On the maximum value of the first coefficients of Kazhdan-Lusztig polynomials for symmetric groups

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Abstract. In this article, we show that \( \max \{ c^{-}(w); w \in S_n \} = \left\lfloor \frac{n^2}{4} \right\rfloor \), where \( c^{-}(w) \) is the number of elements covered by \( w \in S_n \) in the Bruhat order. Using this result, we can see that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials for \( S_n \) equals \( \left\lfloor \frac{n^2}{4} \right\rfloor - n + 1 \).

0. Introduction

Let \((W, S)\) be a Coxeter system and \( \leq_B \) denote the Bruhat order on \( W \). We put

\[
c^{-}(w) = \# \{ y \in W; w \text{ covers } y \text{ in the Bruhat order} \},
\]

\[
g(w) = \# \{ s \in S; s \leq_B w \}.
\]

The purpose of this article is to show that, if \( W \) is the symmetric group \( S_n \) of degree \( n \), the maximum value of \( c^{-}(w) \) (resp. \( c^{-}(w) - g(w) \)) over \( w \in S_n \) is equal to \( \left\lfloor \frac{n^2}{4} \right\rfloor \) (resp. \( \left\lfloor \frac{n^2}{4} \right\rfloor - n + 1 \)), where \( [x] \) denotes the Gaussian symbol, i.e. the greatest integer not exceeding \( x \).

The maximum value of \( c^{-}(w) \) plays a role in solving problems concerning with the Bruhat order with help of computers. Also, by results of Dyer [D] and Irving [I], the maximum value of \( c^{-}(w) - g(w) \) gives the maximum value of the coefficient \( p_1(x, y) \) of \( q \) in the Kazhdan-Lusztig polynomial \( P_{x,y}(q) = \sum_{i \geq 0} p_i(x, y)q^i \).

This article is organized as follows: In Section 1, we associate a poset \( P_x \) to each permutation \( x \in S_n \) and show that \( c^{-}(x) \) (resp. \( c^{-}(x) - g(x) \)) is

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equal to the number of edges of the Hasse diagram of $P_x$ (resp. $n - \text{comp}(P_x)$, where $\text{comp}(P_x)$ is the number of the connected components of the Hasse diagram of $P_x$). In Section 2, we use the Turán’s theorem in the graph theory to evaluate the maximum values of $c^{-}(x)$ and $c^{-}(x) - g(x)$ (Theorem A and B). In Section 3, we combine Theorem A, B with results of Dyer [D] and Irving [I] and prove that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials is given by $[n^2/4] - n + 1$ (Theorem C).

1. **Poset $P_x$ associated to a permutation $x$**

First, we define a poset $P_x$ for $x \in \mathfrak{S}_n$.

**Definition 1.1.** For each integer $n \geq 1$, we put $[n] := \{1, 2, \cdots, n\}$. For $x \in \mathfrak{S}_n$, we define a poset $(P_x, \leq_x)$ as follows:

$$P_x = \{\tilde{i} ; i \in [n]\} \text{ as a set, } \tilde{j} \leq_x \tilde{i} \iff i \leq j \text{ and } x(i) \geq x(j).$$

**Example 1.2.** Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} \in \mathfrak{S}_5$. Then the Hasse diagram of $(P_x, \leq_x)$ is the following.

$$\begin{array}{ccc}
\tilde{1} & \tilde{3} \\
/ & / \\
2 & 4 & \tilde{5}
\end{array}$$

**Remarks 1.3.**

(i) When $n \leq 5$, for any poset $P$ with $n$ elements, there exists $x \in \mathfrak{S}_n$ such that $P_x \simeq P$, where $P \simeq Q$ means that there exists a bijection $f$ from $P$ to $Q$ satisfying $x \leq y$ in $P \iff f(x) \leq f(y)$ in $Q$.

(ii) When $n \geq 6$, the above statement is incorrect. For example, we cannot find $x \in \mathfrak{S}_6$ such that $P_x \simeq P$, where $P$ is a poset with the following Hasse diagram.

