A bifurcation of multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole

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Dedicated to Professor Sigeru Mizohata

Abstract. In the present paper we study the asymptotic expansion of the multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole. We prove the bifurcation of these eigenvalues under certain conditions.

1. Introduction and main results

The purpose of this article is to study asymptotic formula of multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole. Let \( \Omega \) be a bounded domain with smooth boundary in \( \mathbb{R}^3 \) and \( \{0\} \in \Omega \). Let \( B_1 \) be the unit ball in \( \mathbb{R}^3 \). We consider the following problem:

\[
\begin{align*}
\Delta u(x, \varepsilon) + \lambda(\varepsilon)u(x, \varepsilon) &= 0, \quad \text{in } \Omega_\varepsilon = \Omega \setminus \varepsilon B_1 \\
u(x, \varepsilon)|_{\partial \Omega_\varepsilon} &= 0.
\end{align*}
\]

All the eigenvalues of (1)-(2) may be put in non-decreasing order \( 0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \cdots \). The first eigenvalue is always simple (see [1]). The eigenvalue from \( \lambda_2(\varepsilon) \) may be multiple. We shall study the behavior of

1991 Mathematics Subject Classification. Primary 35B20; Secondary 35B32, 35C20, 35P99.

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the functions $\lambda_n(\varepsilon)$ when $\varepsilon \to 0 (n \geq 2)$. The problem (1)-(2) is connected closely with following one in the limit case:

(3) $\Delta u(x) + \lambda u(x) = 0$, in $\Omega$
(4) $u(x)|_{\partial \Omega} = 0.$

All the eigenvalues of (3)-(4) may be also put in non-decreasing order $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$. It is well-known that $\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda_j$ (see[2]). Let $\lambda_j$ be a simple eigenvalue. In the work [2], [3] Ozawa S. obtained the statement:

$$\lambda_j(\varepsilon) = \lambda_j + 4\pi u_j^2(0)\varepsilon + C_j\varepsilon^2 + 0(\varepsilon^{5/2}) \quad (\varepsilon \to 0)$$

where $u_j(x)$ is the normed eigenfunction corresponding to $\lambda_j$ and where $C_j$ is a constant explicitly calculated.

We shall find a full asymptotic formula of $\lambda_j(\varepsilon)$ in a form

$$\lambda_j(\varepsilon) = \sum_{i=0}^{\infty} \lambda_j^{<i>\varepsilon^i}$$

and corresponding eigenfunctions $u_j(x,\varepsilon)$ in a form:

$$u_j(x,\varepsilon) = \sum_{k=0}^{\infty} (m_{kj}(x) + n_{kj}(\xi))\varepsilon^k$$

where $\xi = x\varepsilon^{-1}$. The functions $m_{kj}(x)$ and $n_{kj}(\xi)$ have asymptotic expansions

$$m_{kj}(x) = \sum_{i=0}^{N} m_{kj}^{<i>}(\theta) \cdot |X|^i + \tilde{m}_{kj}^{<N>}(x)$$

$$n_{kj}(\xi) = \sum_{i=1}^{N} n_{kj}^{<i>}(\theta) \cdot |\xi|^{-i} + \tilde{n}_{kj}^{<N>}(\xi)$$

where

$$|D_{x}^{\alpha} \tilde{m}_{kj}^{<N>}(x)| \leq C_{N,k,\alpha,j} |x|^{-N+1-|\alpha|}$$

$$|D_{\xi}^{\alpha} \tilde{n}_{kj}^{<N>}(\xi)| \leq C_{N,k,\alpha,j} |\xi|^{-N-1-|\alpha|}$$

$\theta = (\theta_1, \theta_2)$ denotes coordinates on $S^2$ and $m_{kj}^{<i>}(\theta), n_{kj}^{<i>}(\theta)$ are smooth functions on $S^2$. In the paper [4] Mazia V.G., Nazarov S.A.,
B.A. Plamenevskii found a full asymptotic formula for simple eigenvalues. Let \( \lambda_j \) be a simple eigenvalue of the problem (3)-(4). Then we have the following expansion for \( \lambda_j(\varepsilon) \):

\[
\lambda_j(\varepsilon) = \lambda_j + 4\pi u_j^2(0)\varepsilon + \lambda_j^{<2>}\varepsilon^2 + \ldots + \lambda_j^{<M>}\varepsilon^M + O(\varepsilon^{M+1})
\]

where \( M \) is any positive integer number. In the article [5] the author obtained the

**Theorem.** Let \( \lambda_j \) be a double eigenvalue of (3)-(4). It corresponds two orthonormal eigenfunctions \( u_j(x), u_{j+1}(x) \). Assume that \( u_j^2(0) + u_{j+1}^2(0) > 0 \), then we have a formula for the eigenvalues \( \lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon) \) (respectively)

\[
\lambda_{j+k}(\varepsilon) = \sum_{i=0}^{M} \lambda_j^{<i>}\varepsilon^i + O(\varepsilon^{M+1}) \quad k = 0, 1.
\]

Furthermore \( \lambda_j^{<0>} = \lambda_{j+1}^{<0>} = \lambda_j, \lambda_j^{<1>} = 0, \lambda_{j+1}^{<1>} = 4\pi(u_j^2(0) + u_{j+1}^2(0)) \).

**Remark.** It is easy to see that the sum \((u_j^2(0) + u_{j+1}^2(0))\) is invariant under any orthogonal transformations in the plane \((u_j, u_{j+1})\).

**Corollary.** Assume that \((u_j^2(0) + u_{j+1}^2(0)) > 0\). Then the eigenvalues \( \lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon) \) are simple and different as \( \varepsilon \to 0 \).

In the present paper the author continue the studies in [2]-[5]. We shall consider the case when \( \lambda_j \) is a double or triple eigenvalues. Let \( \lambda_j \) be a double and \( u_j(0) = u_{j+1}(0) = 0 \). We expand \( u_j(x), u_{j+1}(x) \) in series:

\[
u_{j+k}(x) = u_j^{<1>}(-)r + u_j^{<2>}(-)r^2 + \ldots + u_j^{<M>}(-)r^M + 0(r^{M+1}) \quad (r \to 0)
\]

where \( k = 0, 1 \) and \( r = |x| \).

One can write the Laplace operator in the spherical coordinates

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^2}
\]
where $\Delta_{S^2}$ is the Laplace-Beltrami operator on sphere. Since the functions $u_j(x), u_{j+1}(x)$ are the eigenfunctions, it follows that $u_j^{<1>}(\theta), u_{j+1}^{<1>}(\theta)$ satisfy the equations (see [6]):

$$
\Delta_{S^2} u_j^{<1>}(\theta) + 2u_{j+k}^{<1>}(\theta) = 0 \quad (k = 0, 1).
$$

Therefore we have the identities

$$
u_{j+k}^{<1>}(\theta) = a_{j+k}^{<1>} A_1(\theta) + a_{j+k}^{<2>} A_2(\theta) + a_{j+k}^{<3>} A_3(\theta) \quad (k = 0, 1),$$

where $A_1(\theta), A_2(\theta), A_3(\theta)$ denote orthonormal eigenfunctions of $\Delta_{S^2}$ with the eigenvalue 2.

