A Hamiltonian Path Integral for a Degenerate Parabolic Pseudo-Differential Operator

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Dedicated to Professor Hikosaburo Komatsu on his sixtieth birthday

Abstract. The symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator of order \( m (> 0) \) can be described in terms of a Hamiltonian path integral. This Hamiltonian path integral converges in the topology of the symbol class \( S^{2m}_{\lambda,\rho,\delta} \) and in the weak topology of the symbol class \( S^0_{\lambda,\rho,\delta} \).

0. Introduction

In this paper, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator of order \( m (> 0) \) in a different way from that in C. Tsutsumi [10]. In [10], she constructed the fundamental solution by Levi-Mizohata method. On the other hand, in this paper, we construct the fundamental solution by a Hamiltonian path integral. If we use a Hamiltonian path integral, we can actually give an expression of the symbol of the fundamental solution. Furthermore, this Hamiltonian path integral converges in the topology of the symbol class \( S^{2m}_{\lambda,\rho,\delta} \) and in the weak topology of the symbol class \( S^0_{\lambda,\rho,\delta} \).

In Section 1, we introduce some basic properties of pseudo-differential operators, which we use in Section 2. For the details, see Chapter 7 \( \S \) 1 and \( \S \) 2 in H. Kumano-go [6]. In Section 2, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator by a Hamiltonian path integral. Theorem 2.1 is the main theorem in this paper.

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1. Pseudo-Differential Operators

For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n_x \), \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_\xi \) and multi-indices of non-negative integers \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \), we employ the usual notation:

\[
|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad |\beta| = \beta_1 + \cdots + \beta_n,
\]

\[
\alpha! = \alpha_1! \cdots \alpha_n!, \quad \beta! = \beta_1! \cdots \beta_n!,
\]

\[
x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n, \quad \langle x \rangle = (1 + |x|^2)^{1/2}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},
\]

\[
\partial_{\xi_j} = \frac{\partial}{\partial \xi_j}, \quad D_{x_j} = -i \frac{\partial}{\partial x_j}, \quad \partial_{\xi}^{\alpha_1} \cdots \partial_{\xi}^{\alpha_n}, \quad D_{x}^{\beta_1} \cdots D_{x}^{\beta_n}.
\]

\( S \) denotes the Schwartz space of rapidly decreasing \( C^\infty \)-functions on \( \mathbb{R}^n \).

For \( u \in S \), we define semi-norms \( |u|_{l,S} \) by

\[
|u|_{l,S} = \max \sup_{k+|\alpha| \leq l} |\langle x \rangle^k \partial_x^\alpha u(x)| \quad (l = 0, 1, 2, \ldots).
\]

Then, \( S \) is a Fréchet space with these semi-norms.

For simplicity, we set \( d\eta \equiv (2\pi)^{-n} d\eta \) and \( d\xi \equiv (2\pi)^{-n} d\xi \).

Oscillatory integral of a function \( a(\eta, y) \), is defined by the equality

\[
O_s - \iint e^{-iy \cdot \eta} a(\eta, y) dy d\eta \equiv \lim_{\epsilon \to 0} \iint e^{-iy \cdot \eta} \chi(\epsilon \eta, \epsilon y) a(\eta, y) dy d\eta,
\]

where \( \chi(\eta, y) \in S \) in \( \mathbb{R}^{2n}_{\eta, y} \) and \( \chi(0, 0) = 1 \). For the details, see Chapter 1 § 6 in H.Kumano-go [6].

**Definition 1.1 (A weight function \( \lambda(\xi) \)).**

We say that a real-valued \( C^\infty \)-function \( \lambda(\xi) \) on \( \mathbb{R}^n_\xi \) is a weight function, if there exist constants \( A_0, A_\alpha > 0 \) such that

\[
1 \leq \lambda(\xi) \leq A_0 \langle \xi \rangle, \quad (1.1)
\]

\[
|\partial_{\xi}^{\alpha} \lambda(\xi)| \leq A_\alpha \lambda(\xi)^{1 - |\alpha|}. \quad (1.2)
\]

**Examples.**

1° \( \lambda(\xi) = \langle \xi \rangle \).

