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Nondegeneracy and Single-point-blowup for Solution of the Heat Equation with a Nonlinear Boundary Condition

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Abstract. This paper studies the nondegeneracy of the blowup limit and the single-point-blowup for the heat equation $u_t = \Delta u$ with the nonlinear boundary condition $u_n = u^p$ on $\partial\Omega \times [0, T)$. Under certain blowup rate assumption (which was established recently under some assumptions on the initial data), we prove that the blowup limit is nontrivial at the blowup point. We also establish that the single-point-blowup occurs in two space dimensional radially symmetric domain with non-radially symmetric initial data with only one "hill" on the boundary.

1. Introduction

Let us consider the following heat equation with a nonlinear boundary condition:

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{for } x \in \Omega,\ t > 0, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = u^p \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega \qquad (u_0(x) \geq 0), \tag{1.3}$$

where Ω is a bounded domain in R^n with boundary $\partial\Omega \in C^{2+\alpha}$ $(0 < \alpha < 1)$, n is the exterior normal vector on $\partial\Omega$, $p > 1$ and $u_0(x) \geq 0$.

It is known for a long time (cf. [19], [20], [22]) that the solution blows up, for certain $u_0(x)$, and in [6] for all $u_0(x) \not\equiv 0$. If u^p is replaced by a

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general nonlinear function $f(u)$, a necessary and sufficient condition was found in [22] for the problem to have a finite time blowup.

In the one space dimensional case as well as a radially symmetric domain in R^n, the blowup set, blowup rate and asymptotic blowup limit were obtained (see [6] [7]) under certain monotonicity assumptions on the initial data. The blowup rate is also studied in [3] with more general initial data. Similar questions are also studied in [5] for equation (1.1) with an additional competing absorption term $-cu^q$ with the boundary and initial conditions (1.2)–(1.3).

The problem for a general domain in several space dimension is more difficult than the one dimensional case. The local existence theorem for $u_0 \in L^q$ $(q > n(p-1)/2)$ for the system (1.1)–(1.3) is obtained in [17]. Using the integral equation method, partial results on blowup were obtained in [25]. In our recent papers [15] [16], the blowup rate is established for all subcritical p's, namely, for $1 < p < \frac{n}{n-2}$ in the case $n \geq 3$ and $1 < p < \infty$ in the case $n = 2$, under the assumption $\Delta u_0(x) \geq 0$. The asymptotic blowup limit is also discussed.

There are a lot of similarities between the system (1.1)–(1.3) and the equation:

$$u_t - \Delta u = u^p, \qquad (p > 1). \tag{1.4}$$

For this equation, questions like blowup rate, blowup limit, nondegeneracy, single-point-blowup or finite-point-blowup were studied extensively by a number of authors (cf. [1], [2], [4], [8], [11]-[13], [21], [23]-[24], etc.). In [8] various results regarding to the blowup rate and blowup set were obtained. Later, by introducing self-similarity variables, the authors of [12] eliminated the monotonicity condition and obtained the rate estimates for subcritical p's, namely, for $p \in (1, \frac{n+2}{n-2})$. Moreover, the asymptotic behavior near the blowup time was obtained in [11]–[13]. Questions like single-point-blowup or finite-point-blowup were studied in [8], [4], etc..

This work is a continuation of our works [15] [16]. We state the nondegeneracy result as follows.

Suppose that

$$\max_{x \in \overline{\Omega}} u(x,t) \leq \frac{P}{(T-t)^\beta}, \quad \beta = \frac{1}{2(p-1)}, \tag{1.5}$$

for some positive constant P. If for some $K > 0$

$$\liminf_{t\to T}(T-t)^{\beta}\inf_{|y|\le K} u(a+y\sqrt{T-t},t)=0, \tag{1.6}$$

then a is not a blowup point, namely, $u(x,t)$ is uniformly bounded in a neighborhood of the point a. In another words, the blowup limit cannot be 0 if a is a blowup point. As mentioned above, the assumption (1.5) is valid (see [15], [16]) for subcritical p's with monotonicity assumptions on the initial data (i. e., $u_t(x,0)=\Delta u_0(x)\ge 0$).

