A general functional characterization of the microlocal singularities

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1. Introduction

As a generalization of a result of [7], we proved in [1] a functional characterization of the analytic wave front set of an hyperfunction. The main result of [1] is the following.

Denote by \( A(\mathbb{R}^n) \) the set of analytic functions on \( \mathbb{R}^n \). Let \( u \in A'(\mathbb{R}^n \times \mathbb{R}^p) \) be an analytic functional and consider

\[
Sg(x) = \int u(x, y)g(y) \, dy
\]

for \( g \) analytic in a neighbourhood of the projection of \( \text{supp}(u) \) on \( \mathbb{R}^p \). The result states that if \( ((x_0, \xi_0), (y_0, 0)) \notin WF_a(Sg) \) for every \( g \in A(\mathbb{R}^p) \), then \( ((x_0, y_0), (\xi_0, 0)) \notin WF_a(u) \) for every \( y \in \mathbb{R}^p \). It characterizes special points of \( \text{WF}_a(u) \) using the operator \( S \). In this paper, we prove a similar result concerning any point of \( \text{WF}_a(u) \).

Denote by \( B(\mathbb{R}^n) \) the set of all hyperfunctions on \( \mathbb{R}^n \). It is known, [4], [9], that if \( f, g \in B(\mathbb{R}^n) \) and if \( (x, \xi) \in WF_a(f) \) implies \( (x, -\xi) \notin WF_a(g) \), then the product \( fg \) is well defined in \( B(\mathbb{R}^n) \). If in addition \( \text{supp}(f) \cap \text{supp}(g) \) is compact, the product \( fg \) belongs to \( A'(\mathbb{R}^n) \), hence the pairing

\[
\langle f, g \rangle = \int f(x)g(x) \, dx
\]

is also well defined.

Let \( u \in A'(\mathbb{R}^n \times \mathbb{R}^p) \) and define as above

\[
^tSf(y) = \int u(x, y)f(x) \, dx , \quad f \in A(\mathbb{R}^n).
\]

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Let \((y_0, \eta_0) \in \mathbb{R}^p \times \mathbb{R}^p \setminus \{0\}\) and assume that \((x, y_0, 0, -\eta_0) \notin WF_a(u)\) for every \(x \in \mathbb{R}^n\). It follows from the composition properties of the analytic wave front set that \((y_0, -\eta_0) \notin WF_a(\mathcal{I}(Sf))\) for any \(f \in \mathcal{A}(\mathbb{R}^n)\), \([4],[9]\).
Hence, using the formula
\[
(Sg)(f) = \langle g, \mathcal{I}(Sf) \rangle,
\]
the operator \(S\) can be extended to any \(g \in \mathcal{B}(\mathbb{R}^p)\) such that \(WF_a(g) \subset \{(y_0, t\eta_0): t > 0\}\).

In the same way, the operator \(S\) can also be extended to all \(g \in \mathcal{B}(\mathbb{R}^p)\)
such that \((x, y_0, 0, -\eta) \notin WF_a(u)\) for all \(x \in \mathbb{R}^n\) and \((y, \eta) \in WF_a(g)\). Theorem 3 gives an explicit formula for this extension.

In this paper we prove the following result.

**Theorem 1.** Let \(u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)\) and \(y_0 \in \mathbb{R}^p, \eta_0 \in \mathbb{R}^p\) such that
\[
(x, y_0, 0, -\eta_0) \notin WF_a(u)
\]
for every \(x \in \mathbb{R}^n\). If \(x_0 \in \mathbb{R}^n, \xi_0 \in \mathbb{R}^n\) and \((x_0, \xi_0) \notin WF_a(Sg)\) for every \(g \in \mathcal{B}(\mathbb{R}^p)\) satisfying
\[
WF_a g \subset \{(y_0, t\eta_0): t > 0\}
\]
then \((x_0, y_0, \xi_0, -\eta_0) \notin WF_a(u)\).