2° \( \lambda(\xi) = \left\{ 1 + \sum_{j=1}^{n} |\xi_j|^{2m_j} \right\}^{1/(2m)}, \quad (m_j \in \mathbb{N}, \quad m \equiv \max_{1 \leq j \leq n} \{ m_j \}). \)
**Definition 1.2 (Pseudo-differential operators).**

We say that a $C^\infty$-function $p(x, \xi)$ on $\mathbb{R}^{2n}_{x, \xi}$ is a symbol of class $S^m_{\lambda, \rho, \delta}$ $(m \in \mathbb{R}, 0 \leq \delta \leq \rho \leq 1, \delta < 1)$, if for any $\alpha, \beta$, there exists a constant $C_{\alpha, \beta}$ such that

\[(1.3) \quad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_{\alpha, \beta} \lambda(\xi)^{m+\delta|\beta|\rho|\alpha|},\]

where $p^{(\alpha)}_{(\beta)}(x, \xi) \equiv \partial_\xi^\alpha \partial_x^\beta p(x, \xi)$. The pseudo-differential operator $p(X, D_x)$ with the symbol $p(x, \xi)$ is defined by

\[(1.4) \quad p(X, D_x)u(x) \equiv \int\int e^{i(x-x')\cdot \xi} p(x, \xi) u(x') dx' d\xi \quad (u \in S),\]

where $d\xi \equiv (2\pi)^{-n} d\xi$.

**Remark.**

1° For simplicity, we set $p^{(\alpha)}_{(\beta)}(x, \xi) \equiv \partial_\xi^\alpha \partial_x^\beta p(x, \xi)$, $p^{(\alpha)}_{(\beta)}(x, \xi) \equiv \partial_\xi^\alpha p(x, \xi)$ and $p^{(\beta)}_{(\alpha)}(x, \xi) \equiv D^\beta_x p(x, \xi)$ for any $\alpha, \beta$.

2° The symbol class $S^m_{\lambda, \rho, \delta}$ is a Fréchet space with the semi-norms

\[(1.5) \quad |p|^{(m)}_l \equiv \max_{|\alpha+\beta| \leq l} \sup_{(x, \xi)} \{|p^{(\alpha)}_{(\beta)}(x, \xi)| \lambda(\xi)^{-(m+\delta|\beta|\rho|\alpha|)}\} \quad (l = 0, 1, 2, \ldots).\]

3° The continuity of $p(X, D_x) : S \rightarrow S$ is clear. Furthermore, we can extend $p(X, D_x) : S \rightarrow S$ to $p(X, D_x) : S' \rightarrow S'$ by means of

\[(1.6) \quad (p(X, D_x)u, v) \equiv (u, p(X, D_x)^* v) \quad \text{for} \quad u \in S', v \in S.\]

**Theorem 1.3 (Multi-products).**

Let $M$ be a positive constant and let $\{m_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying

\[(1.7) \quad \sum_{j=1}^\infty |m_j| \leq M < \infty.\]
For any $\nu = 1, 2, \ldots$ and $p_j(x, \xi) \in S_{\lambda, \rho, \delta}^{m_j}(j = 1, 2, \ldots, \nu + 1)$, there exists $q_{\nu + 1}(x, \xi) \in S_{\lambda, \rho, \delta}^{\bar{m}_{\nu + 1}}(\bar{m}_{\nu + 1} \equiv m_1 + m_2 + \cdots + m_{\nu + 1})$ such that
\[
q_{\nu + 1}(X, D_x) = p_1(X, D_x)p_2(X, D_x) \cdots p_{\nu + 1}(X, D_x).
\]
Furthermore, for any $l$, there exist a constant $A_l$ and an integer $l'$ such that
\[
|q_{\nu + 1}|_{l}^{(\bar{m}_{\nu + 1})} \leq (A_l)^{\nu} \prod_{j=1}^{\nu + 1} |p_j|_{l'}^{(m_j)},
\]
where $A_l$ and $l'$ depend only on $M$ and $l$, but are independent of $\nu$.

**Proof.** See Theorem 2.4 in Chapter 7 §2 of H. Kumano-go [6]. □

**Theorem 1.4.**
Let $p_j(x, \xi) \in S_{\lambda, \rho, \delta}^{m_j}(j = 1, 2)$. Define $q_\theta(x, \xi)$ ($|\theta| \leq 1$) by
\[
q_\theta(x, \xi) \equiv O_s - \int\int e^{-iy \cdot \eta} p_1(x, \xi + \theta \eta) p_2(x + y, \xi) dy d\eta.
\]
Then $\{q_\theta(x, \xi)\}_{|\theta| \leq 1}$ is a bounded set of $S_{\lambda, \rho, \delta}^{m_1 + m_2}$. Furthermore, for any $l$, there exist a constant $A_l$ and an integer $l'$ independent of $\theta$ such that
\[
|q_\theta|_{l}^{(m_1 + m_2)} \leq A_l |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)}.
\]

**Proof.** See Lemma 2.4 in Chapter 2 §2 or Lemma 2.2 in Chapter 7 §2 of H. Kumano-go [6]. □

**2. The Main Theorem**

**Theorem 2.1 (The main theorem).**
Let $K(t, x, \xi) \in C^0([0, T]; S_{\lambda, \rho, \delta}^{m})$ $(m > 0, 0 \leq \delta < \rho \leq 1)$. Assume that $K(t, x, \xi)$ satisfies the following conditions (a1), (a2):