A natural question is whether it is possible to have a single-point-blowup. For equation (1.4) in one space dimensional case, Chen-Matano [4] studied the number of blowup points by looking at the sign of the u_x (actually, more general $f(u,t)$ in place of u^p is studied in [4]). For the system (1.1)–(1.2), one can also study the one space dimensional problem or the radially symmetric data on a radially symmetric domain (which is essentially one space dimensional). However, such a system must have blowup points *everywhere* on the boundary, if blowup ever occurs, owing to the symmetry of the data. Here, we shall study a two space dimensional problem with radially symmetric domain, but with non-radially symmetric data. Therefore the problem remains two dimensional. We shall establish the single-point-blowup for those monotone initial data with only one *hill* on the boundary. Our result is as follows.

Let $n=2$, $\Omega=B_1(0)=\{(x_1,x_2);\ x_1^2+x_2^2<1\}$ and $1<p<\infty$. We assume that $u_0(r,\theta)=\widetilde{u_0}(x)$ $(x_1=r\cos\theta,\ x_2=r\sin\theta)$ is C^2 and satisfies:

$$u_0\ge 0,\quad \Delta_x\widetilde{u_0}\ge 0\quad \textit{for } x\in B_1(0), \tag{1.7}$$

$$\frac{\partial u_0}{\partial r}=u_0^p\quad \textit{for } |x|=1, \tag{1.8}$$

$$\frac{\partial u_0}{\partial \theta}(r,\theta)<0\quad \textit{for } 0<r<1,\ 0<\theta<\pi, \tag{1.9}$$

$$u_0(r,\theta)=u_0(r,2\pi-\theta)\quad \textit{for } 0\le r\le 1,\ 0\le\theta\le\pi. \tag{1.10}$$

Then $(r,\theta)=(1,0)$ is the only blowup point.

In section 2, we shall establish a local estimate, the proof is purely based on the regularity (Hölder's estimate, etc.) of parabolic equations. This local estimate, together with the energy estimates in section 3, gives us the nondegeneracy result.

It turns out that the nondegeneracy result is a powerful tool to study whether single-point-blowup will occur. By the monotonicity of the solution, all asymptotically self similar solution will have to converge to the one dimensional solution of the limit equation, if we have *more than two blowup points*. This gives us a sharp estimate on the rate of $\partial u/\partial\theta$ as $t \to T-0$, which leads to the single-point-blowup result in section 4.

2. Local estimates

Suppose that u satisfies the equations (1.1)–(1.3) and that

$$(2.1)\ u(x,t) \le \frac{\varepsilon}{(T-t)^\beta} \quad \text{for } (x,t) \in Q_\delta(a) \equiv \left(B_\delta(a) \cap \overline{\Omega}\right) \times (T-\delta^2, T),$$

for some $a \in \partial\Omega$ and $\delta > 0$. (If $a \in \Omega$, then [16, Theorem 4.1] implies that a is not a blowup point). We want to show that a is not a blowup point if ε is small enough.

PROPOSITION 2.1. *There exists $\varepsilon_0 > 0$, depending only on p, n and $C^{2,\alpha}$ norm of $\partial\Omega$ (it is independent of δ), such that if* (2.1) *is valid for some $\varepsilon \le \varepsilon_0$, then*

$$u(x,t) \le C \quad \textit{for } x \in B_{\delta/2^m}(a) \cap \overline{\Omega},\ \ 0 < t < T,$$

for some $m > 1$.