If \(\eta_0 = 0\), this is the result of [1]. If the hypotheses are satisfied for \(\xi_0\) and \(\eta_0\), they are also satisfied for \(t\xi_0\) and \(s\eta_0\) for every \(s, t \geq 0\). Hence, we also have \((x_0, y_0, t\xi_0, -s\eta_0) \notin WF_a(u)\) for every \(s, t \geq 0\).

Conversely, if this last condition is satisfied, the composition properties of the analytic wave front set show that \((x_0, \xi_0) \notin WF_a(Sg)\) for every \(g \in \mathcal{B}(\mathbb{R}^p)\) satisfying \(WF_a g \subset \{(y_0, t\eta_0): t > 0\}\).

In the proof of theorem 1, we use the closed graph theorem from a Fréchet space into a strictly webbed space as in [1].

2. **Extension of the operator**

Let \(u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)\). If \(g\) is analytic in a neighbourhood of the projection of \(\text{supp}(u)\) on \(\mathbb{R}^p\) then
\[
Sg(x) = \int u(x, y)g(y) \, dy
\]
is the element of $A'(\mathbb{R}^n)$ defined by $(Sg)(f) = u(f \otimes g)$ for every $f \in A(\mathbb{R}^n)$.

Using the FBI transform, we give here an explicit formula for the extension of $S$ to all $g \in B(\mathbb{R}^p)$ such that $(x, y, 0, -\eta) \notin WF_a(u)$ for all $x \in \mathbb{R}^n$ and $(y, \eta) \in WF_a(g)$. It will be needed in the proof of theorem 1.

Let us first recall some definitions. If $u \in A'(\mathbb{R}^n)$, the FBI transform of $u$ is the family of holomorphic functions

$$Tu(z, \lambda) = u_{(x)}(e^{i\lambda \varphi(x, z)}), \quad z \in \mathbb{C}^n, \quad \lambda > 0,$$

where $\varphi$ is the quadratic polynomial $\varphi(z, x) = \frac{i}{2}(z - x)^2$. Using the continuity of $u$ as a linear functional, it is easily seen that if $\text{supp}(u) \subset \{x \in \mathbb{R}^n : |x| < a\}$, then for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|Tu(z, \lambda)| \leq C_\varepsilon e^{\frac{\lambda}{2}((3|z|+\varepsilon)^2-|\mathbb{R}|a^2)}$$

if $z \in \mathbb{C}^n$ and $\lambda > 0$. See [5] or [10] for more details.

The analytic wave front set of an hyperfunction $u$ in an open subset $\Omega$ of $\mathbb{R}^n$ is the subset $WF_a(u)$ defined by the condition $(x_0, \xi_0) \notin WF_a(u)$ if and only if there are $C, \varepsilon, r > 0$ and $v \in A'(\mathbb{R}^n)$ that is equal to $u$ near $x_0$ such that

$$|Tv(z, \lambda)| \leq Ce^{\frac{\lambda}{2}(3|z|^2-\varepsilon)}$$

if $|z - (x_0 - i\xi_0)| < r$ and $\lambda > 0$.

**Lemma 2.** Let $u \in A'(\mathbb{R}^n)$, $\rho > 0$ and $f \in \mathcal{O}(\{z \in \mathbb{C}^n : |\mathbb{R}| < \rho\})$. Assume that there are constants $C_k > 0$ such that $|f(z)| \leq C_k(1 + |z|)^{-k}$. Then

$$u(f) = \frac{2^{-n-1}}{\pi^{3n/2}} \int_{\mathbb{R}^n} \frac{|\xi|^{n/2}e^{-|\xi|}}{\xi} \left( \int_{\mathbb{R}^n} Tu(x - i\frac{\xi}{|\xi|}, |\xi|)(1 + i\frac{\xi.D_x}{|\xi|^2}) Tf(x + i\frac{\xi}{|\xi|}, |\xi|) \right) dx.$$

The same formula holds with the differential operator $1 - i\xi.D_x/|\xi|^2$ acting on $Tu$ and no derivatives on $Tf$.