**Theorem 1.** Let $\lambda_j$ be a double eigenvalue and $u_j(0) = u_{j+1}(0) = 0$. Assume that

$$
T_j := \left| \sum_{i=1}^3 [a_{j+k}^{(i)}]^2 - \sum_{i=1}^3 [a_{j+1+k}^{(i)}]^2 \right| + \left| \sum_{i=1}^3 a_{j+k}^{(i)} a_{j+1+k}^{(i)} \right| \neq 0.
$$

Then we have the expansions for $\lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon)$ (resp.)

$$
\lambda_{j+k}(\varepsilon) = \lambda_j + \lambda_{j+k}^{<3>} \varepsilon^3 + \lambda_{j+k}^{<4>} \varepsilon^4 + \cdots + \lambda_{j+k}^{<M>} \varepsilon^M + O(\varepsilon^{M+1}) \quad (\varepsilon \to 0)
$$

where $k = 0, 1$ and $\lambda_j^{<3>} < \lambda_{j+1}^{<3>}$.

**Corollary 1.** Assume that $u_j(0) = u_{j+1}(0) = 0, T_j \neq 0$, then $\lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon)$ are simple and different as $\varepsilon \to 0$.

**Remark.** The condition $T_j \neq 0$ is equivalent to the following condition: the matrix

$$
\left( \begin{array}{cc}
(u_j^{<1>}(\theta), u_j^{<1>}(\theta))_{L^2(\partial B_1)} & (u_j^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)} \\
(u_j^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)} & (u_{j+1}^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)}
\end{array} \right) =: M
$$

has two different eigenvalues. In the future it is easy to see that $3^{-1} \lambda_j^{<3>}, 3^{-1} \lambda_{j+1}^{<3>}$ are the eigenvalues of the matrix $M$.

Now let $\lambda_j$ are a triple eigenvalue of the problems (3)-(4). It corresponds three orthonormal functions $u_j(x), u_{j+1}(x), u_{j+2}(x)$. Assume that
where eigenvalues, then the eigenvalues of the eigenvalues and the eigenfunctions \( A \).

2. A process of finding the full asymptotic formula for \( \lambda \).

Theorem 2. Let \( \lambda_j \) be a triple eigenvalue of (3)-(4). Assume that \( u_{j+2}(0) \neq 0 \) and the matrix

\[
\begin{pmatrix}
(u_{j}^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)} & (u_{j}^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)} \\
(u_{j}^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)} & (u_{j}^{<1>}(\theta), u_{j+1}^{<1>}(\theta))_{L^2(\partial B_1)}
\end{pmatrix} =: M^*
\]

has two different eigenvalues. Then we have the asymptotic formula for \( \lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon) \leq \lambda_{j+2}(\varepsilon) \)

\[
\begin{align*}
\lambda_{j+k}(\varepsilon) &= \lambda_j + \lambda_{j+k}^<<<<3\varepsilon^3 + \lambda_{j+k}^{<4\varepsilon^4} + \cdots + \lambda_{j+k}^{<M\varepsilon^M} + o(\varepsilon^{M+1}) \quad (\varepsilon \to 0) \\
\lambda_{j+2}(\varepsilon) &= \lambda_j + 4\pi |u_{j+2}(0)|^2 \varepsilon + \lambda_{j+2}^{<3}\varepsilon^3 + \cdots + \lambda_{j+2}^{<M\varepsilon^M} + o(\varepsilon^{M+1}) \quad (\varepsilon \to 0)
\end{align*}
\]

where \( k = 0, 1, \) and \( \lambda_{j}^{<3\varepsilon} < \lambda_{j+1}^{<3\varepsilon} \).

Corollary 2. If \( u_{j+2}(0) \neq 0 \) and the matrix \( M^* \) has two different eigenvalues, then the eigenvalues \( \lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon), \lambda_{j+2}(\varepsilon) \) are simple and different when \( \varepsilon \to 0 \).

2. A process of finding the full asymptotic formula of the eigenvalues and the eigenfunctions \( A \).

The case of double eigenvalues:

Put \( \lambda_{j+k}(\varepsilon) \) from (7) into (1) and (2):

\[
[(\Delta + \lambda_j + \lambda_{j+1}^<<<<1\varepsilon + \lambda_{j+2}^{<2}\varepsilon^2 + \lambda_{j+3}^{<3}\varepsilon^3 + o(\varepsilon^4))((u_{j+1} + v_{j+1}) + \varepsilon(u_{j+1} + v_{j+1}) + \\
\varepsilon^2(u_{j+1} + v_{j+1}) + \varepsilon^3(u_{j+1} + v_{j+1}) + \varepsilon^4(u_{j+1} + v_{j+1}) + o(\varepsilon^5)) = 0 \quad \Omega \varepsilon
\]
\[(u_{j0} + v_{j0}) + \varepsilon (u_{j1} + v_{j1}) + \varepsilon^2 (u_{j2} + v_{j2}) + \cdots + O(\varepsilon^5)|_{\partial \Omega} = 0 \]

\[[(\Delta + \lambda_j + \lambda_{j+1}^{<1>} \varepsilon + \lambda_{j+1}^{<2>} \varepsilon^2 + \lambda_{j+1}^{<3>} \varepsilon^3 + O(\varepsilon^4))[(p_{j0} + q_{j0}) + \varepsilon(p_{j1} + q_{j1}) + \varepsilon^2(p_{j2} + q_{j2}) + \cdots + O(\varepsilon^5)] = 0 \text{ in } \Omega_{\varepsilon}\]

\[(p_{j0} + q_{j0}) + \varepsilon(p_{j1} + q_{j1}) + \varepsilon^2(p_{j2} + q_{j2}) + \cdots + O(\varepsilon^5)|_{\partial \Omega} = 0 \]

where

\[u_j(x, \varepsilon) = [(u_{j0} + v_{j0}) + \varepsilon(u_{j1} + v_{j1}) + \varepsilon^2(u_{j2} + v_{j2}) + \cdots] \]

\[u_{j+1}(X, \varepsilon) = [(p_{j0} + q_{j0}) + \varepsilon(p_{j1} + q_{j1}) + \varepsilon^2(p_{j2} + q_{j2}) + \cdots] \]

denote eigenfunctions corresponding to \(\lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon)\). Functions \(u_{j0}(x), u_{j1}(x), \ldots, p_{j0}(x), p_{j1}(x), \ldots\) are defined in \(\Omega\) and they keep an asymptotic expansion as the functions \(m_{kj}(x)\) from (5). Functions \(v_{j0}(\xi), v_{j1}(\xi), q_{j0}(\xi), q_{j1}(\xi), \ldots\) are defined in \(\mathbb{R}^3 \setminus B_1\) and they keep an asymptotic expansions as the function \(n_{kj}(\xi)\) from (6). In the following we shall write \(u_0(x), u_1(x), \ldots, p_0(x), p_1(x), \ldots, v_0(\xi), v_1(\xi), \ldots, q_0(\xi), q_1(\xi), \ldots\) for \(u_{j0}(x), u_{j1}(x), \ldots, p_{j0}(x), p_{j1}(x), \ldots, v_{j0}(\xi), v_{j1}(\xi), \ldots, q_{j0}(\xi), q_{j1}(\xi), \ldots\). Comparing the coefficients in the identical orders of \(\varepsilon\) in (8)-(11) one obtain:

\[\varepsilon^0 \begin{cases} 
\Delta u_0(x) + \lambda_j u_0(x) = 0, & \text{in } \Omega \\
u_0(x)|_{\partial \Omega} = 0 
\end{cases} \]

\[\varepsilon^0 \begin{cases} 
\Delta p_0(x) + \lambda_j p_0(x) = 0, & \text{in } \Omega \\
p_0(x)|_{\partial \Omega} = 0
\end{cases} \]