For the application of this proposition later on, it is important to keep ε_0 to be independent of δ. We divide the proof into two lemmas. In the proof of the following Lemma 2.2, the scaling argument, together with the parabolic Hölder's estimates, gives a function inequality which will imply a better rate estimate than that in (2.1). Similar procedure will then be iterated in Lemma 2.3 to obtain Proposition 2.1. Therefore, Proposition 2.1 can be viewed as a direct consequence of the regularity theory of parabolic equations.

LEMMA 2.2. *There exists $\varepsilon_0 > 0$, depending only on p, n and $C^{2,\alpha}$ norm of $\partial\Omega$, such that if* (2.1) *is valid for some $\varepsilon \le \varepsilon_0$, then*

$$(2.2) \qquad u(x,t) \le \frac{C(\varepsilon,\delta,p,\partial\Omega)}{(T-t)^\eta} \quad \textit{for } x \in B_{\delta/2}(a) \cap \overline{\Omega}, 0 < t < T,$$

where $\eta = \max\Big(\dfrac{\beta}{2}, \beta - \dfrac{1}{2}\Big)$.

PROOF. For simplicity, we let $a = 0$. Take a cutoff function $\zeta(x)$ such that

$$\begin{gathered}\zeta(x) = \begin{cases} 1 & \text{for } |x| \le \delta/2 \\ 0 & \text{for } |x| \ge 3\delta/4 \end{cases}, \qquad \frac{\partial \zeta}{\partial n}(x) = 0 \quad \text{on } \partial\Omega, \\ 0 \le \zeta(x) \le 1, \quad |\nabla \zeta(x)| \le \frac{C}{\delta}, \quad |D^2\zeta(x)| \le \frac{C}{\delta^2}.\end{gathered} \tag{2.3}$$

Then the function $v = \zeta u$ satisfies the equations

$$\frac{\partial v}{\partial t} - \Delta v = -2\nabla\zeta \cdot \nabla u - u\Delta\zeta \equiv f(x,t) \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$\frac{\partial v}{\partial n} = u^{p-1} v \quad \text{on } \partial\Omega \times [0,T]. \tag{2.5}$$

For each $(x^*, t^*) \in \Big(\overline{B_{3\delta/4}(0)} \cap \overline{\Omega}\Big) \times [T - \delta^2, T)$, we set

$$L(t^*) = \max_{\{x \in \overline{B_\delta(0)} \cap \overline{\Omega},\ 0 \le t \le t^*\}} u(x,t)$$

and introduce the scaling

$$\varphi(y,s) = \frac{1}{L(t^*)} u(\lambda y + x^*, \lambda^2 s + t^*), \qquad \lambda = \frac{1}{L^{(p-1)}(t^*)}.$$

The parabolic Schauder's estimate (see [9], [18]) then easily leads to

$$|\nabla u(x^*, t^*)| \le C L^p(t^*) \le \frac{C}{(T - t^*)^{(1/2)+\beta}} \quad \text{for } x^* \in \overline{B_{3\delta/4}} \cap \overline{\Omega},\ 0 < t^* < T.$$

Let

$$M(t) = \max_{x \in \overline{\Omega}, 0 \le \tau \le t} v(x, \tau). \tag{2.6}$$

For each $\widetilde{t} \in [T - \varepsilon^{2(p-1)}\delta^2, T)$ (we assume that $\varepsilon^{2(p-1)} < 1/4$), either

$$M(\widetilde{t}\,) \le M(2\widetilde{t} - T), \tag{2.7}$$

or

$$M(\widetilde{t}\,) > M(2\widetilde{t} - T). \tag{2.8}$$

In the following discussion, we shall assume that (2.8) is valid. It follows that there exists (x^*, t^*) such that $x^* \in \overline{\Omega} \cap \{|x| \leq 3\delta/4\}$, $t^* \in (2\widetilde{t} - T, \widetilde{t}\,]$ and $M(\widetilde{t}\,) = v(x^*, t^*)$.