(iii) It is easy to check that if $P_x = P_y$, then $x = y$.

Let us recall the definition of the Bruhat order on $\mathfrak{S}_n$ and we define some notations.
**Definition 1.4.** Let $a, b$ be elements in $S_n$. We write $a <' b$ if there exist $i,j$ such that $i < j$, $b(i) > b(j)$ and $a = b(i,j)$, where $(i, j)$ is the permutation switching the number $i$ and $j$ and leaving the other numbers fixed. Then the Bruhat order denoted by $\leq_B$ is defined as follows:

$$x \leq_B y \iff \text{there exist } z_0, z_1, \cdots, z_k \in S_n \text{ such that } x = z_0 <' z_1 <' \cdots <' z_k = y.$$  

For $x, y \in S_n$, we put $\langle x, y \rangle := \{z \in S_n; x \leq_B z \leq_B y\}$, $c^-(x) := \sharp\{z \in [e, x]; \ell(z) = \ell(x) - 1\}$, $G(x) := \{s \in [e, x]; \ell(s) = 1\}$, $g(x) := \sharp G(x)$, where $e$ is the identity element and $\ell$ is the length function (cf. [Hu]). In other words, $c^-(x)$ (resp. $g(x)$) is the number of the coatoms (resp. atoms) of the interval $[e, x]$.

We define some more notations.

**Definition 1.5.** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. We write $x <_P y$ if $y$ covers $x$ in $P$ (i.e. $x <_P z \leq_P y \Rightarrow z = y$). If $P \cap Q = \emptyset$, then we define a new poset $(P + Q, \leq_{P+Q})$ as follows: $P + Q = P \cup Q$ as a set and $x \leq_{P+Q} y$ if and only if (i) $x, y \in P$ and $x \leq_P y$ or (ii) $x, y \in Q$ and $x \leq_Q y$. Also we define a new poset $(P \oplus Q, \leq_{P\oplus Q})$ as follows: $P \oplus Q = P \cup Q$ as a set and $x \leq_{P\oplus Q} y$ if and only if (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$ or (iii) $x \in P$ and $y \in Q$. We put

$$h(P) := \sharp\{(x, y) \in P^2; y <_P x\},$$

$$\text{comp}(P) := \text{the number of the connected components of the Hasse diagram of } P.$$  

In other words, $h(P)$ is the number of edges of the Hasse diagram of $P$. We say that $P$ is connected if and only if $\text{comp}(P) = 1$.

**Remark 1.6.** For $x, y \in S_n$, it is well known that $y <_B x$ if and only if there exist $i,j$ such that $y = x(i,j)$, $i < j$, $x(i) > x(j)$ and $x(k) \leq x(j)$ or $x(i) \leq x(k)$ for any $k \in [i, j]$, where $[i, j] := \{i, i + 1, \cdots, j\}$.

Then we have the following.

**Proposition 1.7.** For $x \in S_n$, we have

(i) $c^-(x) = h(P_x),$

(ii) $g(x) = n - \text{comp}(P_x).$

Before the proof of Proposition 1.7, we prepare some more notations.
Definition 1.8. For $x \in S_n$, we put
\[
C(x) := \{(i, j) ; i < j, x(i, j) \leq_B x\}; \\
H(x) := \{\tilde{(i, j)} ; \tilde{j} \in P_x^2 ; \tilde{j} \prec_x \tilde{i}\}.
\]

Remark 1.9. We can check that $\ell(x) = \#\{\tilde{(i, j)} \in P_x^2 ; \tilde{j} \prec_x \tilde{i}\}$ for any $x \in S_n$.