(a1) There exist constants $c > 0$ and $m'(0 \leq m' \leq m)$ such that
\[
Re K(t, x, \xi) \leq -c\lambda(\xi)^{m'} \text{ on } [0, T] \times R^{2n}_{x, \xi}.
\]

(a2) For any $\alpha, \beta$, there exists a constant $C_{\alpha, \beta}$ such that
\[
|K^{(\alpha)}(t, x, \xi)/Re K(t, x, \xi)| \leq C_{\alpha, \beta} \lambda(\xi)^{|\delta| - |\rho|} \text{ on } [0, T] \times R^{2n}_{x, \xi}.
\]
Then we have the following (1) – (5):

(1) Let $\Delta_{t,s} : (T \geq t) \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu + 1} \equiv s(\geq 0)$ be an arbitrary division of interval $[s, t]$ into subintervals, and let $e^{(t_j-t_{j+1})K(t_{j+1})}(X, D_x)$ be an operator defined by

\begin{equation}
(2.3) \quad e^{(t_j-t_{j+1})K(t_{j+1})}(X, D_x)u(x)
= \iint e^{i(x-x')\xi}e^{(t_j-t_{j+1})K(t_{j+1}, x, \xi)}u(x')dx'd\xi.
\end{equation}

Then there exists $p(\Delta_{t,s}; x, \xi) \in S^0_{\lambda, \rho, \delta}$ such that

\begin{equation}
(2.4) \quad p(\Delta_{t,s}; X, D_x) = e^{(t-t_1)K(t_1)}(X, D_x)e^{(t_1-t_2)K(t_2)}(X, D_x)
\quad \cdots e^{(t_\nu-s)K(s)}(X, D_x).
\end{equation}

(2) For any $l$, there exist constants $C_l, C'_l$ and an integer $l'$ such that

\begin{equation}
(2.5) \quad |p(\Delta_{t,s})|^{(0)}_l \leq C_l,
\end{equation}

and

\begin{equation}
(2.6) \quad |p(\Delta_{t,s}) - p(\Delta'_{t,s})|^{(2m)}_l
\leq C'_l(t-s)\left(|\Delta_{t,s}| + \sup_{|t'-t''| \leq |\Delta_{t,s}|} |K(t') - K(t'')|^{(m)}_{l'}\right).
\end{equation}

Here, $\Delta_{t,s} : (T \geq t) \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu + 1} \equiv s(\geq 0)$ is an arbitrary division of interval $[s, t]$ into subintervals, $\Delta'_{t,s}$ is an arbitrary refinement of $\Delta_{t,s}$, $|\Delta_{t,s}|$ denotes the size of division defined by $|\Delta_{t,s}| \equiv \max_{0 \leq j \leq \nu} |t_j - t_{j+1}|$, and the constants $C_l, C'_l$ and the integer $l'$ are independent of $\nu, \Delta_{t,s}$ and $\Delta'_{t,s}$.

(3) There exists $p^*(t, s; x, \xi) \in S^0_{\lambda, \rho, \delta}$ such that $p(\Delta_{t,s}; x, \xi) (\in S^0_{\lambda, \rho, \delta})$ converges to $p^*(t, s; x, \xi) (\in S^0_{\lambda, \rho, \delta})$ in $S^{2m}_{\lambda, \rho, \delta}$ as $|\Delta_{t,s}|$ tends to 0. Furthermore, $p^*(t, s; x, \xi)$ has the following expression:

\begin{equation}
(2.7) \quad p^*(t, s; x, \xi) = \lim_{|\Delta_{t,s}| \to 0} O_s - \iint \cdots \iint e^{-i\sum_{j=1}^\nu y^j \cdot \eta^j}
\times \exp\left(\sum_{j=0}^\nu (t_j - t_{j+1})K(t_{j+1}, x + y^j, \xi + \eta^{j+1})\right)
\times dy^1 d\eta^1 \cdots dy' d\eta'\prime,
\end{equation}
where $\bar{y}^0 \equiv 0$, $\bar{y}^i \equiv y^1 + y^2 + \cdots + y^j$, and $\eta^{\nu+1} \equiv 0$.

(4) For $u \in L^2$, the pseudo-differential operator $U(t, s) \equiv p^*(t, s; X, D_x)$ satisfies the following relation:

\begin{align*}
U(t, s)u(x) &= \lim_{|\Delta t, s| \to 0} e^{(t-t_{j1})K(t_{j1})} (X, D_x) e^{(t_{j1}-t_{j2})K(t_{j2})} (X, D_x) \\
&\quad \quad \cdots e^{(t_{j\nu}-s)K(s)} (X, D_x)u(x) \\
&= \lim_{|\Delta t, s| \to 0} \int \cdots \int \exp \left( \sum_{j=0}^{\nu} i(x^j - x^{j+1}) \cdot \xi^{j+1} \\
&\quad + (t_j - t_{j+1})K(t_{j+1}, x^j, \xi^{j+1}) \right) \\
&\quad \times u(x^\nu) dx^\nu d\xi^{\nu+1} \cdots dx^1 d\xi^1,
\end{align*}

in $L^2$ where $x^0 \equiv x$.