We shall use the scaling argument analogous as in [10]. Introduce the rescaled function

$$(2.9)\; \varphi_\lambda(y, s) = \frac{1}{M(\widetilde{t}\,)} v(\lambda y + x^*, \lambda^2 s + t^*) \quad \text{for } y \in \overline{\Omega_\lambda}, -\frac{\delta^2}{2\lambda^2} \leq s \leq 0,$$

where $\Omega_\lambda = \{y;\ \lambda y + x^* \in \Omega\}$. If we choose

$$(2.10) \qquad \lambda = \frac{\sqrt{T - t^*}}{\varepsilon^{p-1}},$$

then $\lambda \leq \sqrt{2}\delta$. So the function φ_λ solves

$$\begin{aligned}
&\frac{\partial \varphi_\lambda}{\partial s} = \Delta_y \varphi_\lambda + \frac{\lambda^2}{M(\widetilde{t}\,)} \widetilde{f}(y, s) \quad \text{for } y \in \overline{\Omega_\lambda}, -\frac{1}{4} \leq s \leq 0, \\
&\frac{\partial \varphi_\lambda}{\partial n} = b(y, s)\varphi_\lambda \quad \text{for } y \in \partial\Omega_\lambda, -\frac{1}{4} \leq s \leq 0, \\
&0 \leq \varphi_\lambda(y, s) \leq 1, \quad \text{for } y \in \overline{\Omega_\lambda}, -\frac{1}{4} \leq s \leq 0,
\end{aligned}$$

where $\widetilde{f}(y, s) = f(x, t)$ and $b(y, s) = \lambda u^{p-1}(x, t)$. By (2.1),

$$|b(y, s)\varphi_\lambda| \leq |b(y, s)| \leq \frac{1}{\varepsilon^{p-1}} \varepsilon^{p-1} \leq 1;$$

it is clear that for $\widetilde{f}(y, s)$,

$$|\widetilde{f}(y, s)| \leq \frac{C(\delta)}{(T - t^*)^{(1/2)+\beta}} + \frac{C(\delta)}{(T - t^*)^\beta}.$$

Therefore, we can apply the parabolic interior-boundary Hölder's estimates (see [18], or one can simply write the solution in terms of the Green's function for the Neumann data, and obtain Hölder's estimate immediately) to obtain

$$\|\varphi_\lambda\|_{C^{2\sigma,\sigma}(\overline{\Omega_\lambda} \times \{-1/8 \leq s \leq 0\})} \leq C_1\Big(1 + \frac{\lambda^2}{M(\widetilde{t}\,)} \|\widetilde{f}\,\|_{L^\infty}\Big)$$

for some universal constants C_1 and $\sigma \in (0, 1/2)$ depending only on n and $\partial\Omega$. It follows that, in terms of v,

$$\begin{aligned} &v(x^*, t^*) - v(x^*, \lambda^2 s + t^*) \\ &\quad \le C_1 M(\widetilde{t}\,)\Big(1 + \frac{\lambda^2}{M(\widetilde{t}\,)}\|\widetilde{f}\,\|_{L^\infty}\Big)|s|^\sigma \quad \text{for } -\frac{1}{8} < s < 0. \end{aligned}$$

Therefore, for $-1/8 < s < 0$,

$$M(\widetilde{t}\,) \le C_1 M(\widetilde{t}\,)|s|^\sigma + M(\lambda^2 s + t^*) + \frac{C(\delta, \varepsilon)}{(T - t^*)^{\beta - 1/2}}|s|^\sigma. \tag{2.11}$$

We now let

$$s = \frac{1}{\lambda^2}\Big(2\widetilde{t} - T - t^*\Big) = \varepsilon^{2(p-1)}\frac{2\widetilde{t} - T - t^*}{T - t^*}.$$