Proof of Proposition 1.7 (i). We define the map $\eta$ from $C(x)$ to $H(x)$ by $\eta(i, j) := (\tilde{i}, \tilde{j})$. Then, by Remark 1.6 and the definition of $\leq_x$, we have
\[
(i, j) \in C(x) \iff i < j, x(i, j) \leq_B x \\
\iff i < j, x(i) > x(j), x(k) \leq x(j) \text{ or } x(i) \leq x(k) \text{ for any } k \in [i, j] \\
\iff \tilde{j} \prec_x \tilde{i}, x(k) \leq x(j) \text{ or } x(i) \leq x(k) \text{ for any } k \in [i, j] \\
\iff \tilde{j} \leq_x \tilde{i} \\
\iff (\tilde{i}, \tilde{j}) \in H(x).
\]

Hence, $\eta$ is a bijection. It is easy to check that $\#C(x) = c^-(x)$ and $\#H(x) = h(P_x)$. So, we obtain $c^-(x) = h(P_x)$. $\square$

Before the proof of Proposition 1.7 (ii), we will show a lemma.

Lemma 1.10. For $x \in S_n$, we have the following.
(i) If $P_x$ is connected, then $g(x) = n - 1$.
(ii) Let $P_1$ be the connected component of $P_x$ containing $\tilde{1}$. Then $P_1 = \{\tilde{1}, \tilde{2}, \ldots, \tilde{m}\}$ for some $m$ and $x([m]) = [m]$.

Proof. (i) Suppose that $g(x) \neq n - 1$. Then there exists $k \in [n - 1]$ such that $s_1, s_2, \ldots, s_{k-1} \in G(x)$ and $s_k \notin G(x)$, where $s_i := (i, i + 1)$ for each $i \in [n-1]$. If there exist $\tilde{r}, \tilde{m}$ such that $r \in [k], m \in [n] \setminus [k]$ and $\tilde{r}$ and $\tilde{m}$ are comparable, then we have $\tilde{m} <_x \tilde{r}$ (i.e. $r < m$ and $x(r) > x(m)$). On the other hand, since $r \leq k, k + 1 \leq m$ and $s_k \notin G(x)$, we can see that $x(r) \leq k$ and $k + 1 \leq x(m)$. This is a contradiction. So, we can get that every element in $\{\tilde{1}, \tilde{2}, \ldots, \tilde{k}\}$ is incomparable to every element in $\{k + 1, k + 2, \ldots, \tilde{n}\}$. This contradicts the assumption that $P_x$ is connected. Hence, we have
$g(x) = n - 1$. (ii) First, we will show that $P_1 = \{\tilde{1}, \tilde{2}, \ldots, \tilde{m}\}$ as a set. Let $P_1 = \{\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_m\}$, where $1 = i_1 < i_2 < \cdots < i_m$, as a set. Suppose that there exists $k \in [m]$ such that $i_p = p$ for any $p \in [k - 1]$ and $i_k > k$. Then we can see that $\tilde{k} \not\in P_1$ and every element of $P_1$ is incomparable to $\tilde{k}$. Hence, by the inequality $i_1 < i_2 < \cdots < i_{k-1} < k < i_k < \cdots < i_m$, we have $x(i_p) < x(k) < x(i_r)$ for any $p \in [k - 1]$ and for any $r \in [m] \setminus [k - 1]$. This means that every element in $\{\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_{k-1}\}$ is incomparable to every element in $\{\tilde{i}_k, \tilde{i}_{k+1}, \ldots, \tilde{i}_m\}$. This contradicts the assumption that $P_1$ is connected. Next, we will show that $x([m]) = [m]$. Suppose that there exists $k \in [m]$ such that $x(p) \leq m$ for any $p \in [k - 1]$ and $x(k) > m$. Then it follows from $x(k) > m$ that
\[
\#\{\tilde{j}; \tilde{j} \leq x \tilde{k}\} \geq \#\{j; j \geq k, x(j) \leq m\} + 1 = m - k + 2.
\]
On the other hand, we have
\[
\tilde{1}, \tilde{2}, \ldots, \tilde{k} - 1 \not\in \{\tilde{j}; \tilde{j} \leq x \tilde{k}\},
\]
here we use the inequality that $x(p) \leq m < x(k)$ for any $p \in [k - 1]$. Since $P_1$ is connected and $\tilde{k} \in P_1$, we have
\[
P_1 \supset \{\tilde{1}, \tilde{2}, \ldots, \tilde{k} - 1\} \cup \{\tilde{j}; \tilde{j} \leq x \tilde{k}\} \text{ (disjoint union)}.
\]
It follows that we get $\#P_1 \geq m + 1$. This is a contradiction. So, we obtain $x([m]) = [m]$. $\square$

