The $W^{k,p}$-continuity of wave operators for Schrödinger operators III, even dimensional cases $m \geq 4$

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Abstract. Let $H = -\Delta + V(x)$ be the Schrödinger operator on $\mathbb{R}^m$, $m \geq 3$. We show that the wave operators $W_{\pm} = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$, $H_0 = -\Delta$, are bounded in Sobolev spaces $W^{k,p}(\mathbb{R}^m)$, $1 \leq p \leq \infty$, $k = 0, 1, \ldots, \ell$, if $V$ satisfies $\|D^\alpha V(y)\|_{L^p_0(|x-y| \leq 1)} \leq C(1 + |x|)^{-\delta}$ for $\delta > (3m/2) + 1$, $p_0 > m/2$ and $|\alpha| \leq \ell + \ell_0$, where $\ell_0 = 0$ if $m = 3$ and $\ell_0 = [(m-1)/2]$ if $m \geq 4$, $[\sigma]$ is the integral part of $\sigma$. This result generalizes the author’s previous result which appears in J. Math. Soc. Japan 47, where the theorem is proved for the odd dimensional cases $m \geq 3$ and several applications such as $L^p$-decay of solutions of the Cauchy problems for time-dependent Schrödinger equations and wave equations with potentials, and the $L^p$-boundedness of Fourier multiplier in generalized eigenfunction expansions are given.

1. Introduction

Let $H_0 = D_1^2 + \cdots + D_m^2$, $D_j = -i \partial/\partial x_j$, be the free Schrödinger operator on $L^2(\mathbb{R}^m)$ and $H = H_0 + V$ its perturbation by the multiplication operator $V$ with a real valued function $V(x)$. It is well known in the scattering theory (cf. [1], [3], [9]) that, if $V$ is of short range in the sense that $\int_1^{\infty} \|F_R V(H_0 + 1)^{-1}\|dR < \infty$, where $F_R$ is the multiplication with the characteristic function of $\{x \in \mathbb{R}^m : |x| \geq R\}$, then the wave operators $W_{\pm}$ defined by

$$W_{\pm} u = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} u, \quad u \in L^2(\mathbb{R}^m)$$

exist and they are isometries on $L^2(\mathbb{R}^m)$ with the final set $L^2_c(H)$, the continuous spectral subspace for $H$. The wave operators satisfy the intertwining property: $f(H)W_{\pm} = W_{\pm}f(H_0)$ for Borel functions $f$ and they play important roles in the perturbation theory of continuous spectra as well as in the scattering theory ([14]).

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In [21] and [22], we showed that $W_\pm$ are in fact bounded in Sobolev spaces $W^{\ell,p}(\mathbb{R}^m)$:

$$W^{\ell,p}(\mathbb{R}^m) = \{ f \in L^p(\mathbb{R}^m) : \sum_{|\alpha|\leq \ell} \| D^\alpha f \|_{L^p}^p \equiv \| f \|_{W^{\ell,p}}^p < \infty \},$$

if either (1) the spatial dimension $m \geq 3$ is odd, or (2) $m \geq 4$ is even and $V$ is small or $V(x) \geq 0$, where for $\alpha = (\alpha_1, \ldots, \alpha_m)$, $D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. More precisely, we proved the following theorem, where $\ell \geq 0$ is an integer and $m^* = (m - 1)/(m - 2)$. $F$ is the Fourier transform, $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $H^s(\mathbb{R}^m) = W^{s,2}(\mathbb{R}^m)$.

**Theorem 1.1** ([21], [22]). Let $m \geq 3$. Let $V$ be a real valued function such that, for some $\sigma > 2/m^*$, $F(\langle x \rangle^\sigma D^\alpha V) \in L^{m^*}(\mathbb{R}^m)$ for $|\alpha| \leq \ell$, and satisfy one of the following conditions:

1. $\| F(\langle x \rangle^\sigma V) \|_{L^{m^*}(\mathbb{R}^m)}$ is sufficiently small;

2. $m = 2m' - 1$ is odd and, with $\delta > \max(m + 2, 3m/2 - 2)$, $|D^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\delta}$ for $|\alpha| \leq \max\{\ell, \ell + m' - 4\}$;

3. $m$ is even, $V(x) \geq 0$ and, with $\delta > 3m/2 + 1$, $|D^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\delta}$ for $|\alpha| \leq m + \ell$.

Suppose in addition that zero is neither eigenvalue nor resonance of $H$. Then, the wave operators $W_\pm$ are bounded in $W^{k,p}(\mathbb{R}^m)$ for any $k = 0, \ldots, \ell$ and $1 \leq p \leq \infty$.

**Remark 1.** Zero is said to be resonance of $H$ if the equation $-\triangle u(x) + V(x)u(x) = 0$ has a solution $u \notin L^2(\mathbb{R}^m)$ such that $(1 + |x|)^{-1-\varepsilon} u \in L^2(\mathbb{R}^m)$ for any $\varepsilon > 0$. If zero is resonance or eigenvalue of $H$, $W_\pm$ can not be bounded in $L^p$ for all $1 \leq p \leq \infty$ (cf. [21]). It is known that $H$ does not admit zero resonance if $m \geq 5$ or $V(x) \geq 0$.

Theorem 1.1, however, does not cover the case that the spatial dimension $m$ is even and $V(x)$ can be large negative. The main purpose of this paper is to fill this gap and prove the following theorem, where $\ell \geq 0$ is an arbitrarily fixed integer; $p_0 > m/2$ and $\ell_0 = [(m - 1)/2] \ if \ m \geq 4$; and $p_0 = 2$ and $\ell_0 = 0$ if $m = 3$. $[\sigma]$ is the integral part of $\sigma$. 
Theorem 1.2. Let $m \geq 3$. Suppose that $V(x)$ is real valued and, with $\delta > (3m/2) + 1$,

$$\sup_{x \in \mathbb{R}^m} \langle x \rangle^\delta \left( \int_{|x-y| \leq 1} |D^\alpha V(y)|^{p_0} dy \right)^{1/p_0} < \infty \quad (1.1)$$

for $|\alpha| \leq \ell + \ell_0$. Suppose further that zero is neither eigenvalue nor resonance of $H$. Then, $W_\pm$ are bounded in $W^{k,p}(\mathbb{R}^m)$ for any $k = 0, \ldots, \ell$ and $1 \leq p \leq \infty$.

Remark 2. Theorem 1.2 is a generalization of Theorem 1.1 when $m$ is even and $V$ is large, however, none of them is stronger than the other otherwise. We remark that under the condition of Theorem 1.2 it is possible to find $\sigma > 2/\ell_\ast$ such that $\mathcal{F}(\langle x \rangle^\sigma D^\alpha V) \in L^{\ell_\ast}(\mathbb{R}^m)$ for $|\alpha| \leq \ell$.

We refer to [21] for various applications of Theorems and the related reference, and shall be devoted to the proof of Theorem 1.2 in this paper. We shall only prove the $L^p$ boundedness of $W_+$ assuming $\ell = 0$ and $m$ is even $\geq 4$. The odd dimensional cases may be proved by slightly modifying the following argument or by the method of [21]; the proof for $W_-$ is similar; and the extension to general $\ell$ may be done by estimating the multiple commutators $[D_{j_1}, [D_{j_2}, \cdots [D_{j_\ell}, W_+] \cdots]]$ as in section 5 of [21].

We outline the proof here, displaying the plan of this paper and introducing some notations. $B(X, Y)$ is the Banach space of bounded operators from Banach space $X$ to $Y$ and $B(X) = B(X, X)$. $R(z) = (H - z)^{-1}$, $R_0(z) = (H_0 - z)^{-1}$ are resolvents and $R^\pm(\lambda) = R(\lambda \pm i0)$, $R_0^\pm(\lambda) = R_0(\lambda \pm i0)$ are their boundary values on the upper and lower banks of $\mathbb{C} \setminus [0, \infty)$. By using the stationary representation formula ([9], [14]):

$$W_+ u = u - \frac{1}{2\pi i} \int_0^\infty R^-(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}ud\lambda$$

and the identity $R^-(\lambda) = R_0^-(\lambda) - R_0^-(\lambda)VR^-(\lambda)$, we write $W_+ u = u + W_1 u + W_2 u$, where

$$W_1 u = -\frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}ud\lambda, \quad (1.2)$$

$$W_2 u = \frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda)VR^-(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}ud\lambda. \quad (1.3)$$
In the first half of section 2, we study the mapping property of \( R_0^\pm (\lambda) \) and the decay and smoothness properties of the integral kernels of \( R(0) \) and \( \phi(H) \) for \( \phi \in C_0^\infty(\mathbb{R}) \). As we think them of independent interest, these properties will be stated and proved under much weaker assumptions on \( V \) than necessary in what follows. We then recall from [21] the argument that proves \( W_1 \) is bounded in \( L^p \): Express \( W_1 \) explicitly in the form

\[
W_1 u(x) = \int_\Sigma d\omega \int_{2x\omega}^{\infty} \widehat{K}_V(t,\omega)u(t\omega + x_\omega)dt,
\]

where \( \Sigma \) is the unit sphere, \( x_\omega = x - 2(x\omega)\omega \) is the reflection of \( x \) along the \( \omega \)-axis and

\[
\widehat{K}_V(t,\omega) = \frac{i}{2(2\pi)^{m/2}} \int_0^{\infty} \widehat{V}(r\omega) r^{m-2} e^{itr/2} dr;
\]

it follows by Minkowski inequality and the fact that \( x \rightarrow x_\omega \) is measure preserving that for any \( \sigma > 1/2 \),

\[
\|W_1 u\|_{L^p} \leq 2\|\widehat{K}_V\|_{L^1([0,\infty) \times \Sigma)}\|u\|_{L^p}
\leq C\|(x)^\sigma V\|_{H^{(m-3)/2}}\|u\|_{L^p} \leq C'\|u\|_{L^p}.
\]

We wish to show that \( W_2 \) is bounded in \( L^p \) by proving the well known criterion:

\[
\max\left\{ \sup_{x \in \mathbb{R}^m} \int_{\mathbb{R}^m} |W_2(x, y)|dy, \sup_{y \in \mathbb{R}^m} \int_{\mathbb{R}^m} |W_2(x, y)|dx \right\} < \infty
\]

for its integral kernel \( W_2(x, y) \). It can be written as

\[
W_2(x, y) = \frac{1}{2\pi i} \int_0^{\infty} \langle R^- (k^2) V(G_{+,y,k} - G_{-,y,k}), VG_{+,x,k} \rangle dk^2,
\]

where \( \langle \cdot, \cdot \rangle \) is a coupling between suitable function spaces and \( G_{\pm,y,k}(x) = G_{\pm}(x-y, k) \) are the kernels of \( R_0^\pm (k^2) \) or the incoming-outgoing fundamental solutions of \( -\Delta - k^2 \). They satisfy \( G_{\pm}(x, k) \sim Ce^{\pm ik|x|} |x|^{-(m-1)/2}k^{(m-3)/2} \) as \( |x| \rightarrow \infty \) and crude estimations would only yield

\[
|\text{the integrand of (1.7)}| \leq Ck^{m-3} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2}.
\]

Thus we are faced with the two difficulties:

1) **High energy difficulty:** The integral (1.7) does not converge absolutely at \( k = \infty \);
(2) **Low energy difficulty:** If we restrict the integral (1.7) to finite intervals, (1.8) produces only $|W_2(x,y)| \leq C(x)^{-(m-1)/2}(y)^{-(m-1)/2}$ which is insufficient for (1.6). For obtaining improved decay property, we exploit the oscillation property of $G_\pm(x,k)$ and apply integration by parts with respect to the variable $k$. However, the singularity at $k = 0$ of $G_\pm(x,k)$ prevents us from doing this as many times as necessary if $m$ is even.

To separate two difficulties, we decompose $W_2$ into the low and the high energy parts and consider $W_{2,\text{low}} = \phi_1(H)W_2\phi_1(H_0)$ and $W_{2,\text{high}} = \phi_2(H)W_2\phi_2(H_0)$, where cut off functions $\phi_1 \in C_0^\infty(R^1)$ and $\phi_2 \in C^\infty(R^1)$ are such that $\phi_1(\lambda)^2 + \phi_2(\lambda)^2 = 1$, and $\phi_1(\lambda) = 1$ for $|\lambda| \leq 1$ and $\phi_1(\lambda) = 0$ for $|\lambda| \geq 2$. Note that $W_\pm = \sum_{\lambda=1}^2 \phi_j(H)W_\pm\phi_j(H_0)$ thanks to the intertwining property of $W_\pm$ and $\phi_j(H_0)$ and $\phi_j(H)$, $j = 1, 2$, are bounded in $L^p$ as proved in section 2. We show $W_{2,\text{low}}$ and $W_{2,\text{high}}$ are bounded in $L^p$ separately.