Hence \(u_0(x) = a_0^1 u_j(x) + a_0^2 u_{j+1}(x), p_0(x) = b_0^1 u_j(x) + b_0^2 u_{j+1}(x)\). Since \(\Delta_\xi = \varepsilon^2 \Delta_x\) then

\[\varepsilon^{-2} \begin{cases} 
\Delta v_0(\xi) = 0, & \text{in } \mathbb{R}^3 \setminus B_1 \\
v_0(\xi)|_{\partial B_1} = 0 \\
\lim_{|\xi| \to \infty} v_0(\xi) = 0 
\end{cases} \quad \varepsilon^{-2} \begin{cases} 
\Delta q_0(\xi) = 0, & \text{in } \mathbb{R}^3 \setminus B_1 \\
q_0(\xi)|_{\partial B_1} = 0 \\
\lim_{|\xi| \to \infty} q_0(\xi) = 0.
\end{cases} \]

Therefore \(v_0(\xi) = q_0(\xi) = 0\) and

\[\Delta u_1(x) + \lambda_j u_1(x) + \lambda_{j+1}^{<1>} u_0(x) = 0, \text{ in } \Omega \]

\[u_1(X)|_{\partial \Omega} = 0 \]
For the solvability of the problems (12)-(15) we have \( \lambda_{j}^{1>1} = \lambda_{j+1}^{1>1} = 0 \). Hence \( u_{1}(x) = a_{1}^{1}u_{j}(x) + a_{2}^{1}u_{j+1}(x) \) and \( p_{1}(x) = b_{1}^{1}u_{j}(x) + b_{2}^{1}u_{j+1}(x) \). Assume that under some conditions the eigenvalues \( \lambda_{j}(\varepsilon) < \lambda_{j+1}(\varepsilon) \) for sufficiently small \( \varepsilon \). In the process of finding the asymptotic formula that condition will be clear. If it happens, then we have \( u_{0}p_{0} \), i.e. if \( u_{0} = a_{0}^{1}u_{j} + a_{0}^{2}u_{j+1}, \) so \( p_{0} = -a_{0}^{j}u_{j} + a_{0}^{j}u_{j+1} \). Hence one can choose \( u_{1}(x) = c_{1}p_{0}(x) \) and \( p_{1}(x) = d_{1}u_{0}(x) \). Suppose the functions \( u_{1}(x) \) and \( p_{1}(x) \) are found. Then the functions \( v_{1}(\xi), q_{1}(\xi) \) satisfy:

\[
\begin{cases}
\Delta v_{1}(\xi) = 0, & \text{in } \mathbb{R}^{3}\setminus B_{1} \\
v_{1}(\xi)|_{\partial B_{1}} = -(\text{grad } u_{0}(0), \xi) =: -A_{1}(\theta) \\
\lim_{|\xi| \to \infty} v_{1}(\xi) = 0
\end{cases}
\]

\[
\begin{cases}
\Delta q_{1}(\xi) = 0, & \text{in } \mathbb{R}^{3}\setminus B_{1} \\
q_{1}(\xi)|_{\partial B_{1}} = -(\text{grad } p_{0}(0), \xi) =: -A_{2}(\theta) \\
\lim_{|\xi| \to \infty} q_{1}(\xi) = 0
\end{cases}
\]

If \( v_{1}(\xi), q_{1}(\xi) \) are found we can find \( u_{2}(x) \) and \( p_{2}(x) \) from

\[
\begin{align*}
\Delta u_{2}(x) + \lambda j u_{2}(x) + \lambda_{j}^{2>} u_{0}(x) &= 0, & \text{in } \Omega \\
u_{2}(x)|_{\partial \Omega} &= 0 \\
\Delta p_{2}(x) + \lambda j p_{2}(x) + \lambda_{j+1}^{2>} p_{0}(x) &= 0, & \text{in } \Omega \\
p_{2}(x)|_{\partial \Omega} &= 0.
\end{align*}
\]

From the solvability of (16)-(19) we deduce that \( \lambda_{j}^{2>} = \lambda_{j+1}^{2>} = 0 \). Therefore one can choose \( u_{2}(x) = c_{2}p_{0}(x), p_{2}(x) = d_{2}u_{0}(x) \). The functions \( v_{2}(\xi) \) and \( q_{2}(\xi) \) satisfy:

\[
\begin{align*}
\begin{cases}
\Delta v_{2}(\xi) = 0, & \text{in } \mathbb{R}^{3}\setminus B_{1} \\
v_{2}(\xi)|_{\partial B_{1}} = -c_{1}A_{2}(\theta) - B_{1}(\theta), \\
\lim_{|\xi| \to \infty} v_{2}(\xi) = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Delta q_{2}(\xi) = 0, & \text{in } \mathbb{R}^{3}\setminus B_{1} \\
q_{2}(\xi)|_{\partial B_{1}} = -d_{1}A_{1}(\theta) - B_{2}(\theta) \\
\lim_{|\xi| \to \infty} q_{2}(\xi) = 0
\end{cases}
\end{align*}
\]
where \( B_1(\theta) = \sum_{i,k=1}^{3} \frac{\partial^2 u_0(0)}{\partial x_i \partial x_k} \xi_i \xi_k \bigg|_{\partial B_1} \), \( B_2(\theta) = \sum_{i,k=1}^{3} \frac{\partial^2 p_0(0)}{\partial x_i \partial x_k} \xi_i \xi_k \bigg|_{\partial B_1} \).

Note that \( \Delta_s^2 B_1(\theta) + 6B_1(\theta) = 0 \) and \( \Delta_s^2 B_2(\theta) + 6B_2(\theta) = 0 \). It follows that

\[
v_2(\xi) = -c_1 A_2(\theta)|\xi|^{-2} - B_1(\theta)|\xi|^{-3},\quad q_2(\xi) = -d_1 A_1(\theta)|\xi|^{-2} - B_2(\theta)|\xi|^{-3}.
\]

Then \( u_3(x), p_3(x) \) satisfy:

\[
\Delta \{ u_3 - A_1(\theta)|x|^{-2} \} + \lambda_j \{ u_3 - A_1(\theta)|x|^{-2} \} + \lambda_j^{<3>}u_0 = 0
\]

\[
\{ u_3 - A_1(\theta)|x|^{-2} \}|_{\partial\Omega} = 0
\]

\[
\Delta \{ p_3 - A_2(\theta)|x|^{-2} \} + \lambda_j \{ p_3 - A_2(\theta)|x|^{-2} \} + \lambda_j^{(3)}p_0 = 0
\]

\[
\{ p_3 - A_2(\theta)|x|^{-2} \}|_{\partial\Omega} = 0.
\]

For solvability of (20)-(23) we have

\[
\lambda_j^{<3>} = 3 \int_{\partial B_1} A_1^2(\theta)d\theta, \quad \lambda_j^{<3>} = 3 \int_{\partial B_1} A_2^2(\theta)d\theta.
\]

Note that \( A_1(\theta) = a_0^1 u_j^{<1>}(\theta) + a_0^2 u_j^{<1>}(\theta) \) and \( A_2(\theta) = -a_0^2 u_j^{<1>}(\theta) + a_0^1 u_j^{<1>}(\theta) \).

Multiplying (20) by \( u_j(x), u_{j+1}(x) \) and integrating over \( \Omega_\varepsilon \) then turning \( \varepsilon \to 0 \) one obtain:

\[
(M - \frac{\lambda_j^{<3>}}{3}I) \begin{pmatrix} a_0^1 \\ a_0^2 \end{pmatrix} = 0 \text{(see the definition of } M \text{ in the introduction)}.
\]

It means that \( 3^{-1}\lambda_j^{<3>} \) is the eigenvalue of the matrix \( M \) and \( (a_0^1, a_0^2) \) is its eigenvector. By analogy we can prove \( 3^{-1}\lambda_j^{<3>} \) is also eigenvalue of \( M \). Therefore if \( M \) has two different eigenvalues then \( \lambda_j^{<3>}, \lambda_j^{<3>} \) and \( (a_0^1, a_0^2) \) are defined uniquely. So we found \( \lambda_j^{<3>}, \lambda_j^{<3>}, u_0(x)p_0(x), v_0(\xi), \ldots \)
that \(v_0(\xi), v_1(\xi), q_1(\xi)\). Continuing this procedure we can find \(\lambda_j^{<4>}, \lambda_{j+1}^{<4>}, u_1(x), p_1(x), v_2(\xi), q_2(\xi)\).