**Proof of Proposition 1.7 (ii).** Let $P_x = P_1 + P_2 + \cdots + P_k$ be the decomposition into connected components, and put $\#P_i = m_i \geq 1$ and $P_i = \{\tilde{p}_{i,1}, \tilde{p}_{i,2}, \ldots, \tilde{p}_{i,m_i}\}$, where $p_{i,1} < p_{i,2} < \cdots < p_{i,m_i}$. We may assume that $p_{1,1} < p_{2,1} < \cdots < p_{k,1}$. Then, for each $i \in [k]$, it follows from Lemma 1.10 (ii) that there exists $x_i \in \mathfrak{S}_{m_i}$ such that $P_i$ is isomorphic to $P_{x_i}$. Hence, by Lemma 1.10 (i), we have
\[
g(x) = g(x_1) + g(x_2) + \cdots + g(x_k)
= (m_1 - 1) + (m_2 - 1) + \cdots + (m_k - 1)
= m_1 + m_2 + \cdots + m_k - k
= n - \text{comp}(P_x). \square
2. The maximum values of $c^-(w)$ and $c^-(w) - g(w)$

In this section, by the Turán’s theorem, we evaluate the maximum values of $c^-(x)$ and $c^-(x) - g(x)$.

**Theorem (Turán).** The maximum number of the edges in $n$-vertex graphs which has no triangles is $[n^2/4]$.

By the Turán’s theorem, we can easily see the following.

**Corollary 2.1.** If $P$ is a poset with $n$ elements, then we have $h(P) \leq [n^2/4]$.

Hence, we have

**Theorem A.**

$$\max\{c^-(x); x \in S_n\} = [n^2/4].$$

**Proof.** By Proposition 1.7 (i) and Corollary 2.1, we have

$$\max\{c^-(x); x \in S_n\} = \max\{h(P_x); x \in S_n\} \leq [n^2/4].$$

We define $z_n \in S_n$ as follows:

$$(z_n(1), z_n(2), \cdots, z_n(n)) := \begin{cases} (m+1, m+2, \cdots, 2m, 1, 2, \cdots, m) & \text{if } n = 2m, \\ (m+1, m+2, \cdots, 2m+1, 1, 2, \cdots, m) & \text{if } n = 2m+1. \end{cases}$$

Then we can see that $c^-(z_n) = [n^2/4]$. Hence, we obtained this theorem. □

Also, we have the following.

**Proposition 2.2.** For a poset $P$ with $n$ elements, we have

$$h(P) - (n - \text{comp}(P)) \leq [n^2/4] - n + 1.$$
Lemma 2.3. Let $P$ be a poset with $n$ elements. If $P = P_1 + P_2 + \cdots + P_k$ is the decomposition into the connected components, then we have

$$h(P) - (n - \text{comp}(P)) \leq h(P') - (n - \text{comp}(P')),$$

where $P' = (P_1 \oplus P_2) + \cdots + P_k$.

