}}^1 H^5(A, C)$ by Lemma (3.1).

$$Res_{[5]} : Gr_{W_{10}}^1 H^5(A, C) \rightarrow H^0(Y^{[5]}, C)(-5).$$

We may consider $Res_{[5]}$ (the same symbol as above) the canonical map from $\Gamma(\tilde{A}, \Omega_5^5(\log Y))$ to $H^0(Y^{[5]}, C)(-5)$ as before (Lemma (3.2)).

We first consider $\Gamma(A, \Omega_5^5)$. For $f$ is a smooth morphism, one has an exact sequence

$0 \rightarrow f^*\Omega_{V_2(n)}^1 \rightarrow \Omega_A^1 \rightarrow \Omega_A^1_{A/V} \rightarrow 0.$

Then $\Omega_A^5 \cong f^*\Omega_V^1 \otimes \Omega_A^1_{A/V} \cong f^*\Omega_V^3 \otimes \Omega_A^1_{A/V}$. Therefore $\Gamma(A, \Omega_A^5) \cong \Gamma(V, f_*\Omega_A^5) \cong \Gamma(V, \Omega_V^3 \otimes f_*\Omega_A^1_{A/V})$. Since each fiber of $f$ is an abelian variety, its space of invariant differential is identified with the cotangent space at $e$. Hence we have $f^*e^*\Omega_A^1_{A/V} \cong \Omega_A^1_{A/V}$. Moreover as $f$ is proper and its (geometric) fiber is connected, $f_*\mathcal{O}_A \cong \mathcal{O}_V$. Then we conclude the following isomorphisms:

$$f_*\Omega_A^1_{A/V} \cong f_* (f^*e^*\Omega_A^1_{A/V} \otimes \mathcal{O}_A) \cong e^*\Omega_A^1_{A/V}. \quad (*)$$

Here $\omega_{A/V} := e^*\Omega_A^1_{A/V} \cong e^*\Omega_A^1_{A/V}$ is an invertible sheaf which defines an automorphic factor (Chai-Faltings [4]). Since we know $\Omega_V^3 \cong \omega_A^{\otimes 3}$ by Kodaira-Spencer map, $\Gamma(V, \Omega_V^3 \otimes \omega_A^{A/V})$ is isomorphic to the holomorphic Siegel modular forms of weight 4: $M_4(\Gamma_2(n))$. Because every element in $M_4(\Gamma_2(n))$ extends holomorphically to rational boundaries (Koecher principle), by above isomorphism $(*)$, one has $\Gamma(\tilde{A}, \Omega_A^5(\log Y)) \cong M_4(\Gamma_2(n))$. As
in §5, take \( \omega_0 = f(z_1, z_2, z_4)dz_1 \wedge dz_2 \wedge dz_4 \wedge d\zeta_1 \wedge d\zeta_2 \) for \( \omega \in \Gamma(\tilde{A}, \Omega^5_A(\log Y)) \). Here \( z_i, i = 1, 2, 4 \) are coordinates in \( V_2(n) \), \( \zeta_1 = z_3 \) and \( \zeta_2 = z_5 \) are coordinates in fiber variety (use the same symbol in §6 example 2). The function \( f(Z) \) is in \( M_4(\Gamma_2(n)) \). In terms of the local coordinates for the smooth compactification given in §6 example 2, we have the following description of \( \omega_0 \).

\[
\omega_0 = C \cdot \{ a_0(f) + \sum_{\{T\}} a_T(f)(q_1q_3)^{t_1} \cdot q_2^{t_1 + t_4 - 2t_2} \cdot (q_4q_5)^{t_4} \} \wedge \frac{dq_i}{q_i},
\]

where \( q_1, \cdots, q_5 \) are defined as in the example 2, §6 and \( T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix} \) are half integral semipositive matrices. Then \( \tilde{e}_i (\in Y^{[5]}) \)-component of \( \text{Res}_{[5]}^{\omega} \) is equal to \( a_0(f) \): the constant term of a Fourier expansion of \( f(Z) \) at a 0-dimensional Satake boundary.

We consider the Siegel Eisenstein series of weight 4:

\[
E_{e_i}(Z; 4) = \sum_{\sigma \in P_1 \cap \Gamma_2(n) \setminus \Gamma_2(n)} j(\gamma_i^{-1} \sigma, Z)^{-4}.
\]

This series absolutely converges and satisfies the properties of Proposition (7.1). Thus (i) follows as before in the case of modular variety.

We can prove (ii) in the same way as in §8, Theorem (8.1). Indeed, \( Y^{[4]} \) is a union of \( \mathbf{P}^1 \)’s, hence \( H^5(\tilde{A}, Gr^W_4\Omega^*_A(\log Y)) \simeq H^1(Y^{[4]}, C)(-4) = 0 \), which implies (ii) immediately.

We remark that the Leray spectral sequence for \( A \to V_2(n) \),

\[
E_2^{p,q} = H^p(V_2(n), R^q f_* Q) \Rightarrow H^{p+q}(A, Q),
\]

degenerates at \( E_2 \)-terms (Lieberman’s trick). \( \square \)

Remark. To see the situation concretely we discussed only the case of \( g = 2 \) in this section. However after some more work we will also obtain similar results for a Hodge structure of the universal family \( A_g \) over \( V_g(n), g \geq 2 \). The results are the followings.

(i) \( \dim Gr^W_{2M} H^M(A_g, C) = \) the number of 0-dimensional cusps in \( V_g(n)^* \).

(ii) \( Gr^W_{2M-1} H^M(A_g, C) = 0 \).
Here $M = \frac{1}{2}g(g + 1) + g$ is the dimension of $A_g$. The proof is completely similarly as the case of $g = 2$.

References


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