In section 3, we treat the low energy part $W_{2,\text{low}}$. We split $R^-(\lambda) = R^{-}(0) + \tilde{R}^-(\lambda)$ to single out the contribution of $R^{-}(0)$ and decompose as $W_{2,\text{low}} = W^{(1)}_{2,\text{low}} + W^{(2)}_{2,\text{low}}$ accordingly. In virtue of the orthogonality of Hardy functions in the upper and the lower half planes, we have

\begin{equation}
W^{(1)}_{2,\text{low}} u = \phi_1(H) \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} R_0^-(\lambda) V R^{-}(0) V R_0^+(\lambda) d\lambda \right\} \phi_1(H_0) u;
\end{equation}

using the identity $(R_0^+(\lambda) - R_0^-(\lambda)) \phi_1(H_0) = (R_0^+(\lambda) - R_0^-(\lambda)) \phi_1(\lambda)$, we write

\begin{equation}
W^{(2)}_{2,\text{low}} u = \frac{1}{2\pi i} \int_{0}^{\infty} \phi_1(H) R_0^-(\lambda) V \tilde{R}^{-}(\lambda) V (R_0^+(\lambda) - R_0^-(\lambda)) \times \tilde{\phi}_1(\lambda) \phi_1(H_0) u d\lambda,
\end{equation}

where $\tilde{\phi}_1 \in C_0^\infty(R)$ is such that $\tilde{\phi}_1(\lambda) \phi_1(\lambda) = \phi_1(\lambda)$. For dealing with $W^{(1)}_{2,\text{low}}$ it is important to observe the following: If we write the integral kernel of $R^{-}(0)$ by $K(x,y)$ and set $M_y(x) = V(x)K(x,x-y)V(x-y)$, then $W^{(1)}_{2,\text{low}}$ can be expressed as a superposition

\begin{equation}
W^{(1)}_{2,\text{low}} u = - \int_{R^m} \phi_1(H) W_1(M_y) \phi_1(H_0) u_y dy,
\end{equation}

where $u_y(x) = u(x - y)$ and $W_1(M_y)$ is defined by (1.2) with $M_y$ in place of $V$. We show in section 2 that

\begin{equation}
\int_{R^m} \| \langle x \rangle^\sigma M_y \|_{H^{(m-3)/2}(R^m)} dy < \infty
\end{equation}
for some $\sigma > 1/2$. Since (1.5) and (1.11) imply that $\| W_{2, \text{low}}^{(1)} u \|_{L^p}$ is bounded by a constant times

$$
\int_{\mathbb{R}^m} \| W_1 (M_y) \|_{B(L^p)} \| u_y \|_{L^p} dy \leq C \int_{\mathbb{R}^m} \| \langle x \rangle^\sigma M_y \|_{H^{(m-3)/2} (\mathbb{R}^m)} dy \cdot \| u \|_{L^p},
$$

$W_{2, \text{low}}^{(1)}$ is bounded in $L^p$.

We treat $W_{2, \text{low}}^{(2)}$ as follows. Set $G_{\pm, x, k} (y) = e^{\pm ik|x|} \tilde{G}_{\pm, x, k} (y)$ to make oscillation property explicit and write its integral kernel in the form

$$
W_{2, \text{low}}^{(2), \pm} (x, y) = W_{2, \text{low}}^{(2), \pm} (x, y) - W_{2, \text{low}}^{(2), -} (x, y):
$$

(1.13)

$$
W_{2, \text{low}}^{(2), -} (x, y) = \frac{1}{2\pi i} \int_0^\infty e^{-ik(\|x| - \|y\|)} \langle \tilde{R}^\pm (k^2) V \tilde{G}_{\pm, y, k}, V \tilde{G}_{+, x, k} \rangle
\times \tilde{\phi}_1 (k^2) dk^2,
$$

where we ignored the harmless factors $\phi_1 (H_0)$ and $\phi_1 (H)$. We then apply integration by parts with respect to $k$ variable $\ell = (m + 2)/2$ times (when $m$ is even):

(1.14)

$$
= \frac{1}{2\pi i} \int_0^\infty D_k^\ell e^{-ik(\|x| - \|y\|)} \langle \tilde{R}^\pm (k^2) V \tilde{G}_{\pm, y, k}, V \tilde{G}_{+, x, k} \rangle \tilde{\phi}_1 (k^2) dk^2
= \frac{1}{\pi i} \int_0^\infty e^{-ik(\|x| - \|y\|)} D_k^\ell \{ k \langle \tilde{R}^\pm (k^2) V \tilde{G}_{\pm, y, k}, V \tilde{G}_{+, x, k} \rangle \tilde{\phi}_1 (k^2) \} dk,
$$

and gain the addition decay factor $(\|x\| + \|y\|)^{-\ell}$. Here the boundary terms do not appear and the integral converges absolutely because $\tilde{R}^\pm (k^2)$ vanishes at $k = 0$. (Actually we apply the integration by parts in a little more elaborate way. See the text for the details.) In this way we arrive at the estimate

(1.15)

$$
|W_{2, \text{low}}^{(2), \pm} (x, y)| \leq C (1 + \|x\| + \|y\|)^{-m+2/2} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2}
$$

and $W_{2, \text{low}}^{(2)} (x, y)$ indeed satisfies the criterion (1.6). Though the splitting of $R^\pm (\lambda)$ as above is unnecessary when $m$ is odd because of simpler structure of $G_{\pm} (x, k)$, it makes the proof of the theorem simpler even in that case.

In section 4, we prove that the high energy part $W_{2, \text{high}} = \phi_2 (H) W_2 \phi_2 (H_0)$ is also bounded in $L^p$, overcoming the high energy difficulty by the method similar to one that was employed in section 4 of [21]:
We decompose \( W_2 \) into \( 2N + 1 \) summands: 
\[
W_2 = \sum_{n=2}^{2N+2} (-1)^n W^{(n)}
\]
by expanding \( R^-(k^2) \) as
\[
R^-(k^2) = \sum_{n=0}^{2N-1} (-1)^n R_0^-(k^2)(VR_0^-(k^2))^n + (R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N
\]
and inserting (1.16) into (1.3). A repeated application of the argument leading to (1.4) shows that 
\( W^{(2)}(2), \ldots, W^{(2N+1)}(2N+1) \) have expressions similar to (1.4), and the estimate similar to the one used for proving (1.5) implies that they are all bounded in \( L^p \).

To prove \( W^{(2N+2)}(2N+2) \) is bounded in \( L^p \), we let 
\[
F_N(k^2) = (R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N
\]
and define the integral operator 
\[
W^{(2N+2)}_{\text{high}}(x,y) = W^{(2N+2)}_{\text{high}}(x,y) = W^{(2N+2)}_{\text{high}}(x,y) - W^{(2N+2)}_{\text{high}}(x,y) + (R-)(k^2)V
\]
with the integral kernel 
\[
W^{(2N+2)}_{\text{high}}(x,y) = \frac{1}{2\pi i} \int_0^{\infty} e^{-ik(|x| ± |y|)} \langle F_N(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{±,x,k}\rangle \tilde{\phi}_2(k^2) dk^2,
\]
where \( \tilde{\phi}_2 \in C^\infty(\mathbb{R}) \) is such that \( \tilde{\phi}_2(\lambda) = 0 \) near \( \lambda = 0 \) and \( \tilde{\phi}_2(\lambda)\phi_2(\lambda) = \phi_2(\lambda) \). Then we have \( \phi_2(H)W^{(2N+2)}\phi_2(H_0) = \phi_2(H)W^{(2N+2)}_{\text{high}}\phi_2(H_0) \). If \( N \) is sufficiently large \( F_N(k^2) \), as an operator valued function between suitable function spaces, decays rapidly as \( k \to \infty \) and the integrals (1.17) converge absolutely. Moreover, integration parts with respect to \( k \) variable as in the proof of (1.15) yields
\[
|W^{(2N+2)}_{\text{high}}(x,y)| \leq C(1 + ||x| ± |y||)^{-m+1/2} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2},
\]
which shows that \( W^{(2N+2)}_{\text{high}}(x,y) \) satisfies the criterion (1.6). In this way the argument is very much similar to that of the previous section and of section 4 of [21], and therefore, we shall be very sketchy in section 4.

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2. Preliminaries

In this section we first study the mapping property of \( R_0^\pm(\lambda), \lambda \geq 0 \), and the decay and smoothness properties of the integral kernels of \( R^\pm(0) \) and
\( \phi(H) , \phi \in C_0^\infty(\mathbb{R}) \), under the conditions which are more general than in 1.2. We then recall from [21] the argument for proving the \( L^p \) boundedness of \( W_1 \). For \( 1 \leq p,q \leq \infty \) and \( \delta, \ell \in \mathbb{R} \), \( L^p_\delta(\mathbb{R}^m) \) is the weighted \( L^p \)-space:

\[
L^p_\delta(\mathbb{R}^m) = \{ f \in L^p_{loc}(\mathbb{R}^m) : \| f \|_{L^p_\delta} = \| \langle x \rangle^\delta f \|_{L^p} < \infty \};
\]

\( H^\ell_\delta(\mathbb{R}^m) \) is the weighted Sobolev space:

\[
H^\ell_\delta(\mathbb{R}^m) = \{ f \in S'(\mathbb{R}^m) : \| (1 + |x|^2)^{\ell/2}(1 - \Delta)^{\ell/2} f \|_{L^2} = \| f \|_{H^\ell_\delta} < \infty \};
\]

and \( \ell^p_\delta(L^q) \) is the amalgam space:

\[
\ell^p_\delta(L^q) = \{ f \in L^q_{loc}(\mathbb{R}^m) : \| f \|_{\ell^p_\delta(L^q)} = \left( \sum_{n \in \mathbb{Z}^m} \| f \|_{L^q(Q_n)}^p \langle n \rangle^{\delta p} \right)^{1/p} < \infty \},
\]

where for \( n = (n_1, \ldots, n_m) \), \( Q_n = [n_1, n_1 + 1) \times \cdots \times [n_m, n_m + 1) \) is a unit cube.

### 2.1 Resolvent estimate for \( H_0 \)

If \( s > 1 \) and \( t \in \mathbb{R} \), the resolvent \( R_0(z) = (H_0 - z)^{-1} \), which is originally defined as a \( B(L^2) \)-valued analytic function of \( z \in \mathbb{C} \setminus [0, \infty) \), can be extended continuously to the closure \( \overline{\mathbb{C} \setminus [0, \infty)} \) (in the Riemann surface of \( \log z \)) when considered as a \( B(H^s_t, H^{t+2}_{-s}) \)-valued function ([9]). We denote the boundary values on the upper and lower edges by \( \lim_{\epsilon \to 0^+} R_0(\lambda \pm i\epsilon) \equiv R_0^\pm(\lambda) \), \( \lambda \in [0, \infty) \). The following mapping property of \( R_0^\pm(\lambda) \) is well known (cf. Murata [12] and Jensen [4]). In what follows, \( D_k \) will denote \(-i\partial/\partial k\) and should not be confused with \(-i\partial/\partial x_k\). \([\sigma]\) is the largest integer not greater than \( \sigma \in \mathbb{R} \).

**Lemma 2.1.** Let \( \ell = 0, 1, 2, \ldots, t \in \mathbb{R} \) and \( s > \ell + 1/2 \). Then, as a \( B(H^t_s, H^{t+2}_{-s}) \)-valued function of \( k \), \( R_0^\pm(k^2) \) is \( C^\ell \) in \( k \in (0, \infty) \). Moreover:

1. For \( j = 0, 1, \ldots, \ell \) and \( 0 \leq i \leq 2 + [(j + 1)/2] \), \( \| D_k^j R_0^\pm(k^2) \|_{B(H^t_s, H^{t+1}_s)} \leq Ck^{-1+i} \), \( k \geq 1 \).

2. If \( \ell \geq 2 \), then \( R_0^\pm(k^2) \) has the following expansion in \( B(H^t_s, H^{t+2}_{-s}) \) valid for \( k \to 0 \):
Here $F_1, G_j \in B(H^\ell_s, H^{\ell+2}_s)$, and $K_2(k)$ stands for a $B(H^\ell_s, H^{\ell+2}_s)$-valued $C^\ell$-function of $k$ such that, for $0 \leq j \leq \ell$, $\|D^j_k K_2\| = o(k^{2-j})$ as $k \to 0$. Relation (2.1) remains valid if the boundary values $R^\pm_0(k^2)$ are replaced by $R_0(k^2), \text{Im} k > 0.$

In section 4, we shall also use the following mapping property of $D^j_k R^\pm_0(k^2)$ between $L^p$ type spaces. For $0 \leq \ell < (m - 1)/2$, $P^m_\ell$ is the pentagon in the $(x, y)$-plane surrounded by five lines $x = 1, x = 1/2 + (2\ell + 1)/2m, y = 0, y = 1/2 - (2\ell + 1)/2m$ and $y = x - 2(\ell + 1)/(m + 1)$, where the segments $\{(x, 0): 1/2 + (2\ell + 1)/2m < x \leq 1\}$ and $\{(1, y): 0 \leq y < 1/2 - (2\ell + 1)/2m\}$ are included. Note that $(1/2 + (\ell + 1)/m, 1/2 - (\ell + 1)/m) \in P^m_\ell$ as long as $\ell + 1 < m/2$.