Since $t^* \in (2\widetilde{t} - T, \widetilde{t}\,]$, we have $-\varepsilon^{2(p-1)} \le s < 0$. We assume that $\varepsilon^{2(p-1)} < 1/8$. Then from (2.11),

$$\begin{aligned} (1 - C_1\varepsilon^{2(p-1)\sigma})M(\widetilde{t}\,) &\le M(2\widetilde{t} - T) + \frac{C(\delta, \varepsilon)}{(T - t^*)^{\beta - 1/2}} \\ &\le M(2\widetilde{t} - T) + C(\delta, \varepsilon)\Big(1 + \frac{1}{(T - \widetilde{t}\,)^{\beta - 1/2}}\Big) \\ &\qquad \text{for } T - \varepsilon^{2(p-1)}\delta^2 \le \widetilde{t} < T \quad \text{with (2.8) holds.} \end{aligned}$$

It is obvious that the above inequality is valid when (2.7) holds. Let $T - \widetilde{t} = \tau$ and $g(\tau) = M(T - \tau)$, then

$$\begin{aligned} (1 - C_1\varepsilon^{2(p-1)\sigma})g(\tau) &\le g(2\tau) + C(\delta, \varepsilon)\Big(1 + \frac{1}{\tau^{\beta - 1/2}}\Big) \\ &\qquad \text{for } 0 < \tau \le \varepsilon^{2(p-1)}\delta^2. \end{aligned}$$

Recalling that $\eta = \max\Big(\frac{\beta}{2}, \beta - \frac{1}{2}\Big) > 0$, (in fact, the proof works for any $\eta > 0$ if $\beta - \frac{1}{2} \le 0$), we have

$$2^\eta(1 - C_1\varepsilon^{2(p-1)\sigma})\tau^\eta g(\tau) \le (2\tau)^\eta g(2\tau) + C(\delta, \varepsilon, p) \quad \text{for } 0 < \tau \le \varepsilon^{2(p-1)}\delta^2.$$

We now take ε to be small enough such that

$$0 < \varepsilon \le \varepsilon_0, \qquad 2^\eta(1 - C_1\varepsilon_0^{2(p-1)\sigma}) \ge 1 + \frac{2^\eta - 1}{2}, \quad \varepsilon_0^{2(p-1)} < \frac{1}{8}. \tag{2.12}$$

Then

$$\left(1+\frac{2^\eta-1}{2}\right)\tau^\eta g(\tau)\le(2\tau)^\eta g(2\tau)+C(\delta,\varepsilon,p)\quad\text{for }0<\tau\le\varepsilon^{2(p-1)}\delta^2.$$

This inequality, together with the continuity of $\tau^\eta g(\tau)$, gives

$$\tau^\eta g(\tau)\le\frac{2}{2^\eta-1}C(\delta,\varepsilon,p)+\max_{\varepsilon^{2(p-1)}\delta^2/2\le\sigma\le\varepsilon^{2(p-1)}\delta^2}\sigma^\eta g(\sigma),$$

for $0<\tau\le\varepsilon^{2(p-1)}\delta^2$. Rewriting this inequality in terms of $M(t)$, we obtain,

$$M(t)\le\frac{C(\varepsilon,\delta,p)}{(T-t)^\eta},\quad 0<t<T. \tag{2.13}$$

It is clear that ε_0 dependents only on p and C_1. Therefore ε_0 depends only on p, n and the $C^{2+\alpha}$ norm of $\partial\Omega$. The lemma is proved. □

Next, we prove

LEMMA 2.3. *If*

$$u(x,t)\le\frac{C}{(T-t)^\eta}\quad\text{for } x\in B_\delta(a)\cap\overline{\Omega},\ 0<t<T, \tag{2.14}$$

for some $\eta<\beta$ *and some* $C>0$, *then*

$$u(x,t)\le C(\delta,p,\Omega)\quad\text{for } x\in B_{\delta/2^m}(a)\cap\overline{\Omega},\ 0<t<T, \tag{2.15}$$

for some $m>1$.