**Proof.** Using the definition of the FBI transform, we get

$$\int_{\mathbb{R}^n} Tu(x - i\frac{\xi}{|\xi|}, |\xi|)(1 + i\frac{\xi.D_x}{|\xi|^2}) Tf(x + i\frac{\xi}{|\xi|}, |\xi|) dx$$
for every integral is exponentially decreasing with respect to $\xi$
functional supported by $u$
complex shift
For $s > 158 F.$ Bastin for all $x$
compact set containing the support of
that is analytic near $f$
$\pi(−3f) = (2^n/n/3$.
Theorem
We can now give the general definition of the operator $S$
Then the map $f \mapsto (Sg)(f)$ is independent of $g_0$ and is an analytic functional supported by $K$. Moreover, if $g$ is analytic near $M$ then $(Sg)(f) = u(f \otimes g)$. For $s > 0$ small, the integration path $w \mapsto w + is\xi/|\xi|$ shows that the inner integral is exponentially decreasing with respect to $\xi$. Hence, using the complex shift $\xi \to -i|\xi|(y - w)/4$, we get

$$\frac{2^{-n-1}}{\pi^{3n/2}} \int_{\mathbb{R}^n} |\xi|^{n/2}e^{-|\xi|} d\xi \int_{\mathbb{R}^n} Tu(x - i\frac{\xi}{|\xi|}, |\xi|)(1 + i\frac{\xi D_x}{|\xi|^2}) T f(x + i\frac{\xi}{|\xi|}, |\xi|) dx.$$

$$= (2\pi)^{-n} u(y) \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} f(w)e^{-i(y-w)\cdot(\xi - \frac{i|\xi|}{4}(y-w))}(1 - \frac{i(y-w)\cdot\xi}{4|\xi|}) dw)$$

$$= (2\pi)^{-n} u(y) ((\mathcal{F}^-(\mathcal{F}^+f)(y)) = u(f)$$

where $\mathcal{F}^\pm$ is the Fourier transform. $\square$

We can now give the general definition of the operator $S$.

**Theorem 3.** Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$, $g \in \mathcal{B}(\mathbb{R}^p)$ and let $K \times M$ be a compact set containing the support of $u$. Assume that $(x, y, 0, -\eta) \notin WF_a(u)$ for all $x \in \mathbb{R}^n$, $(y, \eta) \in WF_a(g)$ and consider $(Sg)(f)$ given by

$$\frac{2^{-p-1}}{\pi^{3p/2}} \int_{\mathbb{R}^p} |\eta|^{p/2}e^{-|\eta|} d\eta$$

$$\cdot \int_{\mathbb{R}^p} Tg_0(y - i\frac{\eta}{|\eta|}, |\eta|)(1 + i\frac{\eta D_y}{|\eta|^2}) T((Sf)(y + i\frac{\eta}{|\eta|}, |\eta|)) dy$$

for every $g_0 \in \mathcal{A}'(\mathbb{R}^p)$ that is equal to $g$ in a neighbourhood of $M$ and every $f$ that is analytic near $K$.

Then the map $f \mapsto (Sg)(f)$ is independent of $g_0$ and is an analytic functional supported by $K$. Moreover, if $g$ is analytic near $M$ then $(Sg)(f) = u(f \otimes g)$. 
Proof. We first prove that the integral converges and defines an analytic functional supported by $K$. For every $\rho > 0$, let $K_\rho = \{ z \in \mathbb{C}^n : d(z, K) < \rho \}$.