A step of induction : Assume that \(\lambda_j^{<n+3>}, \lambda_{j+1}^{<n+3>}, u_n(x), p_n(x), v_{n+1}(\xi), q_{n+1}(\xi)\) are defined. We show how to find the functions \(\lambda_j^{<n+4>}, \lambda_{j+1}^{<n+4>}, u_{n+1}(x), p_{n+1}(x), v_{n+2}(\xi), q_{n+2}(\xi)\). In previous steps we have already known the equations for \(u_{n+1}(x), p_{n+1}(x)\) and found the condition for their solvability. However, the solutions are defined non-uniquely. Writing once again these equations:

\[
\begin{cases}
\Delta u_{n+1} + \sum_{i=0}^{n+1} \lambda_j^{<i>} u_{n+1-i} + \sum_{i=0}^{n} \lambda_j^{<i>} v_{n-i}^{<1>} (\theta) |x|^{-1} + \\
\sum_{i=0}^{n-1} \lambda_j^{<i>} v_{n-1-i}^{<2>} (\theta) |x|^{-2} = 0 \\
\left\{ u_{n+1} + \sum_{i=1}^{n+1} v_{n+1-i}^{<1>} (\theta) |x|^{-i} \right\} \bigg|_{\partial \Omega} = 0 \\
\Delta p_{n+1} + \sum_{i=0}^{n+1} \lambda_{j+1}^{<i>} p_{n+1-i} + \sum_{i=0}^{n} \lambda_{j+1}^{<i>} q_{n-i}^{<1>} (\theta) |x|^{-1} + \\
\sum_{i=0}^{n-1} \lambda_{j+1}^{<i>} q_{n-1-i}^{<2>} (\theta) |x|^{-2} = 0 \\
\left\{ p_{n+1} + \sum_{i=1}^{n+1} q_{n+1-i}^{<1>} (\theta) |x|^{-i} \right\} \bigg|_{\partial \Omega} = 0.
\end{cases}
\]

Suppose that \(U_{n+1}(x), P_{n+1}(x)\) are the solutions of the above problem such that

\[
\int_{\Omega} U_{n+1} u_0 dx = \int_{\Omega} U_{n+1} p_0 dx = \int_{\Omega} P_{n+1} u_0 dx = \int_{\Omega} P_{n+1} p_0 dx = 0
\]

A general solution must be found in a form:

\[ u_{n+1} = U_{n+1} + c_{n+1} p_0, \quad p_{n+1} = P_{n+1} + d_{n+1} u_0. \]

By analogy we should find \(u_{n+2}(x), p_{n+2}(x), u_{n+3}(x), p_{n+3}(x)\) in form:

\[ u_{n+2} = U_{n+2} + c_{n+2} p_0, \quad p_{n+2} = P_{n+2} + d_{n+2} u_0. \]

\[ u_{n+3} = U_{n+3} + c_{n+3} p_0, \quad p_{n+3} = P_{n+3} + d_{n+3} u_0. \]
Then $v_{n+2}(\xi), q_{n+2}(\xi)$ satisfy:

\[
\begin{cases}
\Delta v_{n+2}(\xi) + \sum_{i=0}^{n} \lambda_{j}^{<i>}\tilde{v}_{n-i}^{<2>}(\xi) = 0 \\
\{v_{n+2}(\theta) + u_0^{<n+2>}(\theta) + \cdots + u_{n+2}^{0>}(\theta)\}|_{\partial B_1} = 0 \\
\lim_{|\xi|\to\infty} v_{n+2}(\xi) = 0
\end{cases}
\]

\[
\begin{cases}
\Delta q_{n+2}(\xi) + \sum_{i=0}^{n} \lambda_{j+1}^{<i>}\tilde{q}_{n-i}^{<2>}(\xi) = 0 \\
\{q_{n+2}(\theta) + p_0^{<n+2>}(\theta) + \cdots + p_{n+2}^{0>}(\theta)\}|_{\partial B_1} = 0 \\
\lim_{|\xi|\to\infty} q_{n+2}(\xi) = 0.
\end{cases}
\]

Therefore $v_{n+2} = V_{n+2} - c_{n+1}A_2(\theta)|\xi|^{-2}$, $q_{n+2} = Q_{n+2} - d_{n+1}A_1(\theta)|\xi|^{-2}$. We denote by $V_{n+2}(\xi)$ and $Q_{n+2}(\xi)$ the solutions of the following problems:

\[
\begin{cases}
\Delta V_{n+2}(\xi) + \sum_{i=0}^{n} \lambda_{j}^{<i>}\tilde{v}_{n-i}^{<2>}(\xi) = 0 \\
\{V_{n+2}(\theta) + E_{n+2}(\theta)\}|_{\partial B_1} = 0 \\
\lim_{|\xi|\to\infty} V_{n+2}(\xi) = 0
\end{cases}
\]

\[
\begin{cases}
\Delta Q_{n+2}(\xi) + \sum_{i=0}^{n} \lambda_{j+1}^{<i>}\tilde{q}_{n-i}^{<2>}(\xi) = 0 \\
\{Q_{n+2}(\theta) + F_{n+2}(\theta)\}|_{\partial B_1} = 0 \\
\lim_{|\xi|\to\infty} Q_{n+2}(\xi) = 0,
\end{cases}
\]

where the functions

\[
E_{n+2}(\theta) = u_0^{<n+2>}(\theta) + \cdots + u_n^{<2>}(\theta) + U_{n+1}^{<1>}(\theta) + U_{n+2}^{0>}(\theta)
\]

\[
F_{n+2}(\theta) = p_0^{<n+2>}(\theta) + \cdots + p_n^{<2>}(\theta) + P_{n+1}^{<1>}(\theta) + P_{n+2}^{0>}(\theta)
\]

are already defined from previous steps.

By analogy we should find $v_{n+3}(\xi), q_{n+3}(\xi)$ in a form

\[
v_{n+3}(\xi) = V_{n+3}(\xi) - c_{n+2}A_2(\theta)|\xi|^{-2} - c_{n+1}B_2(\theta)|\xi|^{-3},
\]

\[
q_{n+3}(\xi) = Q_{n+3}(\xi) - d_{n+2}A_1(\theta)|\xi|^{-2} - d_{n+1}B_1(\theta)|\xi|^{-3}.
\]
Finally we write equations for $u_{n+4}(x), p_{n+4}(x)$:

\[
\begin{align*}
\Delta u_{n+4} + \sum_{i=0}^{n+4} \lambda_j^{<i>} u_{n+4-i} + \sum_{i=0}^{n+3} \lambda_j^{<i>} v_{n-i+3}^{<1>} (\theta) |x|^{-1} + \\
\sum_{i=0}^{n+2} \lambda_j^{<i>} v_{n-i+2}^{<2>} (\theta) |x|^{-2} = 0 \\
\{ u_{n+4} + \sum_{i=1}^{n+4} v_{n+4-i}^{<i>} (\theta) |x|^{-i} \} \bigg|_{\partial \Omega} = 0
\end{align*}
\]

\[
\begin{align*}
\Delta p_{n+4} + \sum_{i=0}^{n+4} \lambda_{j+1}^{<i>} p_{n+4-i} + \sum_{i=0}^{n+3} \lambda_{j+1}^{<i>} q_{n-i+3}^{<1>} (\theta) |x|^{-1} + \\
\sum_{i=0}^{n+2} \lambda_{j+1}^{<i>} q_{n-i+2}^{<2>} (\theta) |x|^{-2} = 0 \\
\{ p_{n+4} + \sum_{i=1}^{n+4} q_{n+4-i}^{<i>} (\theta) |x|^{-i} \} \bigg|_{\partial \Omega} = 0.
\end{align*}
\]