(5) $U(t, s) \equiv p^*(t, s; X, D_x)$ is the fundamental solution for the operator $L \equiv \partial_t - K(t, X, D_x)$ such that

\begin{align*}
\begin{cases}
LU(t, s) = 0 & \text{on } (s, T] \\
U(s, s) = I & (0 \leq s \leq T).
\end{cases}
\end{align*}

**Remark.**

1° It is sufficient to satisfy the conditions (a1) and (a2) for $|\xi| \geq M$, with a constant $M \geq 0$. In fact, in this case, there exists a sufficiently large $R > 0$ such that the symbol $K_R(t, x, \xi) \equiv K(t, x, \xi) - R$ satisfies (a1) and (a2) for any $\xi$. Let $U_R(t, s)$ be the fundamental solution of $L_R \equiv \partial_t - K_R(t, X, D_x)$. Then $U(t, s) \equiv e^{(t-s)R} U_R(t, s)$ is the fundamental solution of $L$.

2° We can replace $(t_j - t_{j+1})K(t_{j+1}, \cdot, \cdot)$ with $\int_{t_{j+1}}^{t_j} K(\tau, \cdot, \cdot)d\tau$. Furthermore, in this case, we can replace (2.6) with

\begin{align*}
|p(\Delta_{t, s}) - p(\Delta'_{t, s})|^{(2m)} \leq C_1'(t - s)|\Delta_{t, s}|,
\end{align*}

and the proof of Theorem 2.1 becomes a little easier.
Example.
Consider

\[ L \equiv \partial_t + a(t)|x|^{2l}(-\Delta)^m + (-\Delta)^{m'} \quad (0 \leq a(t) \in C[0,T], m - m' < l). \]

If we set \( \rho = 1, \delta = (m - m')/l, m \to 2m \) and \( m' \to 2m' \), then the symbol \( a(t)|x|^{2l}|\xi|^{2m} + |\xi|^{2m'} \) satisfies the conditions (a1) and (a2). Therefore, we see that these conditions are satisfied not only by the usual parabolic operators, but also by parabolic operators of a degenerate type.

Before we prove Theorem 2.1, we prepare some lemmas:
To begin with, for \( T \geq t \geq s \geq 0 \), we define \( p(t, s; x, \xi) \) by

\[ p(t, s; x, \xi) \equiv \exp \left( (t - s)K(s, x, \xi) \right). \]

The next lemma is a generalization of asymptotic expansion formulas, and an essential part in this paper. Especially, it is important that all constants are independent of \( \Delta t_0, t_{\nu+1} \) and \( \nu \).

**Lemma 2.2 (A key lemma).**
Let \( \Delta t_{0, t_{\nu+1}} : (T \geq t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1}(\geq 0), \nu = 1, 2, \ldots \), and let \( N_0 \) be a fixed positive integer such that \( \rho - \delta N_0 \geq 2m \). Define \( q(\Delta t_0, t_1; x, \xi) \), \( q(\Delta t_0, t_{\nu+1}; x, \xi) \), and \( r(\Delta t_0, t_{\nu+1}; x, \xi) \) respectively by

\[ q(\Delta t_0, t_1; x, \xi) \equiv p(t_0, t_1; x, \xi), \]

\[ q(\Delta t_0, t_{\nu+1}; x, \xi) \equiv \sum_{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_{\nu}| < N_0} \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_{\nu}!} \]
\[ \times p(\alpha_{\nu})(t_{\nu}, t_{\nu+1}; x, \xi) \partial^\alpha_{\xi} \left( p(\alpha_{\nu-1})(t_{\nu-1}, t_{\nu}; x, \xi) \right) \]
\[ \times \partial^\alpha_{\xi} \left( \cdots p(\alpha_2)(t_2, t_3; x, \xi) \partial^\alpha_{\xi} \left( p(\alpha_1)(t_1, t_2; x, \xi) \right) \right) \times \partial^\alpha_{\xi} \left( p(t_0, t_1; x, \xi) \right) \cdots \right). \]
\begin{equation}
(\Delta_{t_0,t_{\nu+1}}; x, \xi) \equiv \sum_{|\alpha^1| + |\alpha^2| + \cdots + |\alpha^\nu| = N_0, |\alpha^\nu| \neq 0} \frac{|\alpha^\nu|}{\alpha^1! \alpha^2! \cdots \alpha^\nu!} \times \int_0^1 (1 - \theta)^{|\alpha^\nu| - 1} O_s \int e^{-iy\cdot\eta} p(\alpha^\nu)(t_\nu, t_{\nu+1}; x + y, \xi) \times \partial^\nu_\xi (p(\alpha, t_\nu - 1, t_\nu ; x, \xi + \theta\eta)) \\
\times \partial^\nu_{\xi - 1} (\cdots p(\alpha^2)(t_2, t_3 ; x, \xi + \theta\eta)) \times \partial^2_\xi (p(\alpha^1)(t_1, t_2 ; x, \xi + \theta\eta)) \times \partial^1_\xi \right) \left. \left. dy d\eta d\theta. \right)
\end{equation}