**Lemma 2.2.** Let $j = 0, 1, \ldots$ and let $1 \leq p \leq q \leq \infty$ and $1 \leq r \leq \rho \leq \infty$ be such that $1/r \geq 1/q - (j + 2)/m$, where the equality is inclusive only when $1/q - (j + 2)/m > 0$. Then, $D^j_k R^\pm_0(k^2)$ satisfies the following mapping property:

(a) The case $m$ is odd $\geq 3$:

1. If $0 \leq j < (m - 1)/2$, $D^j_k R^\pm_0(k^2) \in B(\ell^p(L^q), \ell^p(L^r))$ for $(1/p, 1/\rho) \in \mathbf{P}^m_j$ and
   $$\|D^j_k R^\pm_0(k^2)\|_{B(\ell^p(L^q), \ell^p(L^r))} \leq C_j k^{m(1/p-1/\rho)-2-j}, \quad k \geq 1.$$  

2. If $(m - 1)/2 \leq j < m - 2$, $D^j_k R^\pm_0(k^2) \in B(\ell^1_{-j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))$ and
   $$\|D^j_k R^\pm_0(k^2)\|_{B(\ell^1_{-j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))} \leq C_j k^{(m-3)/2}, \quad k \geq 1.$$
3. If $j \geq m - 2$, $D^j_k R^\pm_0 (k^2) \in B(L^1_{j-(m-1)/2}, L^\infty_{j-(m-1)/2})$ and
\[ \|D^j_k R^\pm_0 (k^2)\|_{B(L^1_{j-(m-1)/2}, L^\infty_{j-(m-1)/2})} \leq C_j k^{(m-3)/2}, \quad k \geq 1. \]

(b) The case $m$ is even $\geq 4$:

1. If $0 \leq j \leq (m - 2)/2$, $D^j_k R^\pm_0 (k^2) \in B(\ell^p(L^q), \ell^p(L^r))$ for $(1/p, 1/\rho) \in P^m_j$ and
\[ \|D^j_k R^\pm_0 (k^2)\|_{B(\ell^p(L^q), \ell^p(L^r))} \leq C_j k^{m(1/p-1/\rho)-2-j}, \quad k \geq 1. \]

2. If $m/2 \leq j \leq m - 3$, $D^j_k R^\pm_0 (k^2) \in B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{j-(m-1)/2}(L^r))$
and
\[ \|D^j_k R^\pm_0 (k^2)\|_{B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{j-(m-1)/2}(L^r))} \leq C_j k^{(m-3)/2}, \quad k \geq 1. \]

3. If $j = m - 2$, $D^j_k R^\pm_0 (k^2) \in B(\ell^1_{j-(m-1)/2}(L^q), L^\infty_{j+(m-1)/2})$ for any $1 < q \leq \infty$.
\[ \|D^j_k R^\pm_0 (k^2)\|_{B(\ell^1_{j-(m-1)/2}(L^q), L^\infty_{j+(m-1)/2})} \leq C_j k^{(m-3)/2}, \quad k \geq 1. \]

4. If $j \geq m - 1$, $D^j_k R^\pm_0 (k^2) \in B(L^1_{j-(m-1)/2}, L^\infty_{j-(m-1)/2})$ and
\[ \|D^j_k R^\pm_0 (k^2)\|_{B(L^1_{j-(m-1)/2}, L^\infty_{j-(m-1)/2})} \leq C_j k^{(m-3)/2}, \quad k \geq 1. \]

For proving Lemma 2.2, we use the following lemma. We write $u_k(x) = u(x/k)$.

**Lemma 2.3.**

1. If $1 \leq p \leq q \leq \infty$, $\delta \geq 0$ and $k \geq 1$, then
\[ \|u_k\|_{\ell^p_q(L^q)} \leq C k^{m/p+\delta} \|u\|_{\ell^p_q(L^q)} \]

2. If $1 \leq r \leq \rho \leq \infty$, $\delta \geq 0$ and $k \geq 1$, then
\[ \|u_{1/k}\|_{\ell^r_{\rho}(L^r)} \leq C k^{-m/p+b} \|u\|_{\ell^r_{\rho}(L^r)}. \]

**Proof.** We only prove the first statement for integral $k \geq 1$. General case may be proved by a slight modification of the following argument. The
second statement follows from the first by the duality. If $k \geq 1$ is integral, we have by Hölder’s inequality:

\[
\|f_k\|_{L_p^p(L_q)}^p = \sum_{n \in \mathbb{Z}^m} \langle n \rangle^{p\delta} \left( \int_{Q_n} |f(x/k)|^q \, dx \right)^{p/q}
\]

\[
= \sum_{n \in \mathbb{Z}^m} k^{mp/q} \langle n \rangle^{p\delta} \left( \int_{Q_{n/k}} |f(x)|^q \, dx \right)^{p/q}
\]

\[
= k^{mp/q} \sum_{j \in \mathbb{Z}^m} \left\{ \sum_{Q_{n/k} \subset Q_j} \left( \int_{Q_{n/k}} |f(x)|^q \, dx \right)^{p/q} \langle n \rangle^{p\delta} \right\}
\]

\[
\leq k^{mp/q} \sum_{j \in \mathbb{Z}^m} (k^m)^{1-p/q} \left( \sum_{Q_{n/k} \subset Q_j} \int_{Q_{n/k}} |f(x)|^q \, dx \right)^{p/q} \langle j \rangle^{p \delta}
\]

\[
= C^{p\delta} k^{m+p\delta} \sum_{j \in \mathbb{Z}^m} \left( \int_{Q_j} |f(x)|^q \, dx \right)^{p/q} \langle j \rangle^{p \delta} = C^{p\delta} k^{m+p\delta} \|f\|_{L_p^p(L_q)}^p,
\]

where the constant $C$ depends only on the spatial dimension $m$. □

**Proof of Lemma 2.2.** We prove the lemma when $m \geq 3$ is even. The proof for the other case is similar. It is well known that $R_0^\pm(k^2)$, $k \geq 0$, are convolution operators with the outgoing (+) or incoming (−) fundamental solutions $G_\pm(x, k)$ of $-\Delta - k^2$ ([15]):

\[
G_\pm(x, k) = \pm \frac{i}{4(2\pi)^\nu |x|^{m-2}} (k|x|)^\nu H^{(\pm)}_\nu(k|x|), \quad \nu = \frac{m-2}{2}
\]

where $H^{(\pm)}_\nu(z)$ is the Hankel function and by Hankel’s formula ([20])

\[
z^\nu H^{(\pm)}_\nu(z) = \frac{\sqrt{2} e^{\mp i(2\nu+1)\pi/4} e^{\pm iz}}{\sqrt{\pi} \Gamma(\nu+1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} \left( z \pm \frac{it}{2} \right)^{\nu-1/2} \, dt.
\]

Here and hereafter we use the superscript $\pm$ in stead of the traditional 1, 2 for Hankel functions and $\nu = (m-2)/2$. A simple computation shows that $D_k^j R_0^\pm(k^2)$ enjoys the homogeneity property

\[
[D_k^j R_0^\pm(k^2) u](x) = k^{-j-2} \{ D_k^j R_0^\pm(k^2) u_{k=1} \}(kx),
\]

\[
u_k(x) = u(x/k).
\]
We prove the lemma for the case $k = 1$ first. Let $\phi \in C_0^\infty(\mathbb{R}^m)$ be such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Write $G_\pm^{(j)}(x)$ for the convolution kernel of $D_k^j R_0^\pm (k^2)_{k=1}$ and set $G_1^{(j)}(x) = G_\pm^{(j)}(x) \phi(x)$ and $G_2^{(j)}(x) = G_\pm^{(j)}(x)(1 - \phi(x))$. Differentiating (2.2) and (2.3) by $k$ shows that $G_1^{(j)}(x)$ satisfies the following estimate:

$$|G_1^{(j)}(x)| \leq \begin{cases} 
C_j(1 + |x|^{2-m+j}), & \text{if } m \text{ is odd}; \\
C_j((\log |x|) + |x|^{2-m+j}), & \text{if } m \text{ is even and } j \leq m - 2; \\
C_j, & \text{if } m \text{ is even and } j \geq m - 1,
\end{cases}$$

and that $G_2^{(j)}(x)$ can be written as

$$(2.5) \quad G_2^{(j)}(x) = e^{\pm i |x|} a_{j,\pm}(x) |x|^{(2j-m+1)/2},$$

where $a_{j,\pm}(x) \in C^\infty(\mathbb{R}^m)$ is supported by $\{|x| \geq 1\}$ and satisfies for any $\alpha$

$$|D^\alpha a_{j,\pm}(x)| \leq C_j |x|^{-|\alpha|}.$$

Since $G_1^{(j)}(x)$ is supported by the compact set $\{|x| \leq 2\}$, the convolution operator $G_1^{(j)}$ with $G_1^{(j)}(x)$ can be easily estimated by using the fractional integration theory and Young’s inequality:

(i) If $0 \leq j \leq m - 3$, $G_1^{(j)} \in B(\ell^p(\ell^q), \ell^p(\ell^q))$ for any $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$ if $1/q < (j + 2)/m$; $1 \leq r < \infty$ if $1/q = (j + 2)/m$; and $1/q - (j + 2)/m \leq 1/r \leq 1$ if $1/q > (j + 2)/m$.

(ii) If $j = m - 2$, $G_1^{(j)} \in B(\ell^p(\ell^q), \ell^p(\ell^q))$ for any $1 \leq p \leq \infty$, and $1 < q \leq \infty$ (if $m$ is odd $q = 1$ can be included);

(iii) If $j \geq m - 1$, $G_1^{(j)} \in B(\ell^p(L^1), \ell^p(L^\infty))$ for any $1 \leq p \leq \infty$.

On the other hand $G_2^{(j)}(x)$ contains the oscillating factor $e^{\pm i|x|}$ and we estimate the convolution operator $G_2^{(j)}$ with the kernel (2.5) by a theorem of Sogge (cf. [19], Lemma 5.4). We combine the result with the fact $G_2^{(j)} \in B(L^p, L^\infty)$, $1 \leq p < 2m/(m + 2j + 1)$, which follows from Young’s inequality, by using the interpolation theorem and the duality. We obtain the followings:

(iv) If $j \leq (m - 2)/2$, then $G_2^{(j)} \in B(L^p, L^p)$ for any $p$ and $\rho$ such that $(1/p, 1/\rho) \in \mathbb{P}_j^m$ where $\mathbb{P}_j^m$ is the polygon defined as above.

(v) If $j \geq m/2$, then $2j - m + 1 > 0$ and $G_2^{(j)} \in B(L^1_{j -(m-1)/2}, L^\infty_{j -(m-1)/2})$. 

Note here that \( \ell^p_\delta (L^{q_1}) \subset \ell^p_\delta (L^{q_2}) \) whenever \( p_1 \leq p_2 \) and \( q_1 \geq q_2 \). Thus, combing estimates (i) \( \sim \) (v), we obtain the lemma for the case \( k = 1 \).

It remains to estimate the operator norm for \( k \geq 1 \). When \( j \leq (m-2)/2 \) the estimates in the lemma immediately follow from (2.4) and Lemma 2.3. When \( j \geq m/2 \), the direct application of Lemma 2.3 would produce the superfluous power \( k^{j-1} \). Note, however, that in this case \( G_{2,\pm}^{(j)} (x-y) \) satisfies

\[
|G_{2,\pm}^{(j)} (x-y)| \leq C (|x|^{(2j-m+1)/2} + |y|^{(2j-m+1)/2} + 1),
\]

and \( G_{2,\pm}^{(j)} \) is in fact a sum of two operators, one in \( B(L^1_j-(m-1)/2; L^\infty) \) and the other in \( B(L^1_j, L^\infty_{j+(m-1)/2}) \). Hence, say in the case (b.2), \( D^j_k R_{0\pm} (k^2) \) may be written as a sum of two operators, one in \( B(\ell_1^j-(m-1)/2(L^q), \ell^\infty (L^r)) \) and the other in \( B(\ell^1(L^q), \ell^\infty_{j+(m-1)/2}(L^r)) \). Applying Lemma 2.3 to each summand separately and combining the results, we obtain the desired estimates. \( \square \)

### 2.2 Integral kernels of \( \phi(H) \) and \( R(0) \)

In this subsection, we study the integral kernel of \( \phi(H) \) (resp. \( R(0) \)) assuming that \( V \) is of Kato class (resp. very short range). A real valued function \( V(x) \) is said to be of Kato-class if

\[
(2.6) \quad \lim_{\epsilon \to 0} \sup_{x \in \mathbb{R}^m} \int_{|x-y| \leq \epsilon} \frac{|V(y)|}{|x-y|^{m-2}} dy = 0
\]

and to be **very short range** if, for some \( \gamma > 0 \), \( \langle x \rangle^{2+\gamma} V(x) \) satisfies (2.6). In particular, we have for very short range potential that

\[
(2.7) \quad \|V\|_{(\gamma)} \equiv \sup_{x \in \mathbb{R}^m} \langle x \rangle^{2+\gamma} \int_{|x-y| < 1} \frac{|V(y)|}{|x-y|^{m-2}} dy < \infty.
\]

We note that \( V \) which satisfies the assumption of Theorem 1.2 is very short range.