Note that $\lambda_j^{<0>} = \lambda_{j+1}^{<0>} = \lambda_j, \lambda_j^{<1>} = \lambda_{j+1}^{<1>} = \lambda_j^{<2>} = \lambda_{j+1}^{<2>} = 0$. So we have:

\[
\Delta \{ u_{n+4}(x) - c_{n+1} A_2(\theta) |x|^{-2} \} + \lambda_j \{ u_{n+4}(x) - c_{n+1} A_2(\theta) |x|^{-2} \} \\
+ \lambda_j^{<3>} \{ U_{n+1}(x) + c_{n+1} P_0(x) \} + \lambda_j^{<n+4>} u_0(x) = G_n(x) \tag{24}
\]

\[
\{ u_{n+4}(x) - c_{n+1} A_2(\theta) |x|^{-2} \} \bigg|_{\partial \Omega} = H_n(x) \tag{25}
\]

where the functions $G_n(x), H_n(x)$ are defined from previous steps. Multiplying (24) by $u_0(x), p_0(x)$ and integrating over $\Omega_\varepsilon$ as $\varepsilon \to 0$ we obtain immediately $c_{n+1}$ and $\lambda_j^{<n+4>}$. By analogy one can find $d_{n+1}$ and $\lambda_j^{<n+4>}$. Our procedure is ended.

**B. The case of a triple eigenvalue**

We are interested only in the case of a bifurcation, i.e. $\lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon) \leq \lambda_{j+2}(\varepsilon)$ when $\varepsilon$ is sufficiently small.

Suppose:

\[
\lambda_{j+k}(\varepsilon) = \lambda_j + \lambda_j^{<1>} \varepsilon^1 + \lambda_j^{<2>} \varepsilon^2 + \cdots + \lambda_j^{<M>} \varepsilon^M + O(\varepsilon^{M+1}) \quad (k = 0, 1, 2)
\]
and

\[ u_j(x, \varepsilon) = [(u_0 + v_0) + \varepsilon(u_1 + v_1) + \varepsilon^2(u_2 + v_2) + \ldots] \]
\[ u_{j+1}(x, \varepsilon) = [(p_0 + q_0) + \varepsilon(p_1 + q_1) + \varepsilon^2(p_2 + q_2) + \ldots] \]
\[ u_{j+2}(x, \varepsilon) = [(r_0 + s_0) + \varepsilon(r_1 + s_1) + \varepsilon^2(r_2 + s_2) + \ldots]. \]

Putting \( u_j(x, \varepsilon), u_{j+1}(x, \varepsilon), u_{j+2}(x, \varepsilon), \lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon), \lambda_{j+2}(\varepsilon) \) into (1), (2) and comparing the coefficient in the identical order of \( \varepsilon \) we obtain the equations for \( u_0(x), p_0(x), r_0(x) \) as the equations for \( u_0(x), p_0(x) \) in the case of double eigenvalues.

Therefore:

\[ u_0(x) = a_0^1 u_j^*(x) + a_0^2 u_{j+1}^*(x) + a_0^3 u_{j+2}^*(x) \]
\[ p_0(x) = b_0^1 u_j^*(x) + b_0^2 u_{j+1}^*(x) + b_0^3 u_{j+2}^*(x) \]
\[ r_0(x) = c_0^1 u_j^*(x) + c_0^2 u_{j+1}^*(x) + c_0^3 u_{j+2}^*(x) \]

(see the definition of \( u_j, u_{j+1}, u_{j+2} \) in the introduction). Since we are only interested in the case of a bifurcation, it follows that the functions \( u_0, p_0, r_0 \) must be orthogonal. Then we have

\[ v_0(\xi) = -u_0(0)|\xi|^{-1}, q_0(\xi) = -p_0(0)|\xi|^{-1}, s_0(\xi) = -r_0(0)|\xi|^{-1}. \]

Now we write the equations for \( u_1(x), p_1(x), r_1(x) \)

\[
\begin{cases}
\Delta u_1(x) + \lambda_j u_1(x) + \lambda_j^{<1>} u_0(x) - \lambda_j u_0(0)|x|^{-1} = 0 & \text{in } \Omega \\
u_1(x)|_{\partial\Omega} = u_0(0)|x|^{-1}|_{\partial\Omega}
\end{cases}
\]

From the conditions of their solvability and the conditions \( \lambda_j(\varepsilon) < \lambda_{j+1}(\varepsilon) < \lambda_{j+2}(\varepsilon) \) when \( \varepsilon \) is sufficiently small. We have

\[ \lambda_j^{<1>} = \lambda_{j+1}^{<1>} = 0, \lambda_j^{<1>} = 4\pi\{u_{j+2}^*(0)\}^2, c_0^1 = c_0^2 = c_0^3 = b_0^3 = 0, c_0^3 = 1. \]

So the function \( r_0(x) \) is defined. Suppose provisionally the function \( u_0(x), p_0(x) \) are also defined. We show how to find \( \lambda_{j+2}^{<2>}, s_1(\xi), r_1(x) \). Note
the problems for \( r_1(x) \) are solvable. However the solution is defined non-uniquely. Suppose that \( R_1(x) \) is a solution such that \( \int \Omega R_1 u_0 dx = \int \Omega R_1 p_0 dx = \int \Omega R_1 r_0 dx = 0 \). A general solution \( r_1(x) \) may be written as follows: \( r_1(x) = R_1(x) + a_1 u_0(x) + b_1 p_0(x) \). Assume that \( a_1, b_1 \) are found. Then \( s_1(\xi) \) satisfies:

\[
\left\{ \begin{array}{l}
\Delta s_1(\xi) = 0, \text{ in } \mathbb{R}^3 \setminus B_1 \\
\left| s_1(\xi)\right|_{\partial B_1} = -R_1(0) - r_0^{1^>}(\theta) \\
\lim_{|\xi| \to \infty} s_1(\xi) = 0.
\end{array} \right.
\]

Therefore \( s_1(\xi) = -R_1(0)|\xi|^{-1} - r_0^{1^>}(\theta)|\xi|^{-2} \). We obtain the equations for \( r_2(x) \):

\[
\Delta r_2 - \lambda_j R_1(0)|x|^{-1} + \lambda_j r_2 + \lambda_j^{1^>}(R_1 + a_1 u_0 + b_1 p_0) + \lambda_j^{<2^>} r_0 = 0 \quad (26)
\]

\[
\{ r_2(x) - R_1(0)|x|^{-1} \}_{\partial \Omega} = 0. \quad (27)
\]

Multiplying (26) by \( u_0(x), p_0(x), r_0(x) \) and integrating over \( \Omega_\varepsilon \) when \( \varepsilon \to 0 \) one deduce that \( \lambda_j^{<2^>} = 0, a_1 = b_1 = 0 \). So we found \( r_1(x), s_1(\xi), \lambda_j^{<2^>} \). By induction, as in the case of double eigenvalues, we can find all \( r_n(x), s_n(\xi), \lambda_j^{<n+2^>} \). Now, under some conditions, we show how to find \( u_0(x) \) and \( p_0(x) \). In the first step we had:

\[
\begin{aligned}
u_0(x) &= a_0^1 u_0^*(x) + a_0^2 u_{j+1}^*(x), \\
p_0(x) &= b_0^1 u_j^*(x) + b_0^2 u_{j+1}^*(x), \\
\lambda_j^{<1^>} &= \lambda_j^{<1^>} = 0.
\end{aligned}
\]