Choose a compact set $V$ such that $g_0 = g$ near $V$ and $\text{supp}(u)$ is in the interior of $\mathbb{R}^n \times V$. For every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|Tg_0(y - i\frac{\eta}{|\eta|}, |\eta|)| \leq C_\varepsilon e^{\frac{|\eta|}{2}(1+\varepsilon)}.$$  

On the other hand, by the choice of $V$ there are constants $\delta, C_\rho > 0$ such that

$$|(1 + i\frac{\eta}{|\eta|} \cdot D_y)T(t^*f)(y + i\frac{\eta}{|\eta|}, |\eta|)| \leq C_\rho \|f\|_{K_\rho} e^{\frac{|\eta|}{2}(1-\delta(1+|y|^2))}$$

for every $y \notin V$ and $\rho > 0$. Since

$$\int_{\mathbb{R}^p} |\eta|^{p/2} d\eta \int_{\mathbb{R}^p} e^{-\delta(1+|y|^2)}|\eta|(1 + |y|) dy < \infty,$$

for every $\delta > 0$, the integral extended to $\mathbb{R}^p \setminus V \times \mathbb{R}^p$ defines an analytic functional supported by $K$.

It remains to estimate the integral in a conic neighbourhood of every point $(y_0, \eta_0) \in V \times \mathbb{R}^p \setminus \{0\}$.

Assume that $(y_0, \eta_0) \in WF_a(g)$. Then $(x, y_0, 0, -\eta_0) \notin WF_a(u)$ for every $x$ and from the composition properties of the analytic wave front set, we have $(y_0, -\eta_0) \notin WF_a(t^*f)$. As in [1], we can strengthen this property. For every $m \in \mathbb{N}_0$, let

$$E_m = \{ v \in \mathcal{A}'(M) : q_m(v) < \infty \}$$

with

$$q_m(v) = \sup_{|w-w_0| \leq 1/m, \lambda > 0} e^{-\frac{\lambda}{2} |Iw|^2 + \frac{\lambda}{m} |Tw(w, \lambda)|}$$

and $w_0 = y_0 + i\eta_0$. Endowed with the Fréchet semi-norms of $\mathcal{A}'(M)$ and with $q_m$, the linear space $E_m$ is a Fréchet space. It follows that the subspace $E = \cup_mE_m$ of $\mathcal{A}'(M)$ has a strict web in which the sets with one index are the $E_m$. For every $\rho > 0$, $t^*S$ maps $\mathcal{O}(K_\rho)$ in $E$ and has a sequentially closed graph. Two applications of the localization theorem (see [2] or theorem 6 of [1]), show that there are constants $C, \delta > 0$ and $\rho' \in ]0, \rho[$ such that

$$|T(t^*f)(w, \lambda)| \leq C\|f\|_{K_\rho'} e^{\frac{\lambda}{2}(|Iw|^2-\delta)}$$
for every $f \in \mathcal{O}(K\rho)$, $\lambda > 0$ and $|w - w_0| < \delta$. Using the Cauchy’s inequalities, we get

$$|(1 + i \frac{\eta D_y}{|\eta|^2}) T(\mathcal{S} f)(y + i \frac{\eta}{|\eta|}, |\eta|)| \leq \frac{C'}{|\eta|} ||f||_{K\rho} e^{\frac{|\eta|}{2}(1-\delta/2)}$$

in a conic neighbourhood of $(y_0, \eta_0)$. Since

$$|Tg_0(y - i \frac{\eta}{|\eta|}, |\eta|)| \leq C_\varepsilon e^{\frac{|\eta|}{2}(1+\varepsilon)}$$

we conclude in a neighbourhood of $(y_0, \eta_0)$.

In the same way, if $(y_0, \eta_0) \notin WF_a(g)$ then $Tg_0$ is exponentially decreasing in a conic neighbourhood of this point and the other factor can be estimated by continuity.