Then it follows that

\begin{equation}
q(\Delta_{t_0,t_\nu}; X, D_x)p(t_\nu, t_{\nu+1}; X, D_x) = q(\Delta_{t_0, t_{\nu+1}}; X, D_x) + r(\Delta_{t_0, t_{\nu+1}}; X, D_x).
\end{equation}

Furthermore, there exist constants $C_{1,l,1}, C_{2,l,1}, C_{3,l,1}$ such that

\begin{equation}
|q(\Delta_{t_0, t_\nu})|^{(0)}_l \leq C_{1,l,1},
\end{equation}

\begin{equation}
|q(\Delta_{t_0, t_{\nu+1}}) - p(t_0, t_{\nu+1})|^{(2m)}_l \leq C_{2,l,1} |t_0 - t_{\nu+1}| \times \left( (t_0 - t_{\nu+1}) + \sup_{t_0 \geq t' \geq t'' \geq t_{\nu+1}} |K(t') - K(t''){|^{(m)}_l} \right),
\end{equation}

and

\begin{equation}
|r(\Delta_{t_0, t_{\nu+1}})|^{(0)}_l \leq C_{3,l,1} (t_0 - t_\nu)(t_\nu - t_{\nu+1}),
\end{equation}

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \ldots$. 
Proof.

1° For $T \geq t \geq s \geq 0$, we set

\begin{equation}
\eta(t, s; x, \xi) \equiv -(t - s) \Re K(s, x, \xi) \geq 0.
\end{equation}

Furthermore, for $\Delta t_{0, \nu+1} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \geq 0$ and $\nu = 1, 2, \ldots$, we define $d(\Delta t_{0, t_\nu}; x, \xi)$ by

\begin{equation}
d(\Delta t_{0, t_\nu}; x, \xi) \equiv \prod_{j=0}^{\nu-1} p(t_j, t_{j+1}; x, \xi),
\end{equation}

and we set

\begin{equation}
\eta(\Delta t_{0, t_\nu}; x, \xi) \equiv \sum_{j=0}^{\nu-1} \eta(t_j, t_{j+1}; x, \xi).
\end{equation}

Clearly, we have

\begin{equation}
|d(\Delta t_{0, t_\nu}; x, \xi)| = \exp \left( -\eta(\Delta t_{0, t_\nu}; x, \xi) \right).
\end{equation}

2° Define $d_{\alpha, \beta}(\Delta t_{0, t_\nu}; x, \xi)$ by

\begin{equation}
d^{(\alpha)}(\Delta t_{0, t_\nu}; x, \xi) \equiv d_{\alpha, \beta}(\Delta t_{0, t_\nu}; x, \xi)d(\Delta t_{0, t_\nu}; x, \xi).
\end{equation}

Then, by induction, for any $\alpha, \beta$ ($|\alpha + \beta| \geq 1$) and $\alpha', \beta'$, there exists a constant $C_{\alpha, \beta, \alpha', \beta'}$ such that

\begin{equation}
|d_{\alpha, \beta}(\Delta t_{0, t_\nu}; x, \xi)|
\leq C_{\alpha, \beta, \alpha', \beta'} \eta(\Delta t_{0, t_\nu}; x, \xi) \left( \eta(\Delta t_{0, t_\nu}; x, \xi) + 1 \right)^{|\alpha + \beta| - 1}
\times \lambda(\xi)^{\beta |\beta' - \rho| + |\alpha + \alpha'|},
\end{equation}

for any $\Delta t_{0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \geq 0$ and $\nu = 1, 2, \ldots$. 
3° Let \( \tilde{\alpha}^\nu \equiv (\alpha^1, \ldots, \alpha^\nu) \) denote a multi-index of \( R^{\nu n} \). Define \( f_{\tilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}}; x, \xi) \) by

\[
(2.24) \quad f_{\tilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}}; x, \xi) = p(\nu^1)_{t_{\nu^1}, t_{\nu+1}; x, \xi} \partial_\xi^{\nu^1} \left( p(\nu^2)_{t_{\nu^2}, t_{\nu}; x, \xi} \partial_\xi^{\nu^2} \left( \cdots \left( p(\nu^\nu)_{t_{\nu}, t_1; x, \xi} \partial_\xi^{\nu^\nu} \right) \right) \right).
\]