If \( V \) is of Kato class, then, the multiplication operator \( V \) with \( V(x) \) is \( H_0 \)-form bounded with relative bound zero and \( H = H_0 + V \) defined via the form sum is self-adjoint([13]). If we write \( A(x) = |V(x)|^{1/2} \) and \( B(x) = V(x)^{1/2} \equiv |V(x)|^{1/2} \text{sign} V(x) \) and \( A \) and \( B \) for the multiplications by \( A(x) \) and \( B(x) \), respectively, then

\[
(2.8) \quad R(z) = R_0(z) - R_0(z) B(1 + AR_0(z)B)^{-1} AR_0(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
The following lemma solves an open problem in Simon ([17]):

**Lemma 2.4.** Let $V$ be of Kato-class and $\phi(\lambda) \in C_0^\infty(\mathbb{R})$. Then, the integral kernel $\Phi(x, y)$ of $\phi(H)$ satisfies $|\Phi(x, y)| \leq C_\delta (1 + |x - y|)^{-\delta}$ for any $\delta \geq 0$. In particular, $\phi(H)$ is bounded in $L^p$ for any $1 \leq p \leq \infty$.

**Proof.** The following argument which has simplified the original proof is due to Shu Nakamura (private communication). If we set $V_a(x) = V(x + a)$ and $H(a) = H_0 + V_a$, $\Phi(x + a, y + a)$ is the integral kernel of $\phi(H(a))$. Hence, it suffices to show

\[
\sup_{|y| \leq 1} |\Phi(x, y)| \leq C_\delta (1 + |x|)^{-\delta}
\]

with constants $C_\delta$ which is independent of $a$ if $H$ is replaced by $H(a)$. (We say that an estimate holds uniformly in $a$ if it does with the same constant when $H$ is replaced by $H(a)$, $a \in \mathbb{R}^m$). Write $\phi(\lambda) = (\lambda - z)^{-N} \psi(\lambda)(\lambda - z)^{-N}$ so that $\phi(H) = R(z)^N \psi(H) R(z)^N$. By Theorem B.6.3 of [17], $R(z)^N$ is bounded uniformly in $a$ from $L^1_\delta$ to $L^2_\delta$ and from $L^2_\delta$ to $L^\infty_\delta$ for any $\delta \in \mathbb{R}$, if $N$ and real $-z$ are large enough. On the other hand $\psi(H)$ is bounded in $L^2_\delta$ uniformly in $a$ as will be shown below. Hence, $\phi(H)$ is bounded from $L^1_\delta$ to $L^\infty_\delta$ uniformly in $a$ and

\[
\sup_{x \in \mathbb{R}, |y| \leq 1} \langle x \rangle^\delta |\Phi(x, y)|
\]

\[
\leq C_\delta \sup\{\|\phi(H)u\|_{L^\infty_\delta} : \|u\|_{L^1_\delta} = 1, \supp u \subset B(O, 1)\}
\]

\[
\leq C_\delta \|\phi(H)\|_{B(L^1_\delta, L^\infty_\delta)} < \infty.
\]

It remains to show that $\psi(H)$ is bounded in $L^2_\delta$ for any $\delta > 0$ uniformly in $a$. It suffices to show that for any choice of $1 \leq j_k \leq m, k = 1, \ldots, \ell$ and $\ell = 1, 2, \ldots$

\[
\|[[x_{j_1}, [x_{j_2}, \ldots, [x_{j_{\ell}}, \psi(H)] \cdots]]\|_{B(L^2)} \leq C_\ell
\]

uniformly in $a$. Let $\psi(z)$ be an almost analytic extension of $\psi(\lambda)$ which satisfies for any $n$ and $N \geq 0$,

\[
|(\partial \psi/\partial z)(z)| \leq C_{nN} |\text{Im } z|^{n}(1 + |z|)^{-n-N}, \quad z \in \mathbb{C}
\]

and write

\[
\psi(H) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \psi}{\partial z}(z)(H - z)^{-1}d\bar{z} \wedge dz
\]
Then, using inductively the obvious identity
\[ i[x_j, R(z)] = R(z)p_j R(z) \]
and using the fact that \( \| R(z) \| \leq |\text{Im } z|^{-1} \) and \( \| p_j R(z) \| \leq C|\text{Im } z|^{-1} \), where the constant \( C \) is independent of \( a \) (cf. [17]), we immediately obtain the desired boundedness (2.10).

If \( V \) is very short range, then \( V \) is form compact with respect to \( H_0 \); and in virtue of Lemma 2.1, the boundary values
\[
\lim_{\epsilon \to +0} AR_0(\lambda \pm i\epsilon) B \equiv Q_0^\pm(\lambda)
\]
exist in the operator norm of \( L^2 \) and are locally Hölder continuous in \( \lambda \in [0, \infty) \). Moreover, \( 1 + Q_0^\pm(\lambda) \) is an isomorphism of \( L^2(\mathbb{R}^m) \) if and only if \( \lambda \) is not an eigenvalue of \( H \) (\( \lambda \) is not the eigenvalue or resonance of \( H \) if \( \lambda = 0 \)). Thus, if non-negative eigenvalues and zero resonance are absent from \( H \), then the boundary values of the resolvent
\[
(2.12) \quad \lim_{\epsilon \to +0} R(\lambda \pm i\epsilon) \equiv R^\pm(\lambda)
\]
exist for all \( \lambda \in [0, \infty) \) in the operator norm of \( B(L^2_\delta, L^2_{-\delta}) \) and are locally Hölder continuous in \( \lambda \in [0, \infty) \) as well. Note that \( R_0^\pm(0) \) is independent of the sign \( \pm \) and so is \( R^\pm(0) \). We write \( R_0^\pm(0) = R(0) = G_0 \) and \( R^\pm(0) = R(0) \). We have the following lemma on the integral kernel of \( R(0) \).

**Theorem 2.5.** Let \( V(x) \) be very short range. Suppose that zero is not an eigenvalue nor resonance of \( H = H_0 + V \). Then, \( R(0) \) has the integral kernel \( K(x, y) \) which is jointly continuous for \( x \neq y \) and satisfies
\[
|K(x, y)| \leq C|x - y|^{2-m}.
\]

We begin the proof of Theorem 2.5 with the following elementary lemma. In what follows we assume that \( \langle x \rangle^{2+\gamma} V(x) \) satisfies (2.6) for some \( 0 < \gamma < 1 \).

**Lemma 2.6.** Let \( 0 \leq \rho < \gamma < 1 \). Then, with a constant \( C_1 \) depending only on \( m, \rho \) and \( \gamma \),
\[
(2.13) \quad \int_{\mathbb{R}^m} \frac{\langle y \rangle^\rho |V(y)| dy}{|x - y|^{m-2}} \leq C_1 \| V \|_{(\gamma)} \langle x \rangle^{\rho - \gamma};
\]
Since the integral over $|x - y| \geq 1$ as follows:

$$\int_{|x - y| \geq 1} \frac{\langle y \rangle^\rho |V(y)| dy}{|x - y|^{m-2}} \leq 2^{m-2} \int_{\mathbb{R}^m} \frac{\langle y \rangle^\rho |V(y)| \phi(y - z) dy}{|x - y|^{m-2}} dz \leq C_2 \|V\|_{(\gamma)} \|\phi\|_{L^\infty} \int_{\mathbb{R}^m} \frac{\langle y \rangle^\rho |V(y)| \phi(y - z) dy}{(1 + |x - z|)^{m-2}(z)^{2+\gamma - \rho}} \leq C_3 \|V\|_{(\gamma)} \langle x \rangle^{\rho - \gamma}.$$  

Since the integral over $|x - y| \leq 1$ is obviously bounded by a constant times $\|V\|_{(\gamma)} \langle x \rangle^{\rho - \gamma}$, we obtain (2.13).

Write $w = x - y$ and change the variable $z$ by $z + y$. Let $\Omega_1 = \{z : |w|/2 \leq |z|\}$ and $\Omega_2 = \{z : |w|/2 \leq |z - w|\}$. It is clear that $\mathbb{R}^m = \Omega_1 \cup \Omega_2$ and by using (2.13) with $\rho = 0$,

$$\int_{\Omega_1} \frac{|V(z + y)| dz}{|w - z|^{m-2}|z|^{m-2}} \leq 2^{m-2} \int_{\mathbb{R}^m} \frac{|V(z + y)| dz}{|w - z|^{m-2}} \leq C_1 \langle x \rangle^{-\gamma} |w|^{2-m} \|V\|_{(\gamma)};$$

$$\int_{\Omega_2} \frac{|V(z + y)| dz}{|w - z|^{m-2}|z|^{m-2}} \leq 2^{m-2} \int_{\mathbb{R}^m} \frac{|V(z + y)| dz}{|z|^{m-2}} \leq C_1 \langle y \rangle^{-\gamma} |w|^{2-m} \|V\|_{(\gamma)}.$$

Adding these up, we obtain (2.14). □

The following is a corollary of Lemma 2.6 and proves Theorem 2.5 when $V$ is small.

**Lemma 2.7.** There exists a constant $C_0 > 0$ such that, if $\|V\|_{(\gamma)} < C_0$, then the integral kernel $K(x, y)$ of $R(0)$ is continuous for $x \neq y$ and satisfies $|K(x, y)| \leq C|x - y|^{2-m}$.

**Proof.** The integral kernel of $G_0 = R_0^\pm(0)$ is given by the Newton potential $G_0(x - y) = c_m |x - y|^{2-m}$, $c_m = \Gamma(m - 2/2)/4\pi^{m/2}$. By Schwarz
inequality and (2.13) with \(\rho = 0\),
\[
|Q_{0}^{\pm}(0) u, v| \leq c_{m} \int_{\mathbb{R}^{m}} \frac{|A(x)||v(x)||B(y)||u(y)|}{|x-y|^{m-2}} \, dx \, dy
\]
\[
\leq c_{m} \left( \int_{\mathbb{R}^{m}} \frac{|A(x)|^{2}|u(y)|^{2}}{|x-y|^{m-2}} \, dx \, dy \right)^{1/2} \left( \int_{\mathbb{R}^{m}} \frac{|B(y)|^{2}|v(x)|^{2}}{|x-y|^{m-2}} \, dx \, dy \right)^{1/2}
\leq c_{m} C_{1} \|V\|_{(\gamma)} \|u\| \|v\|.
\]
Hence, \(1 + Q_{0}^{\pm}(0)\) is invertible in \(B(L^{2})\) if \(\|V\|_{(\gamma)} < (c_{m}C_{1})^{-1}\), and we may expand \((1 + Q_{0}^{\pm}(0))^{-1}\) into the Neumann series in (2.12) with \(\lambda = 0\) to obtain

\[
R(0) = G_{0} - G_{0} VG_{0} + G_{0} VG_{0} VG_{0} - \cdots.
\]

Since any \(V\) with \(\|V\|_{(\gamma)} < \infty\) may be approximated arbitrarily close by \(C_{0}^{\infty}\) functions in the norm \(\|\cdot\|_{(\gamma')}\), \(\gamma' < \gamma\), it is easy to see that the integral kernels of the summands of the series are continuous for \(x \neq y\). Moreover estimating them inductively by using (2.14), we obtain a majorant series \(\sum_{n=0}^{\infty} c_{m}^{n+1}(2C_{1}\|V\|_{(\gamma)})^{n}|x-y|^{2-m}\) for \(K(x,y)\). The latter series converges uniformly on every compact subset of \(\{(x,y) : x \neq y\}\) and produces the bound \(|K(x,y)| \leq C_{2}|x-y|^{2-m}\) if \(2c_{m}C_{1}\|V\|_{(\gamma)} < 1\). This proves the Lemma. \(\Box\)

For proving Theorem 2.5 for general potentials, we shall use the following lemma. For \(0 < \rho < \min(1, \gamma)\), \(X_{\rho}\) is the Banach space defined by

\[
X_{\rho} = \{u \in C(\mathbb{R}^{m} \setminus \{0\}) : \|u\|_{X_{\rho}} = \sup_{x \in \mathbb{R}^{m} \setminus \{0\}} \langle x \rangle^{-\rho} |x|^{m-2}|u(x)| < \infty\}.
\]

We remark here that if \(K(x,y)\) is as in Lemma 2.7, then \(K_{y}(x) \equiv K(x+y, y)\) belongs to \(X_{\rho}\) and \(y \to K_{y}\) is an \(X_{\rho}\) valued continuous function. This can be easily seen by the proof of the lemma (note that \(K_{y}(x)\) is \(K_{0}(x)\) corresponding to the potential \(V_{y}(x) = V(x+y)\) and \(y \to V_{y}\) is continuous in the \(\|\cdot\|_{(\gamma')}\) norm, \(\gamma' < \gamma\)).