Since : \( u_0(0) = p_0(0) = 0 \) it follows \( v_0(\xi) = q_0(\xi) = 0 \). From the equations for \( u_1(x), p_1(x) \) we can find them in a form:

\[
\begin{aligned}
u_1(x) &= c_1 p_0(x) + d_1 r_0(x), \\
p_1(x) &= e_1 u_0(x) + f_1 r_0(x).
\end{aligned}
\]

Suppose that \( c_1, d_1, e_1, f_1 \) are known. Then, from the equations for \( v_1(\xi), q_1(\xi) \) we obtain immediately:

\[
\begin{aligned}
v_1(\xi) &= -d_1|\xi|^{-1} - u_0^{1^>}(\theta)|\xi|^{-2}, \\
q_1(\xi) &= -f_1|\xi|^{-1} - p_0^{1^>}(\theta)|\xi|^{-2}.
\end{aligned}
\]
Therefore the functions $p_2(x), u_2(x)$ satisfy:

\[
\begin{cases}
\Delta(u_2 - d_1|x|^{-1}) + \lambda_j(u_2 - d_1|x|^{-1}) + \lambda_j^{<2>} u_0(x) = 0 \\
(u_2 - d_1|x|^{-1})|_{\partial\Omega} = 0
\end{cases}
\]

\[
\begin{cases}
\Delta(p_2 - f_1|x|^{-1}) + \lambda_j(p_2 - f_1|x|^{-1}) + \lambda_j^{<2>} p_0(x) = 0 \\
(p_2 - f_1|x|^{-1})|_{\partial\Omega} = 0.
\end{cases}
\]

From the conditions for solvability of this equation we deduce:

\[
\lambda_j^{<2>} = \lambda_j^{<2>} = 0, \ d_1 = f_1 = 0,
\]

\[p_2(x) = P_2(x) + e_2 u_0(x) + f_2 r_0(x), \ u_2(x) = U_2(x) + c_2 p_0(x) + d_2 r_0(x),\]

where $P_2(x), U_2(x)$ denote the solutions such that:

\[
\int\limits_\Omega U_2 u_0 dx = \int\limits_\Omega U_2 p_0 dx = \int\limits_\Omega U_2 r_0 dx = \int\limits_\Omega P_2 u_0 dx = \int\limits_\Omega P_2 p_0 dx = \int\limits_\Omega P_2 r_0 dx = 0.
\]

From the equations for $v_2(\xi), q_2(\xi)$ we have:

\[
v_2(\xi) = -U_2(0)|\xi|^{-1} - d_2 r_0(0)|\xi|^{-1} - c_1 p_0^{<1>}(\theta)|\xi|^{-2} - u_0^{<2>}(\theta)|\xi|^{-3} - f_2 r_0(0)|\xi|^{-1} - e_1 u_0^{<1>}(\theta)|\xi|^{-2} - p_0^{<2>}(\theta)|\xi|^{-3}.
\]

Finally we write the equations for $u_3(x), p_3(x)$

\[
\begin{cases}
\Delta u_3 + \lambda_j u_3 + \lambda_j^{<3>} u_0 - \lambda_j(U_2(0)|x|^{-1} + d_2 r_0(0)|x|^{-1} + d_2 r_0(0)|x|^{-1} + u_0^{<1>}(\theta)|x|^{-2} = 0 \\\n(u_3 - U_2(0)|x|^{-1} - d_2 r_0(0)|x|^{-1} - u_0^{<1>}(\theta)|x|^{-2})|_{\partial\Omega} = 0
\end{cases}
\]

\[
\begin{cases}
\Delta p_3 + \lambda_j p_3 + \lambda_j^{<3>} p_0 - \lambda_j(P_2(0)|x|^{-1} + f_2 r_0(0)|x|^{-1} + f_2 r_0(0)|x|^{-1} + p_0^{<1>}(\theta)|x|^{-2} = 0 \\\n(p_3 - P_2(0)|x|^{-1} - f_2 r_0(0)|x|^{-1} - p_0^{<1>}(\theta)|x|^{-2})|_{\partial\Omega} = 0.
\end{cases}
\]
Bifurcation of multiple eigenvalues and eigenfunctions

From these conditions we have:

\[ \lambda_j^{<3>} = 3 \int_{\partial B_1} |u_0^{<3>} (\theta)|^2 d\theta, \quad \lambda_{j+1}^{<3>} = 3 \int_{\partial B_1} |P_0^{<3>} (\theta)|^2 d\theta \]

\[ d_2 = [r_0(0)]^{-1} U_2(0), \quad f_2 = [r_0(0)]^{-1} P_2(0). \]

As in the case of double eigenvalues we conclude that \( 3^{-1} \lambda_j^{<3>} \) and \( 3^{-1} \lambda_{j+1}^{<3>} \) are the eigenvalues of the matrix \( M^* \) (see the definition in the introduction) and the vector \( (a_0^1, a_0^2) \) is its eigenvector. So we found \( \lambda_j^{<3>}, \lambda_{j+1}^{<3>}, u_0(x), p_0(x), v_0(\xi), q_0(\xi), v_1(\xi), q_1(\xi). \)

A step of induction: Suppose that \( \lambda_j^{<n+3>}, \lambda_{j+1}^{<n+3>}, u_n(x), p_n(x), v_{n+1}(\xi), q_{n+1}(\xi) \) are found. We shall find \( \lambda_j^{<n+4>}, \lambda_{j+1}^{<n+4>}, u_{n+1}(x), p_{n+1}(x), v_{n+2}(\xi), q_{n+2}(\xi) \) as follows. In previous steps we have known the equations for \( u_{n+1}(x), p_{n+1}(x) \) and found the conditions for their solvability. However, the solutions are defined non-uniquely. Assume that \( U_{n+1}(x), P_{n+1}(x) \) are the solutions such that

\[ \int_\Omega U_{n+1} u_0 dx = \int_\Omega U_{n+1} p_0 dx = \int_\Omega U_{n+1} r_0 dx = 0 \]

\[ \int_\Omega P_{n+1} u_0 dx = \int_\Omega P_{n+1} p_0 dx = \int_\Omega P_{n+1} r_0 dx = 0. \]

The functions \( u_{n+1}(x), p_{n+1}(x) \) may be found in a form:

\[ u_{n+1} = c_{n+1} p_0 + d_{n+1} r_0 + U_{n+1}, \quad p_{n+1} = e_{n+1} u_0 + f_{n+1} r_0 + P_{n+1}. \]

By analogy we have:

\[ u_{n+2} = c_{n+2} p_0 + d_{n+2} r_0 + U_{n+2}, \quad p_{n+2} = e_{n+2} u_0 + f_{n+2} r_0 + P_{n+2} \]

\[ u_{n+3} = c_{n+3} p_0 + d_{n+3} r_0 + U_{n+3}, \quad p_{n+3} = e_{n+3} u_0 + f_{n+3} r_0 + P_{n+3}. \]

From the equations for \( v_{n+2}(\xi), q_{n+2}(\xi) \) we claim that:

\[ v_{n+2}(\xi) = V_{n+2}(\xi) - d_{n+1} A_3(\theta) |\xi|^{-2} - d_{n+2} r_0(0) |\xi|^{-1} - c_{n+1} A_2(\theta) |\xi|^{-2} \]
\[ q_{n+2}(\xi) = Q_{n+2}(\xi) - f_{n+1}A_3(\theta)\abs{\xi}^{-2} - f_{n+2}r_0(0)\abs{\xi}^{-1} - e_{n+1}A_1(\theta)\abs{\xi}^{-2}, \]

where \( A_1(\theta) = u_0^{\leq 1}(\theta), A_2(\theta) = p_0^{\leq 1}(\theta), A_3(\theta) = r_0^{\leq 1}(\theta) \) and \( V_{n+2}(\xi), Q_{n+2}(\xi) \) are defined by the equations as in the case of double eigenvalues.