Then, by induction, for any \( N = 1, 2, \ldots \) and \( \alpha, \beta \), there exists a constant \( C_{N,\alpha,\beta} \) such that

\[
(2.25) \quad |f_{\tilde{\alpha}^\nu}(\alpha^\nu, \beta^\nu)(\Delta_{t_0,t_{\nu+1}}; x, \xi)| \leq C_{N,\alpha,\beta} \left( \prod_{j=1}^{J} \eta(t_{j_k}, t_{j_k+1}; x, \xi) \right) \eta(\Delta_{t_0,t_{\nu+1}}; x, \xi)
\]

\[
\times \left( \eta(\Delta_{t_0,t_{\nu+1}}; x, \xi) + 1 \right)^{2(N-1)} \lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|},
\]

where

\[
1 \leq j_1 < j_2 < \cdots < j_J \leq \nu, \quad |\alpha^{j_k}| \neq 0 (k = 1, 2, \ldots, J),
\]

and

\[
\sum_{j=1}^{\nu} |\alpha^j| = \sum_{k=1}^{J} |\alpha^{j_k}| = N,
\]

for any \( \Delta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1} (\geq 0) \) and \( \nu = 1, 2, \ldots \).

4° For \( N = 1, 2, \ldots \), define \( g_N(\Delta_{t_0,t_{\nu+1}}; x, \xi) \) by

\[
(2.26) \quad g_N(\Delta_{t_0,t_{\nu+1}}; x, \xi) = \sum_{|\alpha^1| + |\alpha^2| + \cdots + |\alpha^\nu| = N} \frac{1}{\alpha^1! \alpha^2! \cdots \alpha^\nu!} f_{\tilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}}; x, \xi).
\]
By (2.25), we have

\[
|g_N^{(\alpha)}(\Delta t_0, t_{\nu+1}; x, \xi)| \leq \sum_{J=1}^{N} \sum_{1 \leq j_1 < j_2 < \cdots < j_J \leq \nu} \sum_{j=1}^{J} |\alpha|^{j} \lambda(j) \eta(\Delta t_0, t_{\nu+1}; x, \xi) \eta(\Delta t_0, t_{\nu+1}; x, \xi)
\]

\[
\times \left( \eta(\Delta t_0, t_{\nu+1}; x, \xi) + 1 \right)^{2(N-1)} \lambda(\xi)^{-(\rho - \delta)N + \delta|\beta| - \rho|\alpha|}
\]

\[
\times \left( \sum_{\sum_{j=1}^{J} \sum_{1 \leq j_1 < j_2 < \cdots < j_J \leq \nu} \prod_{k=1}^{J} \eta(t_{j_k}, t_{j_k+1}; x, \xi) \right).
\]

Hence, for any \( N = 1, 2, \ldots \) and \( \alpha, \beta \), there exists a constant \( C'_{N,\alpha,\beta} \) such that

\[
|g_N^{(\alpha)}(\Delta t_0, t_{\nu+1}; x, \xi)| \leq C'_{N,\alpha,\beta} \left( \eta(\Delta t_0, t_{\nu+1}; x, \xi) \right)^2
\]

\[
\times \left( \eta(\Delta t_0, t_{\nu+1}; x, \xi) + 1 \right)^{3(N-1)} \lambda(\xi)^{-(\rho - \delta)N + \delta|\beta| - \rho|\alpha|},
\]

for any \( \Delta t_0, t_{\nu+1} : (T \geq t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1} \geq 0) \) and \( \nu = 1, 2, \ldots \).

5° Set

\[
h_N(\Delta t_0, t_{\nu+1}; x, \xi) \equiv g_N(\Delta t_0, t_{\nu+1}; x, \xi)d(\Delta t_0, t_{\nu+1}; x, \xi).
\]

Here we note that

\[
\sup_{\eta > 0} \eta^k e^{-\eta} < \infty \quad (k = 0, 1, 2, \ldots).
\]
By (2.21), (2.23) and (2.28), there exist constants $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C'''_{N,\alpha,\beta}$, $C''''_{N,\alpha,\beta}$ such that

\begin{equation}
|d^{(a)}_{(\beta)}(\Delta t_0, t_\nu; x, \xi)| \leq \begin{cases}
  C'_{\alpha,\beta} \lambda(|\xi|)^{|\delta| \beta} - \rho|\alpha| \\
  C''_{\alpha,\beta} (t_0 - t_\nu) \lambda(|\xi|)^{m+|\delta| \beta} - \rho|\alpha| & (|\alpha + \beta| \geq 1),
\end{cases}
\end{equation}