**Lemma 2.8.** Let \(V_{1} \in C_{0}^{\infty}(\mathbb{R}^{m})\). Let \(K_{0}(x,y)\) be continuous for \(x \neq y\) and satisfy \(|K_{0}(x,y)| \leq C|x-y|^{2-m}\). Define the integral operator \(Z_{y}\) for \(y \in \mathbb{R}^{m}\) by

\[
Z_{y}u(x) = \int_{\mathbb{R}^{m}} K_{0}(x+y, z+y)V_{1}(z+y)u(z) \, dz.
\]
Then, $Z_y$ is a compact operator in $\mathcal{X}_\rho$ and is norm continuous with respect to $y \in \mathbb{R}^m$.

PROOF. We prove the lemma for $m \geq 5$. The proof for $m = 3, 4$ may be given by slightly modifying the following argument. Let $S$ be the unit ball of $\mathcal{X}_\rho$. Then for $u \in S$, we have as in (2.14)

\begin{equation}
|Z_y u(x)| \leq C \int_{\mathbb{R}^m} \frac{|V_1(z + y)| \langle z \rangle^\rho dz}{|x - z|^{m-2}|z|^{m-2}}
\end{equation}

where $C_y$ is a constant bounded for bounded $y$. Let $\psi \in C_0^\infty(\mathbb{R}^m)$ be such that $\psi(x) = 1$ for $|x| \geq 2$ and $\psi(x) = 0$ for $|x| \leq 1$. Set, for $\epsilon > 0$, $\psi_\epsilon(x) = \psi(x/\epsilon)$ and let $Z_{y,\epsilon}$ be the integral operator defined by (2.16) with $K_{0\epsilon}(x, y) = \psi_\epsilon(x - y)K_0(x, y)$ in place of $K_0(x, y)$. Because of the estimate (2.17) and the fact that $K_{0\epsilon}(x, y)$ is jointly continuous with respect to $(x, y)$, it can be easily seen via Ascoli-Arzela’s lemma that $Z_{y,\epsilon}$ is a compact operator in $\mathcal{X}_\rho$ and is norm continuous with respect to $y$. On the other hand, for $y$ in a compact subset of $\mathbb{R}^m$, $Z_{y,\epsilon}u(x) = Z_y u(x)$ for $|x| \geq C_0$ and we have for $u \in S$ and $\epsilon \to 0$

\begin{equation}
\sup_{x \in \mathbb{R}^m} |x|^{m-2} |Z_{y,\epsilon}u(x) - Z_y u(x)|
\end{equation}

\begin{equation}
\leq c_m \sup_{|x| \leq C_0} |x|^{m-2} \int_{|x-z|<2\epsilon} \frac{\langle z \rangle^\rho |V_1(z + y)| dz}{|x - z|^{m-2}|z|^{m-2}}
\end{equation}

\begin{equation}
\leq \sup_{|x| \leq C_0} C \int_{|x-z|<2\epsilon} \frac{|x|^{m-2} dz}{|x - z|^{m-2}|z|^{m-2}}
\end{equation}

\begin{equation}
\leq C_\epsilon^2 \sup_{x \in \mathbb{R}^m} \int_{|z|<2/|x|} \frac{|x|^2 dz}{|\hat{x} - z|^{m-2}|z|^{m-2}} \to 0
\end{equation}

uniformly with respect to $y$, where $\hat{x} = x/|x|$. This shows that $Z_{y,\epsilon}$ converges to $Z_y$ in the operator norm of $\mathcal{X}_\rho$ locally uniformly with respect to $y$. Hence $Z_y$ is compact and is norm continuous. \(\square\)

PROOF OF THEOREM 2.5. Decompose $V(x) = V_0(x) + V_1(x)$ in such a way that $\|V_0\|_{(\gamma)} < C_0$ and $V_1 \in C_0^\infty(\mathbb{R}^m)$, where $C_0$ is the constant
appeared in Lemma 2.7. Denote by $K_0(x, y)$ the integral kernel of $K_0 \equiv \lim_{\epsilon \to 0} (H_0 + V_0 + i \epsilon)^{-1}$. In virtue of Lemma 2.7, $K_0(x, y)$ is continuous for $x \neq y$ and satisfies $|K_0(x, y)| \leq C|x - y|^{2-m}$. Thus, by Lemma 2.8, the integral operator $Z_y$ defined in $\mathcal{X}_\rho$ by (2.16) with this $K_0(x, y)$ and $V_1(x)$ is compact and is norm continuous with respect to $y$.

We show that $1 + Z_y$ is an isomorphism of $\mathcal{X}_\rho$. Suppose that $u(x) + Z_y u(x) = 0$, $u \in \mathcal{X}_\rho$. Then $|u(x)|$ is bounded by a constant times the RHS of (2.17) and repeating the similar estimate implies that $u(x)$ is continuous and satisfies $|u(x)| \leq C \langle x \rangle^{2-m}$. (This may also be seen by the elliptic regularity theorem for Schrödinger operators with Kato class potentials, see e.g. [16]). Set $u_y(x) = u(x - y)$. $u_y$ is continuous, $|u_y(x)| \leq \langle x - y \rangle^{2-m}$, and it satisfies the integral equation

$$u_y(x) + \int_{\mathbb{R}^m} K_0(x, z)V_1(z)u_y(z)dz = 0. \tag{2.18}$$

By applying $-\triangle + V_0(x)$ to (2.18), we see $-\triangle u_y(x) + V(x)u_y(x) = 0$. It follows that $u(x) \equiv 0$, since $u_y \in L^2_{-1-\epsilon}(\mathbb{R}^m)$ (or $u_y \in L^2(\mathbb{R}^m)$ if $m \geq 5$), and since we are assuming that zero is not resonance nor eigenvalue of $H = H_0 + V$. Thus $1 + Z_y$ is an isomorphism of $\mathcal{X}_\rho$.

Set $K_{0y}(x) = K_0(x + y, y)$. By the remark after the definition (2.15) of $\mathcal{X}_\rho$, $K_{0y}$ is an $\mathcal{X}_\rho$ valued continuous function. Hence, $K_y = (1 + Z_y)^{-1}K_{0y}$ is well defined and is also an $\mathcal{X}_\rho$ valued continuous function. Set $K(x, y) = K_y(x - y)$. $K(x, y)$ is jointly continuous for $x \neq y$; $|K(x, y)| \leq C_y \langle x - y \rangle^\rho \langle x - y \rangle^{2-m}$ with $C_y$ bounded for bounded $y$; and it satisfies the integral equation

$$K(x, y) = K_0(x, y) - \int_{\mathbb{R}^m} K_0(x, z)V_1(z)K(z, y)dz. \tag{2.19}$$

Note that (2.19) and (2.17) imply that $K(x, y)$ in fact satisfies the estimate $|K(x, y)| \leq C_y \langle x - y \rangle^{2-m}$, where $C_y$ is again bounded for bounded $y$.

We show that $K(x, y)$ is the integral kernel of $R(0)$ and it satisfies the estimate mentioned in the theorem. Denote by $K$ the integral operator with the integral kernel $K(x, y)$. Then, for $u \in C_0^\infty(\mathbb{R})$, $Ku(x)$ is continuous, $|Ku(x)| \leq C \langle x \rangle^{2-m}$ and, in virtue of (2.19), $Ku = K_0u - K_0V_1 Ku$. Subtract $R(0)u = K_0u - K_0V_1R(0)u$ from this equation side by side and write $v = R(0)u - Ku$. Then $v \in L^2_{-1-\epsilon}$, $\epsilon > 0$, and it satisfies $v + K_0V_1v = 0$. Applying $H_0 + V_0$ to both sides of this equation implies $-\triangle v(x) + V(x)v(x) = 0$ and
Applying Theorem 2.5 and Lemma 2.6 and using the assumptions on $wu$ have $K(x, y) = K(y, x)$ and $|K(x, y)| \leq Cx|x - y|^{2-m}$ with $Cx$ bounded for bounded $x$. Going back to (2.19), we conclude $|K(x, y)| \leq C|x - y|^{2-m}$. This completes the proof of Theorem 2.5. $\blacksquare$

Since $K(x, y)$ satisfies $-\Delta_x K(x, y) + V(x)K(x, y) = \delta(x - y)$, we expect from the elliptic regularity that $K(x, y)$ is smooth where $V$ is. We prove the following result.

**Lemma 2.9.** Suppose $V$ is as in Theorem 2.5 and, in addition, $D^\alpha V(x)$ satisfies (2.7) for $|\alpha| \leq \ell$. Let $K(x, y)$ be the integral kernel of $R(0)$. Then, for $y \neq 0$, $K(x, x - y)$ is $C^\ell$ with respect to $x \in \mathbb{R}^m$ and $|D^\alpha_x K(x, x - y)| \leq C_\alpha |y|^{2-m}$, $|\alpha| \leq \ell$.

**Proof.** Let $\tau_h$ be the translation by $h$ and $V_h(x) = V(x + h)$. Then $K(x + h, y + h)$ is the integral kernel of $\tau_h R(0) \tau_h^{-1} = (-\Delta + V_h)^{-1} \equiv R_h(0)$ and the resolvent equation $R_h(0) - R(0) = -R_h(0)(V_h - V)R(0)$ implies that

$$K(x + h, y + h) - K(x, y) = -\int_{\mathbb{R}^m} K(x + h, z + h)(V(z + h) - V(z))K(z, y)dz.$$ 

Hence Theorem 2.5, Lemma 2.6 and the assumption on $DV$ together imply

$$(\partial/\partial h_j)K(x + h, y + h)|_{h=0} = -\int_{\mathbb{R}^m} K(x, z)(\partial V/\partial z_j)(z)K(z, y)dz.$$ 

Repeating this argument, we obtain

$$D^\alpha_h K(x + h, y + h)|_{h=0} = \sum_{|\alpha|} \sum_{\ell=1}^{\alpha_1+\ldots+\alpha_\ell = \alpha} C_{\alpha_1,\ldots,\alpha_\ell} G_{\alpha_1,\ldots,\alpha_\ell}(x, y),$$

where $G_{\alpha_1,\ldots,\alpha_\ell}(x, y)$ is the integral kernel of $R(0)V^{(\alpha_1)}(0)\ldots V^{(\alpha_\ell)}(0)$. Applying Theorem 2.5 and Lemma 2.6 and using the assumptions on $D^\alpha V$ for estimating $G_{\alpha_1,\ldots,\alpha_\ell}(x, y)$, we obtain the lemma immediately. $\blacksquare$

We need the following lemma.

**Lemma 2.10.** Let $1 \leq p, q, r \leq \infty$ satisfy $r^{-1} \geq p^{-1} + q^{-1} - 1$. Then:

1. If $\rho, \sigma < m$ and $\rho + \sigma > m$. Then $\|f \ast g\|_{\ell^\infty_\rho+\sigma-m(L^r)} \leq C\|f\|_{\ell^p_\rho(L^r)}$. 


\[ \|g\|_{\ell^\infty(L^q)}. \]

(2) If \( \rho \) or \( \sigma > m \), then
\[ \|f * g\|_{\ell^\infty} \leq C \|f\|_{\ell^\infty(L^p)} \cdot \|g\|_{\ell^\infty(L^q)}. \]

**Proof.** Take \( \phi \in C_0^\infty(|x| < 1/2) \) such that \( \int \phi(x)dx = 1 \) and set \( f_y(x) = \phi(x - y)f(x) \) and etc. Clearly \( f_y \) is supported by \( y + B(O, 1/2) \), \( f(x) = \int f_y(x)dy \) and we may write
\[ (f * g)(x) = \int (f_y * g_z)(x)dydz. \]

Note that \( f_y * g_z \) is supported by \( y + z + B(O, 1) \). It follows by Young’s inequality that, if \( Q^* \) is the cube of side 4 with center at the origin,
\[ \|f * g\|_{L^r(Q_n)} \leq C \int_{y+z-n \in Q^*} \|f_y\|_{L^p(R^m)} \|g_z\|_{L^q(R^m)} dydz \]
\[ \leq C \|f\|_{\ell^\infty(L^p)} \|g\|_{\ell^\infty(L^q)} \int_{y+z-n \in Q^*} \langle y \rangle^{-\rho} \langle z \rangle^{-\sigma} dydz. \]

Estimating the last integral in a standard fashion, we obtain the lemma. □

The following lemma implies the estimate (1.12) in the introduction.

**Lemma 2.11.** Let \( V \) satisfy (1.1) for \( |\alpha| \leq [(m-2)/2] \) and \( \delta > (m + 3)/2 \). Then:
\[ \int_{R^m} \left\{ \int \langle x \rangle^{2\sigma} |D^\alpha V(x)D_x^\beta K(x, x - y)D^\gamma V(x - y)|^2 dx \right\}^{1/2} dy < \infty, \]
for \( |\alpha + \beta + \gamma| \leq [(m-2)/2] \) and \( \sigma < \delta - 2 \).