By analogy we have

\[ v_{n+3}(\xi) = V_{n+3}(\xi) - \{d_{n+1}\{B_3(\theta) - 6^{-1}\lambda_jr_0(0)\} + c_{n+1}B_2(\theta)\}\abs{\xi}^{-3} - \{c_{n+2}A_2(\theta) + d_{n+2}A_3(\theta)\}\abs{\xi}^{-2} - \{d_{n+3}r_0(0) + 6^{-1}d_{n+1}\lambda_jr_0(0)\}\abs{\xi}^{-1} \]

\[ q_{n+3}(\xi) = Q_{n+3}(\xi) - \{f_{n+1}\{B_3(\theta) - 6^{-1}\lambda_jr_0(0)\} + e_{n+1}B_1(\theta)\}\abs{\xi}^{-3} - \{e_{n+2}A_1(\theta) + f_{n+2}A_3(\theta)\}\abs{\xi}^{-2} - \{f_{n+3}r_0(0) + 6^{-1}f_{n+1}\lambda_jr_0(0)\}\abs{\xi}^{-1}, \]

where the functions \( B_1(\theta) = u_0^{\leq 2}(\theta), B_2(\theta) = p_0^{\leq 2}(\theta), B_3(\theta) = r_0^{\leq 2}(\theta), V_{n+3}(\xi), Q_{n+3}(\xi), d_{n+1}, d_{n+2}, f_{n+1}, f_{n+2} \) are defined.

Finally, we write the equations for \( u_{n+4}(x), p_{n+4}(x) : \)

\[ \Delta \bar{u}_{n+4}(x) + \lambda_j\bar{u}_{n+4}(x) + \lambda_j^{\leq n+4}u_0 + \lambda_j^{\leq 3}(U_{n+1} + c_{n+1}p_0 + d_{n+1}r_0) = G_{n+4}(x) \]

(28) \[ \bar{u}_{n+4}|_{\partial\Omega} = H_{n+4}(x) \]

\[ \Delta \bar{p}_{n+4}(x) + \lambda_j\bar{p}_{n+4}(x) + \lambda_j^{\leq n+4}p_0 + \lambda_j^{\leq 3}(P_{n+1} + e_{n+1}u_0 + f_{n+1}r_0) = I_{n+4}(x) \]

(30) \[ \bar{p}_{n+4}|_{\partial\Omega} = K_{n+4}(x) \]

where \( \bar{u}_{n+4}(x) := (u_{n+4} - d_{n+3}r_0(0))\abs{x}^{-1} - c_{n+1}A_2(\theta)\abs{x}^{-2} \)

and \( \bar{p}_{n+4}(x) := (p_{n+4} - f_{n+3}r_0(0))\abs{x}^{-1} - e_{n+1}A_1(\theta)\abs{x}^{-2}. \)
From the conditions for solvability of (28) - (31) we have:

\[ c_{n+1} = (\lambda_j^{<3>} - \lambda_{j+1}^{<3>})^{-1} \left[ \int_{\Omega} G_{n+4} p_0 dx + \int_{\partial \Omega} H_{n+4} \frac{\partial p_0}{\partial n} ds \right] \]

\[ e_{n+1} = (\lambda_j^{<3>} - \lambda_{j+1}^{<3>})^{-1} \left[ \int_{\Omega} I_{n+4} u_0 dx + \int_{\partial \Omega} K_{n+4} \frac{\partial u_0}{\partial n} ds \right] \]

\[ \lambda_j^{<n+4>} = - \left[ \int_{\Omega} G_{n+4} p_0 dx + \int_{\partial \Omega} H_{n+4} \frac{\partial p_0}{\partial n} ds - \lambda_j^{<3>} c_{n+1} \right] \]

\[ \lambda_{j+1}^{<n+4>} = - \left[ \int_{\Omega} I_{n+4} u_0 dx + \int_{\partial \Omega} K_{n+4} \frac{\partial u_0}{\partial n} ds - \lambda_{j+1}^{<3>} e_{n+1} \right] \]

\[ d_{n+3} = (4\pi r_0^2(0))^{-1} \left[ \int_{\Omega} G_{n+4} r_0 dx + \int_{\partial \Omega} H_{n+4} \frac{\partial r_0}{\partial n} ds - \lambda_j^{<3>} d_{n+1} \right] \]

\[ f_{n+3} = (4\pi r_0^2(0))^{-1} \left[ \int_{\Omega} I_{n+4} r_0 dx + \int_{\partial \Omega} K_{n+4} \frac{\partial r_0}{\partial n} ds - \lambda_{j+1}^{<3>} f_{n+1} \right] \]

Our procedure is ended.

3. Proof

We shall prove our results only in the case of double eigenvalues. The case of triple eigenvalues may be proved similarly. Suppose

\[ \alpha_N(x, \varepsilon) = \sum_{i=0}^{N} \varepsilon^i (u_i(x) + v_i(x\varepsilon^{-1})) \]

\[ \beta_N(x, \varepsilon) = \sum_{i=0}^{N} \varepsilon^i (p_i(x) + q_i(x\varepsilon^{-1})) \]

\[ \lambda_i^{(N)}(\varepsilon) = \sum_{i=0}^{N} \lambda_j^{(i)} \varepsilon^i, \lambda_{j+1}^{(N)}(\varepsilon) = \sum_{i=0}^{N} \lambda_{i+1}^{(i)} \varepsilon^i. \]
We have
\[
\Delta \alpha_N(x, \varepsilon) + \lambda_j^{N>}(\varepsilon) \alpha_N(x, \varepsilon) = \sum_{i=0}^{N} \varepsilon_i [\Delta u_i + \sum_{p=0}^{i-1} \lambda_j^{p>} u_{i-p-1} + \sum_{p=0}^{i-2} \lambda_j^{p>}] \\
|x|i-1v_{i-p-2}(\theta) \\
+ \sum_{p=0}^{i-3} \lambda_j^{p>} |x|i-2v_{i-p-3}(\theta) + \sum_{i=0}^{N} \varepsilon_i^i [\Delta \xi v_i(\varepsilon) + \sum_{p=0}^{i-3} \lambda_j^{p>} \tilde{v}_{i-p-3}(\varepsilon)] \\
+ \varepsilon \sum_{i=0}^{N-1} \varepsilon_i^i \lambda_j^{N>}[\sum_{p=N-i}^{N} \varepsilon^p u_p + \sum_{p=N-i-2}^{N} \varepsilon^p \tilde{v}_p^{2>}(x\varepsilon^{-1})} \\
+ \sum_{p=N-i-1}^{N} \varepsilon^{p+1} |x|i-1v_{i-p-2}(\theta) + \sum_{p=N-i-2}^{N} \varepsilon^{p+2} |x|i-2v_{i-p-2}(\theta)].
\]

Obviously
\[
|\Delta \alpha_N(x, \varepsilon) + \lambda_j^{N>}(\varepsilon) \alpha_N(x, \varepsilon)| = 0(\varepsilon^{N+1}|x|-2) = 0(\varepsilon^{N-1}) \quad (x \in \Omega_\varepsilon) \\
\alpha_N|_{\partial \Omega_\varepsilon} = 0(\varepsilon^{N+1}).
\]