and

\begin{equation}
|h^{(a)}_{(\beta)}(\Delta t_0, t_{\nu+1}; x, \xi)| \leq \begin{cases}
  C''_{N,\alpha,\beta} \lambda(|\xi|)^{-(\rho - \delta)N + |\delta| \beta} - \rho|\alpha| \\
  C'''_{N,\alpha,\beta} (t_0 - t_{\nu+1}) \lambda(|\xi|)^{m-(\rho - \delta)N + |\delta| \beta} - \rho|\alpha| \\
  C''''_{N,\alpha,\beta} (t_0 - t_{\nu+1})^2 \lambda(|\xi|)^{2m-(\rho - \delta)N + |\delta| \beta} - \rho|\alpha|,
\end{cases}
\end{equation}

for any $\Delta t_0, t_{\nu+1} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \ldots$

6° Now we note that

\begin{equation}
q(\Delta t_0, t_{\nu+1}; x, \xi) = d(\Delta t_0, t_{\nu+1}; x, \xi) + \sum_{N=1}^{N_0-1} h_N(\Delta t_0, t_{\nu+1}; x, \xi),
\end{equation}

and

\begin{equation}
d(\Delta t_0, t_{\nu+1}; x, \xi) - p(t_0, t_{\nu+1}; x, \xi) = \sum_{j=0}^{\nu} (t_j - t_{j+1}) \left( K(t_{j+1}, x, \xi) - K(t_{\nu+1}, x, \xi) \right) \\
\times \int_0^1 \exp \left( \theta \sum_{j=0}^{\nu} (t_j - t_{j+1}) K(t_{j+1}, x, \xi) \right) \\
\times \exp \left( (1 - \theta)(t_0 - t_{\nu+1}) K(t_{\nu+1}, x, \xi) \right) d\theta.
\end{equation}
By (2.31) and (2.32), we get (2.15) and (2.16). Furthermore, we note that

\begin{equation}
(2.35) \quad r(\Delta_{t_0,t_{\nu+2}}; x, \xi) \\
= \sum_{0 < |\alpha^\nu+1| < N_0} \frac{|\alpha^\nu+1|}{\alpha^\nu+1!} \int_0^1 (1 - \theta)^{|\alpha^\nu+1| - 1} \\
\times O_s - \int \int e^{-iy\cdot\eta} h_{N_0 - |\alpha^\nu+1|}(\Delta_{t_0,t_{\nu+1}}; x, \xi + \theta\eta) \\
\times d(\alpha^\nu+1)(\Delta_{t_{\nu+1},t_{\nu+2}}; x + y, \xi) dyd\eta d\theta \\
+ \sum_{|\alpha^\nu+1| = N_0} \frac{|\alpha^\nu+1|}{\alpha^\nu+1} \int_0^1 (1 - \theta)^{|\alpha^\nu+1| - 1} \\
\times O_s - \int \int e^{-iy\cdot\eta} d(\alpha^\nu+1)(\Delta_{t_0,t_{\nu+1}}; x, \xi + \theta\eta) \\
\times d(\alpha^\nu+1)(\Delta_{t_{\nu+1},t_{\nu+2}}; x + y, \xi) dyd\eta d\theta.
\end{equation}

By (2.31), (2.32) and Theorem 1.4, we get (2.17).

By induction, we get (2.14). □

The idea of the next lemma is found in Fujiwara [3].

**Lemma 2.3 (Fujiwara’s skip).**

Define $\mathcal{Y}(\Delta_{t_0,t_{\nu+1}}; x, \xi) \in \mathcal{S}_0^{0,0,\delta}$ by

\begin{equation}
(2.36) \quad p(t_0, t_1; X, D_x)p(t_1, t_2; X, D_x) \cdots p(t_{\nu}, t_{\nu+1}; X, D_x) \\
\equiv q(\Delta_{t_0,t_{\nu+1}}; X, D_x) + \mathcal{Y}(\Delta_{t_0,t_{\nu+1}}; X, D_x).
\end{equation}

Then it follows that

\begin{equation}
(2.37) \quad \mathcal{Y}(\Delta_{t_0,t_{\nu+1}}; X, D_x) \\
= \sum_{j_1 < j_1 + 1 < j_2 + 1 < \cdots < j_{\nu-1} < j_{\nu-1} + 1 < j_{\nu}} r(\Delta_{t_{j_1},t_{j_1+1}}; X, D_x) r(\Delta_{t_{j_{\nu}},t_{j_{\nu}+1}}; X, D_x) \\
\cdots r(\Delta_{t_{j_{\nu-1}},t_{j_{\nu-1}+1}}; X, D_x) q(\Delta_{t_{j_{\nu-1}},t_{j_{\nu}-1}}; X, D_x),
\end{equation}