**Proof.** In virtue of Lemma 2.9, the left hand side of (2.20) is bounded by a constant times
\[ \int_{R^m} \left\{ \int \langle x \rangle^{2\sigma} |D^\alpha V(x)D^\gamma V(x - y)|^2 dx \right\}^{1/2} dy \]
\[ \frac{dy}{|y|^{m-2}}. \]

We estimate (2.21) by applying Lemma 2.10. We denote the function \( \{\ldots\}^{1/2} \) in (2.21) by \( W_{\alpha\gamma}(y) \). If \( m = 3 \), we have only the case \( \alpha = \beta = \gamma = 0 \). By using Lemma 2.10, (2), we have
\[ W_{00}(y) = \left\{ \int \langle x \rangle^{2\sigma} |V(x)V(x - y)|^2 dx \right\}^{1/2} \in \ell^\infty_{\delta-\sigma}(L^2). \]
Hence, if $\sigma < \delta - 2$, we have (2.21) $\leq \int_{\mathbb{R}^m} (|W_{\alpha \gamma}(y)|/|y|)dy < \infty$.

When $m = 4$ or $= 5$, we only prove (2.20) for the case $|\alpha| = 1$ and $\beta = \gamma = 0$. We may assume $p_0 > m/2$ is close to $m/2$. We have $|V|^2 \in \ell_{2\delta}^\infty (L^{q_0/2})$, $1/q_0 = 1/p_0 - 1/m$, by Sobolev’s lemma. Thus Lemma 2.10 implies $W_{\alpha \gamma} \in \ell_{\delta - \sigma}^\infty (L^r)$, $1/r = 2/p_0 - 1/m - 1/2 < 2/m$, and $\int_{\mathbb{R}^m} (|W_{\alpha \gamma}(y)|/|y|^{m-2})dy < \infty$, if $\sigma < \delta - 2$. The proof for $m \geq 6$ is similar (in fact easier) and we omit the details. $\square$

### 2.3 $L^p$ boundedness of $W_1$

We close this section by recalling the argument in [21] that shows that $W_1$ defined by (1.2):

$$W_1 u(x) \equiv -\frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} R_0(\lambda - i\varepsilon)V R_0(\lambda + i\varepsilon)u(x)d\lambda$$

is bounded in $L^p$. We begin with the following lemma (Lemma 2.3 of [21]), which may be proved by computing the inverse Fourier transform of essentially one dimensional function $\xi \rightarrow (2\pi \xi - \eta^2 + i\varepsilon)^{-1}$.

**Lemma 2.12.** Let $\eta \in \mathbb{R}^m \setminus \{0\}$ and $\tilde{\eta} = \eta/|\eta|$. Then

$$(2.22) \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^m/2} \int_{\mathbb{R}^m} \frac{e^{ix\xi} f(\xi)}{2\eta \xi - \eta^2 + i\varepsilon} d\xi = \frac{1}{2i|\eta|} \int_0^\infty e^{-it|\eta|^2/2} f(x + t\tilde{\eta})dt.$$ 

The following proposition proves that $W_1$ is bounded in $L^p$ under a rather mild condition on $V(x)$. $\Sigma$ is the unit sphere of $\mathbb{R}^m$ and $d\omega$ is its surface element.

**Proposition 2.13.** Set for $t \in \mathbb{R}$ and $\omega \in \Sigma$

$$(2.23) \hat{K}_V(t, \omega) = \frac{i}{2(2\pi)^{m/2}} \int_0^\infty \hat{\nabla}(r\omega) r^{m-2} e^{itr^2/2} dr.$$ 

We write $x\omega = x - 2(x\omega)\omega$ for the reflection of $x$ along the $\omega$-axis. Then:

1. The operator $W_1$ can be expressed as follows:

$$(2.24) W_1 u(x) = \int\int_{2x\omega} \hat{K}_V(t, \omega) u(t\omega + x\omega) dt.$$
2. For any $1 \leq p \leq \infty$, we have

\begin{equation}
\|W_1u\|_{L^p(\mathbb{R}^m)} \leq 2\|\hat{K}_V\|_{L^1([0,\infty) \times \Sigma)}\|u\|_{L^p(\mathbb{R}^m)}.
\end{equation}

3. Let $\sigma > 1/2$ and $\rho > m/2 + \sigma$. Then, there exist constants $C_1, C_2$ such that

\begin{equation}
\|\hat{K}_V\|_{L^1([0,\infty) \times \Sigma)} \leq C_1\|\langle x \rangle^\sigma V\|_{H^{(m-3)/2}} \leq C_2 \sum_{|\alpha| \leq \ell_0} \|D^\alpha V\|_{\ell^\infty_p(L^{p_0})},
\end{equation}

where $p_0, \ell_0$ are as in Theorem 1.2.

**Proof.** We compute the Fourier transform of $W_1u$. Performing the $\lambda$-integration first via the residue theorem, we see that it is equal to

\begin{equation}
\frac{-1}{(2\pi i)^m} \frac{1}{(2\pi)^{m/2}} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \left\{ \int_{\mathbb{R}^m} \frac{\hat{V}(\eta)\hat{u}(\xi - \eta)d\eta}{(\xi^2 - \lambda + i\varepsilon)((\xi - \eta)^2 - \lambda - i\varepsilon)} \right\} d\lambda
\end{equation}

\begin{equation}
= \lim_{\varepsilon \to 0} \frac{-1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \hat{V}(\eta)\hat{u}(\xi - \eta) d\eta.
\end{equation}

We then invert the Fourier transform. Applying (2.22), we deduce

\begin{equation}
W_1u(x) = \frac{-1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \hat{V}(\eta) \left\{ \int_{0}^{\infty} e^{-it|\eta|/2+i\eta(x+t\eta)}u(x+t\eta)dt \right\} d\eta.
\end{equation}

Introducing the polar coordinates $\eta = r\omega, r > 0, \omega \in \Sigma$, and changing the order of integration, we obtain

\begin{equation}
W_1u(x) = \int_{\Sigma} d\omega \int_{0}^{\infty} dt \left\{ \frac{i}{2(2\pi)^m} \int_{0}^{\infty} \hat{V}(r\omega)e^{i(t+2x\omega)r/2}r^{m-2}dr \right\} u(x + t\omega).
\end{equation}

The identity (2.24) follows from this by the change of variable $t \to t - 2(x\omega)$. Observing that $x \to x\omega$ is measure preserving, we apply Minkowski’s inequality to (2.24) and obtain (2.25).

By Parseval-Plancherel formula we have

\begin{equation}
\int_{0}^{\infty} |\hat{K}_V(t,\omega)|^2dt = \frac{1}{2(2\pi)^{m-1}} \int_{0}^{\infty} |\hat{V}(r\omega)|^2r^{2m-4}dr.
\end{equation}
Integrating both sides with respect to $\omega$ over $\Sigma$ gives
\[
\|\widehat{K}_V\|_{L^2([0,\infty)\times\Sigma)}^2 = \frac{1}{2(2\pi)^{m-1}} \int_{\mathbb{R}^m} |\xi|^{m-3} \hat{V}(\xi)^2 \, d\xi \leq C \|V\|_{H^2(m-3/2)}^2.
\]
Similarly we have
\[
\|t\widehat{K}_V\|_{L^2([0,\infty)\times\Sigma)}^2 \leq C \int_{\mathbb{R}^m} |\xi|^{m-3} (|\nabla_\xi \hat{V}(\xi)|^2 + |\xi|^{-2} \hat{V}(\xi)^2) \, d\xi
\leq C \|\langle x \rangle V\|_{H^2(m-3/2)}^2.
\]
Interpolating these two estimates by the complex interpolation method, we deduce that for any $\sigma > 1/2$,
\[
\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)} \leq C_\sigma \|\langle t \rangle^\sigma \widehat{K}_V\|_{L^2([0,\infty)\times\Sigma)} \leq C_\sigma \|\langle x \rangle^\sigma V\|_{H^2(m-3/2)}.
\]
The second inequality of (2.26) is obvious since $p_0 \geq 2$. $\square$

3. Estimate at low energy

In what follows we assume that $V$ satisfies the condition of Theorem 1.2 with $\ell = 0$. In this section, we prove that the low energy part $W_\pm \phi_1(H_0)^2 = \phi_1(H)W_\pm \phi_1(H_0)$ of $W_\pm$ is bounded in $L^p$, where $\phi_1 \in C_0^\infty(R^4)$ is such that $\phi_1(\lambda) = 1$ for $|\lambda| \leq 1$ and $\phi_1(\lambda) = 0$ for $|\lambda| \geq 2$. We prove this for the case $m \geq 4$ is even only. Nevertheless, we state some results for the case $m \geq 3$ is odd as well when we think them of independent interest.

Since $V$ is clearly very short range and $H = H_0 + V$ admits no positive eigenvalues ([2]), all statements in the previous section hold. Moreover, writing $V(x) = A(x)B(x)$ as before, we have the following properties which are all well known in scattering theory (cf. [1], [7], [14]):

1. $AR_0(\lambda \pm i0)B \equiv Q_0^\pm(\lambda) \in B(L^2)$ is uniformly bounded on $[0, \infty)$ and $1 + Q_0^\pm(\lambda)$ has a bounded inverse in $B(L^2)$ for all $\lambda \in [0, \infty)$. We have the resolvent equation (2.12):
\[
R^\pm(\lambda) = R_0^\pm(\lambda) - R_0^\pm(\lambda)B(1 + Q_0^\pm(\lambda))^{-1}AR_0^\pm(\lambda).
\]

2. $AR^\pm(\lambda)B$ are uniformly bounded in $B(L^2)$ and locally H"older continuous on $[0, \infty)$.

3. $A$ and $B$ are $H_0$- as well as $H$-smooth in the sense of Kato:
\[ \sup_{\epsilon > 0} \int_{-\infty}^{\infty} \| AR_0(\lambda \pm i\epsilon)u \|^2 d\lambda \leq C \| u \|^2; \]

(3.30)

\[ \sup_{\epsilon > 0} \int_{0}^{\infty} \| AR(\lambda \pm i\epsilon)u \|^2 d\lambda \leq C \| u \|^2. \]

4. The wave operators \( W_{\pm} \) exist and have the stationary expression (1.2) \( \sim \) (1.3).

In virtue of Proposition 2.13 the \( L^p \) boundedness of \( \phi_1(H)W_{\pm} \phi_1(H_0) \) is equivalent to that of \( W_{2,\text{low}} = \phi_1(H)W_2 \phi_1(H_0) \). We decompose \( W_{2,\text{low}} = W_{2,\text{low}}^{(1)} + W_{2,\text{low}}^{(2)} \) by splitting the resolvent as \( R(\lambda) = \tilde{R}(\lambda) + R(0) \) in the formula (1.3):

(3.31) \( W_{2,\text{low}}^{(1)} u = \phi_1(H) \)

\[ \times \left\{ \frac{1}{2\pi i} \int_{0}^{\infty} R_0^-(\lambda)V R(0) V(R_0^+(\lambda) - R_0^-(!)) d\lambda \right\} \phi_1(H_0)u, \]

(3.32) \( W_{2,\text{low}}^{(2)} u = \phi_1(H) \)

\[ \times \left\{ \frac{1}{2\pi i} \int_{0}^{\infty} R_0^-(\lambda)V \tilde{R}^-(\lambda) V(R_0^+(\lambda) - R_0^-(!)) d\lambda \right\} \phi_1(H_0)u. \]

We prove that \( W_{2,\text{low}}^{(1)} \) and \( W_{2,\text{low}}^{(2)} \) are both bounded in \( L^p \) separately.