By analogy we can see :
\[
|\Delta \beta_N(x, \varepsilon) + \lambda_{j+1}^{N>}(\varepsilon) \beta_N(x, \varepsilon)| = 0(\varepsilon^{N+1}|x|-2) = 0(\varepsilon^{N-1}) \quad (x \in \Omega_\varepsilon) \\
\beta_N|_{\partial \Omega_\varepsilon} = 0(\varepsilon^{N+1}).
\]

Suppose that \( \alpha^*_N(x, \varepsilon) = g_N(\varepsilon)[\alpha_N(x, \varepsilon) - \Gamma_1(x) \sum_{i=0}^{N} \varepsilon^i \tilde{v}_i^{(N-i)}(x\varepsilon^{-1}) - \Gamma_2(x\varepsilon^{-1}) \sum_{i=0}^{N} \varepsilon^i \tilde{u}_i^{(N-i)}(x)] \),

\[
\beta^*_N(x, \varepsilon) = k_N(\varepsilon)[\beta_N(x, \varepsilon) - \Gamma_1(x) \sum_{i=0}^{N} \varepsilon^i \tilde{v}_i^{(N-i)}(x\varepsilon^{-1})} \\
- \Gamma_2(x\varepsilon^{-1}) \sum_{i=0}^{N} \varepsilon^i \tilde{u}_i^{(N-i)}(x)]
\]
where $\Gamma_1(x) \in C^{\infty}(\mathbb{R}^3), \Gamma_1(x) \equiv 1$ in a neighborhood of $\partial \Omega$ and $\Gamma_1(x) = 0$ in a neighborhood of $\{0\}$ and $\Gamma_2(x) \in C_{0}^{\infty}(\mathbb{R}^3)$, $\Gamma_2(x) \equiv 1$ in a neighborhood of $\bar{B}_1$. The constants $g_N(\varepsilon), k_N(\varepsilon)$ are chosen such that

$$\|\alpha^*_N(x, \varepsilon)\|_{L^2(\Omega_\varepsilon)} = \|\beta^*_N(x, \varepsilon)\|_{L^2(\Omega_\varepsilon)} = 1.$$ 

It is easy to see

$$\begin{cases} 
\Delta \alpha^*_N(x, \varepsilon) + \lambda_j^{<N>}(\varepsilon)\alpha^*_N(x, \varepsilon) = L_N(x, \varepsilon) \quad \text{in } \Omega_\varepsilon \\
\alpha^*_N(x, \varepsilon)|_{\partial \Omega_\varepsilon} = 0 
\end{cases}$$

$$\begin{cases} 
\Delta \beta^*_N(x, \varepsilon) + \lambda_{j+1}^{<N>}(\varepsilon)\beta^*_N(x, \varepsilon) = M_N(x, \varepsilon) \quad \text{in } \Omega_\varepsilon \\
\beta^*_N(x, \varepsilon)|_{\partial \Omega_\varepsilon} = 0. 
\end{cases}$$

Expand $\alpha^*_N(x, \varepsilon)$ and $\beta^*_N(x, \varepsilon)$ in the series of orthonormal eigenfunctions $u_1(x, \varepsilon), u_2(x, \varepsilon), \ldots$ in $\Omega_\varepsilon$ one have:

$$\alpha^*_N(x, \varepsilon) = \sum_{i=1}^{\infty} \alpha_i(\varepsilon) u_i(x, \varepsilon) \quad \text{where } \sum_{i=1}^{\infty} \alpha_i^2(\varepsilon) = 1$$

$$\beta^*_N(x, \varepsilon) = \sum_{i=1}^{\infty} \beta_i(\varepsilon) u_i(x, \varepsilon) \quad \text{where } \sum_{i=1}^{\infty} \beta_i^2(\varepsilon) = 1.$$ 

We claim that

$$\Delta \alpha^*_N(x, \varepsilon) = -\sum_{i=1}^{\infty} \lambda_i(\varepsilon) \alpha_i(\varepsilon) u_i(x, \varepsilon) = -\lambda_j^{<N>}(\varepsilon) \sum_{i=1}^{\infty} \alpha_i(\varepsilon) u_i(x, \varepsilon) + L_N(x, \varepsilon).$$

Obviously $|D^\alpha L_N(x, \varepsilon)|_{\Omega_\varepsilon} = 0(\varepsilon^{N+1}|x|^{-|\alpha|})$.

Therefore $|\lambda_j^{<N>}(\varepsilon) - \lambda_j(\varepsilon)| \sim |\lambda_{j+1}^{<N>}(\varepsilon) - \lambda_j^{<N>}(\varepsilon)| = 0(\varepsilon^{N-1}).$

Since we have known $\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda_j$ ($j = 1, \ldots, \infty$) it follows that

$$\|\alpha^*_N(x, \varepsilon) - u_j(x, \varepsilon)\|_{L^2(\Omega_\varepsilon)} \sim \|\beta^*_N(x, \varepsilon) - u_{j+1}(x, \varepsilon)\|_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{N-1}).$$
We have also:

\[ \Delta \{ \alpha^*_N(x, \varepsilon) - u_j(x, \varepsilon) \} + \lambda_j(\varepsilon) \{ \alpha^*_N(x, \varepsilon) - u_j(x, \varepsilon) \} = L_N(x, \varepsilon) - \{ \lambda_j^{<N>}(\varepsilon) - \lambda_j(\varepsilon) \} \alpha^*_N(x, \varepsilon) \quad \text{in } \Omega_\varepsilon, \]

\[ |D^\alpha \{ \alpha^*_N(x, \varepsilon) - u_j(x, \varepsilon) \}|_{\partial \Omega_\varepsilon} = 0(\varepsilon^{N-1-|\alpha|}) \quad \text{for } |\alpha| \leq N - 1, \]

\[ \Delta \{ \beta^*_N(x, \varepsilon) - u_{j+1}(x, \varepsilon) \} + \lambda_{j+1}(\varepsilon) \{ \beta^*_N(x, \varepsilon) - u_{j+1}(x, \varepsilon) \} = M_N(x, \varepsilon) - \{ \lambda_{j+1}^{<N>}(\varepsilon) - \lambda_{j+1}(\varepsilon) \} \beta^*_N(x, \varepsilon) \quad \text{in } \Omega_\varepsilon, \]

\[ |D^\alpha \{ \beta^*_N(x, \varepsilon) - u_{j+1}(x, \varepsilon) \}|_{\partial \Omega_\varepsilon} = 0(\varepsilon^{N-1-|\alpha|}) \quad \text{for } |\alpha| \leq N - 1. \]

From a priori estimates for elliptic boundary value problems we conclude that:

\[ \max_{\overline{\Omega}_\varepsilon} |D^\alpha \{ \alpha^*_N(x, \varepsilon) - u_j(x, \varepsilon) \}| \leq C \varepsilon^{N-1} |x|^{-|\alpha|} \]

\[ \max_{\overline{\Omega}_\varepsilon} |D^\alpha \{ \beta^*_N(x, \varepsilon) - u_{j+1}(x, \varepsilon) \}| \leq C \varepsilon^{N-1} |x|^{-|\alpha|} \]

which completes the proof.

4. The final remark

The author of this note think we can study a bifurcations of any eigenvalues by our method under some conditions (for a bifurcation). These conditions are necessary because the bifurcation may be not occured when \( \Omega \) is the ball (in general, when \( \Omega \) is a domain with some symmetries).

References


(Received January 14, 1994)
(Revised April 25, 1994)

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