where $\sum$ stands for the summation with respect to the sequences of integers $(j_1, j_2, \ldots, j_{\nu})$ with the property

\begin{equation}
(2.38) \quad 0 < j_1 < j_1 + 1 < j_2 < j_2 + 1 < \cdots < j_{\nu-1} < j_{\nu-1} + 1 < j_{\nu} \leq \nu,
\end{equation}
and, in the special case of $jJ = \nu$, we set $q(\Delta_{t_{j_j+1},t_{\nu+1}}; X, D_x) \equiv I$. Furthermore, there exists a constant $C_{4,l}$ such that

$$\| \Upsilon(\Delta_{t_0,t_{\nu+1}}) \|_{l}^{(0)} \leq C_{4,l}(t_0 - t_{\nu+1})^2,$$

for any $\Delta_{t_0,t_{\nu+1}}: (T \geq t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1} \geq 0)$ and $\nu = 1, 2, \ldots$.

**Proof.** Using (2.14) inductively, we get (2.37). Now let $A_l, l'$ be the same constants in Theorem 1.3, and let $C_{1,l}, C_{3,l}$ be the same constants in Lemma 2.2. By (2.15), (2.17) and Theorem 1.3, we have

$$\| \Upsilon(\Delta_{t_0,t_{\nu+1}}) \|_{l}^{(0)} \leq \sum_{j=1}^{J} (A_l)^j \left| r(\Delta_{t_0,t_{j_j+1}}) \right|_{l'}^{(0)} \left| r(\Delta_{t_{j_j+1},t_{j_{j+1}+1}}) \right|_{l'}^{(0)} \cdots \left| r(\Delta_{t_{j_{j-1}+1},t_{j_{j}+1}}) \right|_{l'}^{(0)} \left| q(\Delta_{t_{j_{j+1}},t_{\nu+1}}) \right|_{l'}^{(0)}$$

$$\leq \sum_{j=1}^{J} (A_l)^j \left( \prod_{k=1}^{J} C_{3,l'}(t_0 - t_{\nu+1})(t_{j_k} - t_{j_k+1}) \right) C_{1,l'}$$

$$\leq C_{1,l'} \left( \prod_{j=0}^{\nu} \left( 1 + A_l C_{3,l'}(t_0 - t_{\nu+1})(t_j - t_{j+1}) \right) - 1 \right) \leq C_{4,l}(t_0 - t_{\nu+1})^2. \qed$$

Now we prove Theorem 2.1:

**Proof of Theorem 2.1.**

1° Define $p(\Delta_{t,s}; x, \xi)$ by

$$p(\Delta_{t,s}; x, \xi) \equiv q(\Delta_{t,s}; x, \xi) + \Upsilon(\Delta_{t,s}; x, \xi).$$

Then (1) is clear.

2° By (2.14) and (2.39), we get (2.5). Next, we note that

$$p(\Delta'_{t_j,t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi)$$

$$= \left( q(\Delta'_{t_j,t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi) \right) + \Upsilon(\Delta'_{t_j,t_{j+1}}; x, \xi),$$
Hence, by (2.16) and (2.39), there exists a constant $C_{5,l}$ such that

\[(2.43)\quad |p(t_j, t_{j+1}) - p(\Delta'_{t_j, t_{j+1}})|^{(2m)}_l \leq C_{5,l}(t_j - t_{j+1}) + \sup_{t_j \geq t' \geq t'' \geq t_{j+1}} |K(t') - K(t'')|^{(m)}_l.\]

Here we can write

\[(2.44)\quad p(\Delta_{t,s}; X, D_x) - p(\Delta'_{t,s}; X, D_x) = \sum_{j=0}^{\nu} p(\Delta'_{t_0, t_j}; X, D_x) \circ (p(t_j, t_{j+1}; X, D_x) - p(\Delta'_{t_j, t_{j+1}}; X, D_x)) \circ p(\Delta_{t_{j+1}, t_{\nu+1}}; X, D_x).\]

By (2.5), (2.43) and Theorem 1.3, we get (2.6).

3° By (2.6) and (2.5), there exists $p^*(t, s; x, \xi) \in S^0_{\lambda, \rho, \delta}$ such that

\[(2.45)\quad |p^*(t, s)|^{(0)}_l \leq C_l,\]

and

\[(2.46)\quad |p(\Delta_{t,s}) - p^*(t, s)|^{(2m)}_l \leq C'_l(t - s) \left(|\Delta_{t,s}| + \sup_{|t' - t''| \leq |\Delta_{t,s}|} |K(t') - K(t'')|^{(m)}_l \right).\]

Hence we get (3).

4° By the result of (3), we get (4). See Chapter 3 § 7 in H.Kumano-go [6].

5° Using the results of (2) and (3), it is easy to check (5). \(\square\)

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References


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