We rewrite (3.31) as follows. By using that \( R_0^+(\lambda) = R_0^-(\lambda) \) for \( \lambda \leq 0 \), we extend the region of integration to the whole line and write

(3.33)

\[ (W_{2,\text{low}}^{(1)} u, v) = \frac{1}{2\pi i} \int_{0}^{\infty} (AR(0) B \cdot A(R_0^+(\lambda) - R_0^-(\lambda)) \phi_1(H_0)u, BR_0^+(\lambda) \phi_1(H) v) d\lambda. \]

Here, in virtue of (3.30), \( AR_0^-(\lambda) \phi_1(H_0)u \) and \( BR_0^+(\lambda) \phi_1(H) v \) are boundary values of \( L^2 \)-valued Hardy functions in the lower and upper half planes respectively. Hence they are orthogonal to each other and we obtain

Recall that \( \phi_1(H_0), \phi_1(H) \) are bounded in \( L^p \) as shown in section 2. Denote the integral kernel of \( R(0) \) by \( K(x, y) \), the multiplication with the function
\[ M_y(x) = V(x)K(x, x - y)V(x - y) \] by \( M_y \), and the translation by \( y \in \mathbb{R}^m \) by \( \tau_y \). Then we write \( VR(0)V \) in the form

\[
(3.34) \quad VR(0)Vu(x) = \int_{\mathbb{R}^m} V(x)K(x, x - y)V(x - y)u(x - y)dy = \int_{\mathbb{R}^m} M_y\tau_yu(x)dy,
\]

and inserting (3.34) into (3.33), we obtain

\[
(3.35) \quad (W_{2,\text{low}}^{(1)}u, v) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\mathbb{R}^m} \langle M_y R_0^+(\lambda) \phi_1(H_0) \tau_y u, R_0^+(\lambda) \phi_1(H) v \rangle dyd\lambda.
\]

Here the integral is absolutely convergent with respect to \( dyd\lambda \). Indeed, for \( \sigma > 1/2 \) we have \( \langle x \rangle^\sigma M_y(x) \in H^{(m-3)/2}(\mathbb{R}^m) \) for some \( \sigma > 1/2 \) in virtue of Lemma 2.11 and \( \|M_y\|_{L^{m/2}(\mathbb{R}^m)} \leq C \|\langle x \rangle^\sigma M_y(x)\|_{H^{(m-3)/2}(\mathbb{R}^m)} \) by Sobolev’s lemma. Hence \( |M_y|^{1/2} \) is \( H_0 \)-smooth for every \( y \in \mathbb{R}^m \) ([7]):

\[
\int_{\mathbb{R}} \| |M_y|^{1/2} R_0^+(\lambda)u \|^2 d\lambda \leq C \|\langle x \rangle^\sigma M_y(x)\|_{H^{(m-3)/2}(\mathbb{R}^m)}^2 \|u\|_{L^2}^2
\]

and, thanks to (2.20) we have

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^m} |\langle M_y R_0^+(\lambda) \phi_1(H_0) \tau_y u, R_0^+(\lambda) \phi_1(H) v \rangle| dyd\lambda
\]

\[
\leq C \|\phi_1(H_0)u\|_{L^2} \|\phi_1(H)v\|_{L^2} \int_{\mathbb{R}^m} \|\langle x \rangle^\sigma M_y\|_{H^{(m-3)/2}} dy < \infty.
\]

It follows by changing the order of integration in (3.35) that

\[
(3.36) \quad (W_{2,\text{low}}^{(1)}u, v) = \int_{\mathbb{R}^m} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \langle R_0^-(\lambda) M_y R_0^+(\lambda) \phi_1(H_0) \tau_y u, \phi_1(H) v \rangle d\lambda \right\} dy
\]

and the application of Proposition 2.13 and (2.20) to (3.36) yields, with \( \sigma > 1/2 \) and \( 1/p + 1/q = 1 \) that

\[
|(W_{2,\text{low}}^{(1)}u, v)| \leq C \int_{\mathbb{R}^m} \|\langle x \rangle^\sigma M_y\|_{H^{(m-3)/2}} dy \cdot \|u\|_{L^p} \|v\|_{L^q} \leq C_1 \|u\|_{L^p} \|v\|_{L^q}
\]

Thus, we have proved the following lemma.
Lemma 3.14. $W_{2,\text{low}}^{(1)}$ is bounded in $L^p$ for any $1 \leq p \leq \infty$.

Before starting the proof of the $L^p$ boundedness of $W_{2,\text{low}}^{(2)}$, we record some results about the differentiability of $R^\pm(\lambda)$ that are necessary in what follows. They are simple consequences of the resolvent equation (3.29), Lemma 2.1 and the decay property of the potential $D^\alpha V \in \ell_2^\infty(L^p_{0})$, and we omit the proof.

Lemma 3.15. Let $0 \leq j \leq (m+2)/2$ and $\epsilon > 0$. Then $R^\pm(\lambda)$ is $j$ times differentiable as a $B(L^2_{0+j+1/2+\epsilon},L^2_{-j-1/2-\epsilon})$ valued function of $\lambda \in (0,\infty)$.

Lemma 3.16. Let $2 \leq \rho \leq (m+2)/2$ and $s > \rho+1/2$. Then, for $0 < k < 1$,

$$
(3.37) \quad \| (d/dk)^j \tilde{R}^\pm(k^2) \|_{B(L^2_k,L^2_{-s})} \leq \left\{ \begin{array}{ll} C_j k^{2-j} \langle \log k \rangle, & \text{if } m \geq 4; \\ C_j k^{1-j}, & \text{if } m = 3, \end{array} \right.
$$

for $0 \leq j \leq \rho$.

We show that the integral kernel $W_{2,\text{low}}^{(2)}(x,y)$ of $W_{2,\text{low}}^{(2)}$ satisfies the criterion (1.6). Using the identity $(R_0^+(\lambda) - R_0^-(\lambda))\phi_1(H_0) = (R_0^+(\lambda) - R_0^-(\lambda))\phi_1(\lambda)$ and changing the variable $\lambda = k^2$, we write

$$
(3.38) \quad W_{2,\text{low}}^{(2)} = \frac{1}{\pi i} \int_0^\infty \phi_1(H) R_0^-(k^2) V \tilde{R}^-(k^2) V (R_0^+(k^2) - R_0^-(k^2)) \times \phi_1(H_0) \tilde{\phi}_1(k^2) dk,
$$

where $\tilde{\phi}_1 \in C_0^\infty(\mathbb{R})$ is such that $\tilde{\phi}_1(\lambda) \phi_1(\lambda) = \phi_1(\lambda)$, Hence, if we denote the integral kernels of $R_0^\pm(k^2)\phi_1(H_0)$ and $R_0^\pm(k^2)\phi_1(H)$ respectively by $G_\pm^{(*)}(x,y,k)$ and $G_\pm^{(**)}(x,y,k)$, and if we set $G_\pm^{(*)}(x) = G_\pm^{(**)}(x,y,k)$ and $G_\pm^{(**)}(x) = G_\pm^{(**)}(x,y,k)$, then $W_{2,\text{low}}^{(2)}(x,y)$ is given by $W_{2,\text{low}}^{(2)}(x,y) = W_{2,\text{low}}^{(2),+}(x,y) - W_{2,\text{low}}^{(2),-}(x,y)$, where

$$
(3.39) \quad W_{2,\text{low}}^{(2),\pm}(x,y) = \frac{1}{\pi i} \int_0^\infty \tilde{\phi}(k^2) \langle \tilde{R}^- (k^2) V G_\pm^{(*)}, V G_\pm^{(**)} \rangle dk,
$$

Recall that the integral kernel of $R_0^\pm(k^2)$ is given by $G_\pm(x-y,k)$ (see (2.2)) and that we are assuming $m$ is even. Expanding $(z \pm (it/2))^\nu$ in the
Hankel formula (2.3):

\[
\pm i z^\nu H^{(j)}_{\nu}(z) = \sum_{s=0}^\nu C^{\pm}_{\nu s} e^{\pm is} z^s H^\pm_{\nu s}(z),
\]

\[
H^\pm_{\nu s}(z) = \int_0^\infty e^{-t} t^{2\nu-s-1/2} \left( z \pm \frac{it}{2} \right)^{-1/2} dt.
\]

and introducing \( \phi(x,y) = |x - y| - |x| \), we decompose

\[
G^{\pm, x, k}(y) = e^{\pm ik|x|} \sum_{s=0}^\nu k^s C^{\pm}_{\nu s} \frac{e^{\pm is} \phi(x,y) H^\pm_{\nu s}(k|x - y|)}{|x - y|^{m-2-s}}
\]

\[
\equiv e^{\pm ik|x|} \sum_{s=0}^\nu k^s G^{\pm, x, k, s}(y),
\]

where \( C^{\pm}_{\nu s} \) are constants and the definition of \( G^{\pm, x, k, s}(y) \) should be obvious.

We have obvious inequality \( |\phi(x,y)| \leq |y| \). We decompose \( G^{(*)}(x,y,k) \) and \( G^{(**)}(x,y,k) \) accordingly: Write \( \Phi_0(x,y) \) and \( \Phi(x,y) \) for the kernels of \( \phi(H_0) \) and \( \phi(H) \) respectively, and define

\[
G^{(*)}_{\pm, x, k, s}(y) = \int_{\mathbb{R}^m} e^{\pm ik(|z| - |x|)} G^{\pm, x, k, s}(y) \Phi_0(z,x) dz;
\]

\[
G^{(**)}_{\pm, x, k, s}(y) = \int_{\mathbb{R}^m} e^{\pm ik(|z| - |x|)} G^{\pm, x, k, s}(y) \Phi(z,x) dz.
\]

We have

\[
G^{(*)}_{\pm, x, k}(y) = e^{\pm ik|x|} \sum_{s=0}^\nu k^s G^{(*)}_{\pm, x, k, s}(y),
\]

\[
G^{(**)}_{\pm, x, k}(y) = e^{\pm ik|x|} \sum_{s=0}^\nu k^s G^{(**)}_{\pm, x, k, s}(y),
\]

and inserting (3.43) into (3.39) yields

\[
W^{(2), \pm}_{2, \text{low}}(x,y) = \sum_{s,s' = 0}^\nu \frac{1}{\pi i} \int_0^\infty e^{-ik(|x| + |y|)}
\]

\[
\times \tilde{\phi}_1(k^2) \langle \tilde{R}^{-}(k^2) VG^{(*)}_{\pm, y, k, s}, VG^{(**)}_{+, x, k, s'} \rangle k^{s+s'+1} dk.
\]

We write each summand in the RHS of (3.44)

\[
T^{\pm}_{ss}(x,y) = \int_0^\infty e^{-ik(|x| + |y|)} \tilde{\phi}_1(k^2) L^{s}_{ss'}(x,y,k) k^{s+s'+1} dk,
\]

\[
L^{\pm}_{ss'}(x,y,k) = (1/\pi i) \langle \tilde{R}^{-}(k^2) VG^{(*)}_{\pm, y, k, s}, VG^{(**)}_{+, x, k, s'} \rangle.
\]
Lemma 3.17. Let \( \alpha + \beta = 0, 1, \ldots, (m + 2)/2 \) and \( s = 0, \ldots, (m - 2)/2 \). Then, for some \( \epsilon > 0 \),

\[
\| V D_k^\beta G_{\pm, x, k, s} \|_{L^2_{\alpha + 1 + \epsilon}} \leq \begin{cases} C \langle x \rangle^{-m+s+3/2} k^{-1/2-\beta}, & \text{if } m \text{ is even;} \\ C \langle x \rangle^{-m+2+s}, & \text{if } m \text{ is odd,} \end{cases}
\]

for \( 0 < k \leq 2 \). The estimate (3.47) remains true if \( G_{\pm, x, k, s}^{(*)} \) is replaced by \( G_{\pm, x, k, s}^{(**)} \).

Proof. We prove only the case \( m \) is even. We have \( |k| |x| (|k| + (it/2))^{-1}| \leq 1 \) and

\[
|D_k^\beta H_{\pm s}^\beta (k|x|)| \leq C |x|^\beta \left| \int_0^\infty e^{-t} t^{2\nu-1/2} (k|x| + (it/2))^{-1/2-\beta} dt \right| 
\leq C |x|^\beta (k|x|)^{-1/2-\beta} = C k^{-1/2-\beta} |x|^{-1/2}
\]

It follows that \( |D_k^\beta G_{\pm, x, k, s}^\beta (y)| \leq C k^{-1/2-\beta} |x - y|^{3/2-m+s} \langle y \rangle^\beta \). On the other hand we know from Lemma 2.4 that \( |\Phi_0(z, x)| \leq C N \langle z - x \rangle^{-N} \) for any \( N \). Using these, we deduce from (3.42) that

\[
|D_k^\beta G_{\pm, x, k, s}^\beta (y)| \leq C k^{-1/2-\beta} \langle x - y \rangle^{3/2-m+s} \langle y \rangle^\beta.
\]

Since \( \| V(y) \langle y \rangle^\beta \langle y \rangle^{\alpha+1+\epsilon} \|_{L^2(Q_n)} \leq C (n)^{\alpha+\beta+1+\epsilon-\delta} \) and \( \delta - (\alpha + \beta + 1 + \epsilon) > m - 1 \) for sufficiently small \( \epsilon > 0 \), the estimate (3.47) for \( G_{\pm, x, k, s}^{(*')} \) follows.