Let us show that the definition of $Sg$ is independent of $g_0$. We have to show that the right hand side is equal to 0 if supp$(g_0) \cap M = \emptyset$. If $\varepsilon > 0$, consider

$$g_\varepsilon(x) = (2\pi \varepsilon)^{-p/2} g_0(y) (e^{-\frac{\pi}{16} |x-y|^2}).$$

By lemma 2, we get

$$u(f \otimes g_\varepsilon) = (\mathcal{S} f)(g_\varepsilon) = \frac{2^{-p-1}}{\pi^{3p/2}} \int_{\mathbb{R}^p} |\eta|^{p/2} e^{-|\eta|} d\eta \cdot \int_{\mathbb{R}^p} Tg_\varepsilon(y - i \frac{\eta}{|\eta|}, |\eta|) (1 + i \frac{\eta D_y}{|\eta|^2}) T(\mathcal{S} f)(y + i \frac{\eta}{|\eta|}, |\eta|) dx.$$ 

This expression converges to 0 with $\varepsilon$ since $f \otimes g_\varepsilon$ uniformly converges to 0 in a complex neighbourhood of supp$(u)$. Let us show that the integral converges to the right hand side in the definition. Choose a compact neighbourhood $W$ of $M$ such that supp$(g_0) \cap W = \emptyset$. Since

$$Tg_\varepsilon(z, \lambda) = (1 + \varepsilon \lambda)^{-p/2} Tg_0(z, \frac{\lambda}{1 + \varepsilon \lambda}),$$

we have the estimation

$$|Tg_\varepsilon(y - i \frac{\eta}{|\eta|}, |\eta|)| \leq C_\delta e^{\frac{|\eta|}{2}(1+\delta)}.$$
There are constants $C, \rho > 0$ such that
\[
|(1 + i \frac{\eta \cdot D_y}{|\eta|^2}) T(tSf)(y + i \frac{\eta}{|\eta|}, |\eta|)| \leq C(1 + |y|) e^{\frac{|\eta|}{2} (1 - \rho (1 + |y|^2))}.
\]
if $y \notin W$. So we can conclude in this case. In the same way, for the integral on $W$, we use the estimates
\[
|Tg_\varepsilon(y - i \frac{\eta}{|\eta|}, |\eta|)| \leq C \varepsilon \frac{|\eta|}{2(1 + \varepsilon |\eta|)} (1 - \rho) \leq C \varepsilon \frac{|\eta|}{2(1 - \delta)}
\]
and
\[
|(1 + i \frac{\eta \cdot D_y}{|\eta|^2}) T(tSf)(y + i \frac{\eta}{|\eta|}, |\eta|)| \leq C_\delta (1 + |y|) e^{\frac{|\eta|}{2} (1 + \delta)}.
\]

Finally, assume that $g$ is analytic in a neighbourhood of $M$ and apply lemma 2 to $(tSf)(g_\varepsilon)$ with
\[
g_\varepsilon(x) = (2\pi\varepsilon)^{-p/2} \int_\omega g(y) e^{-\frac{1}{4\varepsilon} |x - y|^2} dy
\]
where $\omega$ is a relatively compact open set such that $M \subset \omega$ and $g$ is analytic near $\varpi$. The function that is equal to 1 in $\omega$ and 0 outside is denoted by $\chi_\omega$. Using a complex deformation to evaluate the integral over $\omega$, we obtain the estimate
\[
|Tg_\varepsilon(y - i \frac{\eta}{|\eta|}, |\eta|)| = (1 + \varepsilon |\eta|)^{-p/2} \left| \int_\omega g(x) e^{-\frac{1}{4\varepsilon} |y - i \frac{\eta}{|\eta|} - x|^2} dx \right| \leq C e^{\frac{|\eta|}{2(1 + \varepsilon |\eta|)} (1 - \delta)} \leq C e^{\frac{|\eta|}{2} (1 - \delta)}
\]
for some $\delta > 0$ and $y$ near $M$. Hence, as before we can say that $u(f \otimes g_\varepsilon)$ converges to $S(g\chi_\omega)(f) = (Sg)(f)$ when $\varepsilon$ goes to 0. Moreover from lemma 9.1.2 of [4], it follows that $g_\varepsilon$ uniformly converges to $g$ in a complex neighbourhood of $M$. This proves that theorem 3 extends the previous definition of $S$. □