Proof. Since $P_1, P_2 \neq \emptyset$, we have $h(P_1 + P_2) + 1 \leq h(P_1 \oplus P_2)$. Hence, we can see $h(P) + 1 \leq h(P')$. So, by the equality $n - \text{comp}(P) = n - \text{comp}(P') - 1$, we obtained this lemma. □

Proof of Proposition 2.2. Let $P = P_1 + P_2 + \cdots + P_k$ be the decomposition into the connected components. Then, by Corollary 2.1 and Lemma 2.3, we have

$$h(P) - (n - \text{comp}(P)) = h(P_1 + P_2 + \cdots + P_k) - (n - k) \leq h((P_1 \oplus P_2) + \cdots + P_k) - (n - k + 1) \leq h((P_1 \oplus P_2 \oplus P_3) + \cdots + P_k) - (n - k + 2) \leq h(P_1 \oplus P_2 \oplus \cdots \oplus P_k) - (n - 1) \leq \lfloor n^2/4 \rfloor - n + 1.$$ 

Hence, we have the following.

Theorem B.

$$\max\{c^-(x) - g(x); x \in \mathfrak{S}_n\} = \lfloor n^2/4 \rfloor - n + 1.$$

Proof. By Proposition 1.7 and Proposition 2.2, we have

$$\max\{c^-(x) - g(x); x \in \mathfrak{S}_n\} = \max\{h(P_x) - (n - \text{comp}(P_x)); x \in \mathfrak{S}_n\} \leq \lfloor n^2/4 \rfloor - n + 1.$$ 

On the other hand, for $z_n$ defined in the proof of Theorem A, we can see that

$$c^-(z_n) - g(z_n) = \lfloor n^2/4 \rfloor - n + 1.$$ 

Hence, we proved Theorem B. □
3. The maximum value of the first coefficient of Kazhdan-Lusztig polynomials

Here, we combine Theorem A, B with results of Dyer [D] and Irving [I] and prove that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials is given by $\lfloor n^2/4 \rfloor - n + 1$.

First, we define Kazhdan-Lusztig polynomials.

**Definition 3.1.** Let $(W, S)$ be a Coxeter system. For $x, w \in W$, we define the Kazhdan-Lusztig polynomial for $x, w$ denoted by $P_{x, w}(q) = \sum_{i \geq 0} p_i(x, w)q^i \in \mathbb{Z}[q]$ as follows:

$$P_{x, x}(q) = 1 \text{ for all } x \in W, \quad P_{x, w}(q) = 0 \text{ if } x \nleq w.$$  

If $x < w$, then choose $s \in S$ satisfying $\ell(sw) < \ell(w)$ and set

$$c := \begin{cases} 
0 & \text{if } x < sx, \\
1 & \text{if } sx < x.
\end{cases}$$  

Then $P_{x, w}(q)$ is defined inductively as follows:

$$P_{x, w}(q) = q^{1-c}P_{sx, sw}(q) + q^c P_{x, sw}(q) - \sum_{sz < z < sw} \mu(z, sw)q^{(\ell(w)-\ell(z))/2}P_{x, z}(q),$$  

where $\mu(z, sw)$ is the coefficient of $q^{(\ell(sw)-\ell(z)-1)/2}$ of $P_{z, sw}(q)$.

**Remark 3.2.** This definition is independent of the choice of $s$ and is equivalent to the original definition in [KL]. See [Hu].

We can obtain the following.

**Theorem C.**

$$\max\{p_1(x, w); x, w \in \mathfrak{S}_n\} = \lfloor n^2/4 \rfloor - n + 1.$$
First coefficients of Kazhdan-Lusztig polynomials

Proof. First, the following statements are valid. $p_1(e, w) = c^-(w) - g(w)$ for any $w \in \mathfrak{S}_n([D])$. $P_{x,z}(q) - P_{y,z}(q)$ has non-negative coefficients for any $x, y, z \in \mathfrak{S}_n$ with $x \leq_B y \leq_B z([I])$. Hence, by virtue of Theorem B, we have

$$\max\{p_1(x, w); x, w \in \mathfrak{S}_n\} \leq \max\{p_1(e, w); w \in \mathfrak{S}_n\}$$
$$= \max\{c^-(w) - g(w); w \in \mathfrak{S}_n\}$$

In particular, for $z_n$ defined in the proof of Theorem A, we have

$$p_1(e, z_n) = [n^2/4] - n + 1. \quad \square$$

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References


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