The proof for \( G_{\pm, x, k, s}^{(**')} \) is similar. \( \Box \)

Applying Lemma 2.1 and Lemma 3.17 with \( \beta = 0 \), we obtain that

\[
|L_{ss'}^\pm (x, y, k)| \leq C k^{-1} \langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2}
\]

and by integration

\[
|T_{ss'}^\pm (x, y)| \leq C \langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2}.
\]
For improving the decay estimate of (3.48), we apply integrations by parts with respect to the variable $k \mu_{ss'} = \max\{s,s'\} + 2$ times in (3.45). A computation with Leibniz’ formula shows that

$$D_k^{\mu_{ss'}} \tilde{\phi}(k^2) k^{s+s'+1} L_{ss'}^\pm(x,y,k)$$

(3.49)

$$= \sum_{\alpha+\beta+\gamma=\mu_{ss'}} C_{\alpha\beta\gamma} \langle D_k^\alpha (\tilde{\phi}(k^2) k^{s+s'+1} \tilde{R}^-(k^2)) V D_k^\beta G_{\pm,y,k,s}^{(s)} V D_k^\gamma G_{+,x,k,s'}^{(**)}(x,y,k,s') \rangle$$

and applying Lemma 3.17 and Lemma 3.16, we see that each summand in (3.49) is bounded in modulus by a constant times

$$k^{s+s'+3-\alpha} \langle \log k \rangle k^{1/2-\beta} \langle y \rangle^{-m+s+3/2} k^{-1/2-\gamma} \langle x \rangle^{-m+s'+3/2} \leq C \langle \log k \rangle \langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2}, \quad 0 \leq k \leq 2. \quad (3.50)$$

It follows that no boundary terms appear in the following integration by parts:

$$T_{ss'}^\pm(x,y) = \int_0^\infty \frac{(-D_k)^{\mu_{ss'}} (e^{-ik(|x|\mp|y|)})}{(|x| \mp |y|)^{\mu_{ss'}}} \tilde{\phi}(k^2) L_{ss'}^\pm(x,y,k) k^{s+s'+1} dk$$

$$= \frac{1}{(|x| \mp |y|)^{\mu_{ss'}}} \int_0^\infty e^{-ik(|x|\mp|y|)} D_k^{\mu_{ss'}} (\tilde{\phi}(k^2) L_{ss'}^\pm(x,y,k) k^{s+s'+1}) dk$$

and, in virtue of (3.49) $\sim$ (3.50),

$$|T_{ss'}^\pm(x,y)| \leq C_{s,s'} \langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2} |x| \mp |y|^{-\mu_{ss'}} \quad (|x| \mp |y|)$$

Combining this with (3.48) and summing up for $0 \leq s,s' \leq \nu = (m-2)/2$, we obtain

$$|W_{2,low}^{(2),\pm}(x,y)| \leq \sum_{s,s'=0}^\nu C_{s,s'} \langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2} |x| \mp |y| \quad (|x| \mp |y|)$$

(3.51)

Now we can complete the proof of the following

**Lemma 3.18.** The functions $W_{2,low}^{(2),\pm}(x,y)$ satisfy the estimates (1.6) and the operator $W_{2,low}^{(2)}$ is bounded in $L^p$ for any $1 \leq p \leq \infty$. 
**Proof.** We integrate (3.51) with respect to the variable $x$ by using the polar coordinates: The $(s, s')$-summand in the RHS produces a constant times
\[
\int_{\mathbb{R}^m} \frac{\langle x \rangle^{-m+s'+3/2} \langle y \rangle^{-m+s+3/2}}{\langle |x| \pm |y| \rangle^{ss'}} \, dx
\]
(3.52)
\[
\leq C \int_0^\infty \frac{\langle r \rangle s' + 1/2}{\langle |r| \rangle^m \langle |y| \rangle^{m-s-3/2}} \, dr
\]
Here $s' + 1/2 \leq m - s - 3/2$, since $s + s' \leq m - 2$, and the sup$_{y \in \mathbb{R}^m}$ of the RHS is finite. Hence, \[
\sup_{y \in \mathbb{R}^m} \int_{\mathbb{R}^m} \left| W_{2, \text{low}}^\pm (x, y) \right| \, dx < \infty.
\]
We may likewise prove the other relation of (1.6) and the lemma follows. $\square$

4. **Estimate at high energy**

In this section we prove that the high energy part $\phi_2(H)W_2\phi_2(H_0)u$ of $W_2$ is also bounded in $L^p$. Recall that $W_2$ is given by (1.3):
\[
W_2u = \frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda)V R^- (\lambda) V \{ R_0^+(\lambda) - R_0^- (\lambda) \} u d\lambda
\]
and that $\phi_2 \in C^\infty (\mathbb{R})$ is such that $\phi_2 (\lambda) = 1$ for $\lambda \geq 2$ and $\phi_2 (\lambda) = 0$ for $\lambda \leq 1$. As the argument in this section is very much similar to that of the previous section as well as of section 4 of [21], we shall be rather sketchy here.

Expand $R^-(\lambda)$ via the repeated use of the resolvent equation (3.29):
\[
R^+(\lambda) = \sum_{n=0}^{2N-1} (-1)^n R_0^- (\lambda) (V R_0^- (\lambda))^n + \left( R_0^- (\lambda) V \right)^N R^+(\lambda) (V R_0^- (\lambda))^N,
\]
and decompose $W_2 = \sum_{n=2}^{2N+2} (-1)^n W^{(n)}$ accordingly, where $W^{(n)}$ is given by
\[
W^{(n)}u = \frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda) (V R_0^- (\lambda))^n V \{ R_0^+(\lambda) - R_0^- (\lambda) \} u d\lambda,
\]
for $n = 2, \ldots, 2N + 1$;
\[
W^{(2N+2)}u = \frac{1}{2\pi i} \int_0^\infty R_0^- (\lambda) V F_N (\lambda) V \{ R_0^+(\lambda) - R_0^- (\lambda) \} u d\lambda.
\]
Here we wrote $F_N(\lambda) = (R_0^-(\lambda)V)^N R^-(\lambda)(VR_0^-(\lambda))^N$. It is shown in section 2 of [21] by repeated application of the argument similar to the one used in the proof of Proposition 2.13 that $W^{(n)}u$, $n = 2, \ldots, 2N + 1$, has the following expression: Set for $s_1, \ldots, s_n \in \mathbb{R}^1$ and $\omega_1, \ldots, \omega_n \in \Sigma$, $\Sigma$ being the unit sphere of $\mathbb{R}^m$,

$$K_n(s_1, \ldots, s_n, \omega_1, \ldots, \omega_n) = C^n(s_1 \cdots s_n)^{m-2} \prod_{j=1}^{n} \hat{V}(s_j \omega_j - s_{j-1} \omega_{j-1}),$$

where $C$ is an absolute constant, whose precise value is not important here, and $s_j \omega_j = 0$ if $j = 0$; and denote its “Fourier transform” with respect to the radial variables $(s_1, \ldots, s_n)$ by

$$\hat{K}_n(t_1, \ldots, t_n, \omega_1, \ldots, \omega_n) = \int_{[0, \infty)^n} e^{i \sum_{j=1}^{n} t_j s_j / 2} K_n(s_1, \ldots, s_n, \omega_1, \ldots, \omega_n) ds_1 \cdots ds_n.$$

Then $W^{(n)}u$, $n = 2, \ldots, 2N + 1$, can be written in the form

$$W^{(n)}u(x) = \int_{[0, \infty)^{n-1} \times I \times \Sigma^n} \hat{K}_n(t_1, \ldots, t_{n-1}, \tau, \omega_1, \ldots, \omega_n) u(x_{\omega_n} + \rho) dt_1 \cdots dt_{n-1} d\tau d\omega_1 \cdots d\omega_n$$

where $I = (2x \cdot \omega_n, \infty)$ is the range of the integration by the variable $\tau$, $x_{\omega_n} = x - 2(\omega_n \cdot x)\omega_n$, is the reflection of $x$ along $\omega_n$, and $\rho = t_1 \omega_1 + \cdots + t_{n-1} \omega_{n-1} + \tau \omega_n$. Since $x \rightarrow x_{\omega_n}$ is measure preserving and $\rho$ is independent of $x$, Minkowski’s inequality implies as in section 2 that

$$\|W^{(n)}u\|_{L^p} \leq 2 \|\hat{K}_n\|_{L^1([0, \infty)^{n} \times \Sigma^n)} \|f\|_{L^p}, \quad 1 \leq p \leq \infty. \tag{4.53}$$

We showed in Lemma 2.5 of [21] that for any $\sigma > 1$

$$\|\hat{K}_n\|_{L^1([0, \infty)^{n} \times \Sigma^n)} \leq C^n \|\mathcal{F}(\langle x \rangle^\sigma V)\|_{L^{m\sigma}}.$$
for any $\sigma$ and this holds obviously if $m = 3$. On the other hand it is clearly possible to find $1 < \sigma < \delta$ such that

$$\| \langle x \rangle^{\sigma} V \|_{H^p} \leq C_1 \sum_{|\alpha| \leq \ell_0} \| D^{\alpha} V \|_{L^\infty(L^0)}.$$  

This proves that $W^{(n)}$ hence $\phi_2(H) W^{(n)} \phi_2(H_0)$ are bounded in $L^p$ if $n = 2, \ldots, 2N + 1$.

For completing the proof of Theorem 1.2, it remains only to prove that the operator $\phi_2(H) W^{(2N+2)} \phi_2(H_0)$ is bounded in $L^p$. We write it in the following form:

$$\phi_2(H) \frac{1}{2\pi i} \left( \int_0^\infty R^+_0(\lambda) V F_N(\lambda) V \{ R^+_0(\lambda) - R^-_0(\lambda) \} \tilde{\phi}_2(\lambda) d\lambda \right) \phi_2(H_0).$$

Here $\tilde{\phi}_2 \in C^\infty(\mathbb{R})$ is such that $\tilde{\phi}_2(\lambda) \phi_2(\lambda) = \phi_2(\lambda)$ and $\tilde{\phi}_2(\lambda) = 0$ for $\lambda \leq 1/2$. We need only prove that the operator inside the parenthesis

$$T_\pm = \int_0^\infty R^-_0(k^2) V F_N(k^2) V R^-_0(k^2) \tilde{\phi}_2(k^2) k dk$$

is bounded in $L^p$. The integral kernel $T_\pm(x,y)$ of $T_\pm$ can be computed as in the previous section and are given by

$$T_\pm(x,y) = \int_0^\infty (F_N(k^2) V G_{\pm,y,k}, V G_{\pm,x,k}) \tilde{\phi}_2(k^2) k dk$$

$$= \int_0^\infty e^{-ik(|x|+|y|)} (F_N(k^2) V \tilde{G}_{\pm,y,k}, V \tilde{G}_{\pm,x,k}) \tilde{\phi}_2(k^2) k dk,$$

where we wrote as in (3.41):

$$G_{\pm,x,k}(y) = e^{\pm ik|x|} \sum_{s=0}^\nu k^s G_{\pm,x,k,s}(y) = e^{\pm ik|x|} \tilde{G}_{\pm,x,k}(y).$$

Here, as can be easily see from (2.2) and (2.3), we have for $k \geq 1/4$:

$$|D^\rho_k \tilde{G}_{\pm,x,k}(y)| \leq C_\rho \langle y \rangle^{\rho} |x - y|^{2-m(1+k|x-y|)/(m-3)/2}.$$

Using Lemma 2.1 and Lemma 2.2 for the mapping property and the decay of the resolvent in the $k$ variable, we obtain as in section 4 of [21] that, for sufficiently large $N$,

$$|\tilde{\phi}_2(k^2) (F_N(k^2) V G_{\pm,y,k}, V G_{\pm,x,k})| \leq C \langle k \rangle^{-3} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2}.$$
Integrating with respect to the variable $k$ gives

\begin{equation}
|T_\pm(x, y)| \leq C \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2}.
\end{equation}

which is, however, is not sufficient for $T_\pm(x, y)$ to satisfy the criterion (1.6). For proving that $T_\pm(x, y)$ enjoys better decay property, we perform integrations by parts $\mu = (m + 2)/2$ times in (4.54) as in the previous section:

\begin{equation}
T_\pm(x, y) = \int_0^\infty (|y| + |x|)^{-\mu} (D_k^\mu e^{-ik(|x|\pm|y|)})
\cdot (F_N(k^2) V \tilde{G}_{\pm, y, k, +, x, k}) \tilde{\phi}(k^2) dk
= \sum_{\alpha+\beta+\gamma+\delta=\mu} \int_0^\infty e^{-ik(|x|-|y|)}
(D_\alpha^\alpha F_N(k^2) V D_\beta^\beta \tilde{G}_{\pm, y, k, +, x, k})
\times (D_\gamma^\gamma \tilde{G}_{\pm, y, k, +, x, k}) (D_\delta^\delta \tilde{\phi}(k^2) ) dk.
\end{equation}

Note that we do not have to worry about singularities at $k = 0$ because $\tilde{\phi}(k^2) = 0$ for $0 \leq k \leq 1/4$. By using again Lemma 2.1 and Lemma 2.2, we see that

\begin{equation}
|(D_\alpha^\alpha F_N(k^2) V D_\beta^\beta \tilde{G}_{\pm, y, k, +, x, k})|
\leq C \langle k \rangle^{-3} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2}.
\end{equation}

Thus applying (4.59) to (4.58), and combining the result with (4.57), we obtain

\begin{equation}
|T_\pm(x, y)| \leq C \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2} \langle |x| \mp |y| \rangle^{-(m+2)/2}.
\end{equation}

Thus the estimation as in the final paragraph of section 3 implies that $T_\pm(x, y)$ satisfies (1.6). Thus $\phi_2(H) W^{(2N+2)} \phi_2(H_0)$ is also bounded in $L^p$. This completes the proof of Theorem 1.2.

References


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