3. Proof of theorem 1

We may assume that $\xi_0 \neq 0, \eta_0 \neq 0$ and $(x_0, y_0) \in \text{supp}(u)$. Let $B(r)$ be an open ball such that $\text{supp}(u) \subset \mathbb{R}^n \times B(r)$ and $V = B(R)$ with $R > r$. Let $C_m, m \in \mathbb{N}_0$, be a fundamental sequence of compact sets of
\[
C = \{ w \in \mathbb{C}^p : |\Re w| \leq r, |\Im w| \geq 1 \} \setminus \{ y_0 - it\eta_0 : t > 0 \}
\]
and \( w_0 = y_0 + i\eta_0 \). There is a sequence \( \delta_m > 0 \) such that
\[
\lambda\mu|\Re w - \Re w'|^2 + |\lambda\Im w + \mu\Im w'|^2 \geq \delta_m(\lambda + \mu)^2
\]
if \( w' \in C_m, \lambda, \mu > 0 \) and \( |w - w_0| \leq \delta_m \). Choose a decreasing sequence \( r_m \) such that \( 0 < r_m < \delta_m \) and consider
\[
F = \{ g \in \mathcal{A}'(V) : q_m(g) < \infty \text{ for every } m \geq 1 \}
\]
with
\[
q_m(g) = \sup_{w \in C_m, \mu > 0} |Tg(w, \mu)| e^{-\frac{\mu}{2}(|\Im w|^2 - r_m)}.
\]
As the balls \( \{ g \in \mathcal{A}'(V) : q_m(g) \leq r \} \) are closed in \( \mathcal{A}'(V) \) for every \( r \) and \( m \), the space \( F \) endowed with the semi-norms induced by \( \mathcal{A}'(V) \) and the semi-norms \( q_m \) is a Fréchet space. Moreover, if \( g \in F \) then \( WF_a(g) \cap (B(r) \times \mathbb{R}^p) \subset \{ (y_0, t\eta_0) : t > 0 \} \).

Let \( K \) be the projection of \( \text{supp}(u) \) on \( \mathbb{R}^n \) and \( E_m = \{ f \in \mathcal{A}'(K) : p_m(f) < +\infty \} \) with
\[
p_m(f) = \sup_{|z - z_0| < 1/m, \lambda > 0} e^{-\frac{\lambda}{2}|\Im z|^2} |Tf(z, \lambda)|
\]
and \( z_0 = x_0 - i\xi_0 \). For every \( m \in \mathbb{N}_0, E_m \) is a Fréchet space for the semi-norms induced by \( \mathcal{A}'(K) \) and \( p_m \). Hence, it has a strict web for which the sets with one index are closed balls of \( p_m \). Let \( E = \bigcup_{m=1}^{\infty} E_m \) with the topology induced by \( \mathcal{A}'(K) \). This space has a strict web for which the sets with one index are the spaces \( E_m \).

The operator \( S \) maps \( F \) into \( E \). Indeed, since the sheaf of microfunctions is flabby hence supple, any \( g \in F \) can be written \( g = g_0 + g_1 \) with \( g_0, g_1 \in \mathcal{B}(\mathbb{R}^p) \) and
\[
WF_a(g_0) \subset \{ (y_0, t\eta_0) : t > 0 \}, \quad WF_ag_1 \subset \overline{V} \setminus B(r) \times \mathbb{R}^p.
\]
By hypothesis, we have \( Sg_0 \in E \). On the other hand, by the main result of [1], the point \( ((x_0, y), (\xi_0, 0)) \) does not belong to \( WF_a(u) \) for every \( y \in \mathbb{R}^p \). Since \( g_1 \) is analytic near the projection of \( \text{supp}(u) \), it follows that \( (x_0, \xi_0) \not\in WF_a(Sg_1) \).

Using the explicit expression of \( Sg \) given in the previous paragraph, it is easily seen that \( S : F \to E \) is linear and has a sequentially closed graph.
With two applications of the localization theorem of [2], we get \( C, \varepsilon > 0, m \in \mathbb{N}_0 \) and a neighbourhood \( W \) of \( V \) in \( \mathbb{C}^p \) such that
\[
|T(Sg)(z, \lambda)| \leq C e^{\frac{\lambda}{2}|z|^2} (q_m(g) + \sup_{h \in \mathcal{O}(\mathbb{C}^p), \|h\| \leq 1} |g(h)|)
\]
if \( |z - z_0| < \varepsilon, \lambda > 0 \) and \( g \in F \).

With
\[
f_{\lambda, z}(x) = e^{-\lambda(z-x)^2/2}, \quad g_{\lambda, w}(y) = e^{-\lambda(w-y)^2/2}
\]
we have
\[
Tu(z, w, \lambda) = T(Sg_{\lambda, w})(z, \lambda).
\]
It follows that
\[
|Tu(z, w, \lambda)| \leq C e^{\frac{\lambda}{2}|z|^2} (q_m(g_{\lambda, w} \chi_V) + \sup_{\|h\| \leq 1} |\int_V g_{\lambda, w}(y) h(y) \, dy|)
\]
if \( |z - z_0| < \varepsilon \).

We have to show that the right hand side is exponentially decreasing with respect to the weight \( \frac{\lambda}{2}(|\Im z|^2 + |\Im w|^2) \) near \((z_0, w_0)\).

We first evaluate \( q_m(g_{\lambda, w} \chi_V) \). We have
\[
T(g_{\lambda, w} \chi_V)(w', \mu) = \left( \frac{2\pi}{\lambda + \mu} \right)^{n/2} e^{-\frac{\lambda_{\mu}}{2(\lambda + \mu)} (w-w')^2}
\]
\[
- \int_{\mathbb{R}^n \setminus V} e^{-\frac{\mu}{2} (w'-y)^2 - \frac{\lambda}{2} (w-y)^2} \, dy.
\]
In the estimation of \( Tu(z, w, \lambda) \), the first term of the previous equality gives the exponential of
\[
\frac{\lambda}{2} |\Im z|^2 - \frac{\mu}{2} (|\Im w'|^2 - r_m) - \frac{\lambda \mu}{2(\lambda + \mu)} |\Re w - \Re w'|^2 + \frac{\lambda \mu}{2(\lambda + \mu)} |\Im w - \Im w'|^2
\]
\[
= \frac{\lambda}{2} (|\Im z|^2 + |\Im w|^2) - \frac{\lambda \mu}{2(\lambda + \mu)} |\Re w - \Re w'|^2 - \frac{\lambda |\Im w + \mu \Im w'|^2}{2(\lambda + \mu)} + \frac{r_m \mu}{2}
\]
\[
\leq \frac{\lambda}{2} (|\Im z|^2 + |\Im w|^2) - \frac{\delta_m}{2} (\lambda + \mu) + \frac{r_m \mu}{2}
\]
if \( |w - w_0| < \delta_m, w' \in C_m \) and \( \mu > 1 \). This term is exponentially decreasing with respect to \( \lambda \). Consider now the integral over \( \mathbb{R}^n \setminus V \). Since \( |\Re w|, |\Re w'| \leq r \) and \( |y| > R \), the exponent can be estimated by
\[
\frac{\lambda}{2} (|\Im w|^2 + \frac{\mu}{2} |\Im w'|^2 - \frac{\lambda + \mu}{2} (R - r)^2).
\]
If \( r_m < (R - r)^2 \), this shows that this term is also exponentially decreasing.

Finally, using a complex shift \( y \to y + it\Im w\chi(y) \) where \( \chi \in C_0^\infty(V) \) is equal to 1 in a neighbourhood of \( \Re w_0 \) and \( t > 0 \) is small, we get the estimate

\[
\sup_{\|h\|_W \leq 1} \left| \int_V g_{\lambda,w}(y) h(y) \, dy \right| \leq C' e^{\frac{1}{2}(\|\Im w\|^2 - \varepsilon')}. 
\]

This proves the theorem. \( \square \)

References


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