PROOF. For each $(x^*,t^*)\in\left(\overline{B_{3\delta/4}(a)}\cap\overline{\Omega}\right)\times[T/2,T)$, we let

$$L(t^*)=\max_{\{x\in\overline{B_\delta(a)}\cap\overline{\Omega},\ 0\le t\le t^*\}}u(x,t).$$

Similar to Lemma 2.2, the parabolic Schauder's estimate (see [9], [18]) implies that

$$|\nabla u(x^*,t^*)|\le CL^p(t^*)\le\frac{C}{(T-t^*)^{p\eta}}. \tag{2.16}$$

Define function ζ as in (2.3), and define $v=\zeta u$ as before. We apply the same procedure as in (2.4)–(2.9) except that this time we define

$$\lambda=(T-t^*)^{\eta(p-1)}. \tag{2.17}$$

The function $b(y,s) = \lambda u^{p-1}(x,t)$ then satisfies

$$|b(y,s)\varphi_\lambda| \le (T-t^*)^{\eta(p-1)} \frac{C}{(T-t^*)^{\eta(p-1)}} \le C.$$

Similarly, $\widetilde{f}(y,s)$ satisfies

$$|\widetilde{f}(y,s)| \le \frac{C(\delta)}{(T-t^*)^{p\eta}}.$$

Similar to (2.11), we now have

$$M(\widetilde{t}\,) \le C_1 M(\widetilde{t}\,)|s|^\sigma + M(\lambda^2 s + t^*) + \frac{C(\delta,p)}{(T-t^*)^{\eta(2-p)}}|s|^\sigma. \tag{2.18}$$

Define s the same way as before, namely,

$$s = \frac{1}{\lambda^2}\Big(2\widetilde{t} - T - t^*\Big) = \frac{2\widetilde{t} - T - t^*}{(T-t^*)^{2\eta(p-1)}}.$$

Since $\eta < \beta$, we easily conclude

$$-\frac{1}{8} \le -\Big(T-\widetilde{t}\Big)^{1-2\eta(p-1)} = -\Big(T-\widetilde{t}\Big)^{1-\eta/\beta} \le s < 0,$$

provided $T - \widetilde{t}$ is small enough. Substitute this into (2.18), we obtain

$$\begin{aligned} M(\widetilde{t}\,) \le\ & C_1 M(\widetilde{t}\,)|T-\widetilde{t}|^{\sigma(1-\eta/\beta)} \\ & + M(2\widetilde{t} - T) \\ & + C(\delta,p)\bigg(1 + \frac{1}{(T-\widetilde{t}\,)^{\eta-(p-1)\eta-\sigma(1-\eta/\beta)}}\bigg). \end{aligned} \tag{2.19}$$

Similar to the proof of Lemma 2.2, we now obtain a better rate estimate as follows.

$$u(x,t) \le M(t) \le \frac{C(\delta,p,\eta)}{(T-t)^\theta} \quad \text{for } x \in B_{\delta/2}(a) \cap \overline{\Omega},\ \ 0 < t < T,$$

where

$$\theta = \max\bigg(\frac{1}{2(2p-1)}, \eta - (p-1)\eta - \sigma(1-\eta/\beta)\bigg).$$

For any $\eta < \beta$, (notice that $0 < \sigma < 1/2$)

$$(p-1)\eta + \sigma(1-\eta/\beta) = \sigma + \eta\big[(p-1) - 2\sigma(p-1)\big] \ge \sigma.$$

Therefore by repeating this procedure finitely many times (the exponent will be reduced by at least σ each time we apply the procedure, if the resulting exponent is still $\geq 1/(4p-2)$), we obtain

$$(2.20)\ u(x,t) \leq \frac{C}{(T-t)^{1/(4p-2)}} \quad \text{for } x \in B_{\delta/2^{m-1}}(a) \cap \overline{\Omega},\ \ 0<t<T,$$

for some $m > 1$. But then the regularity obtained from the scaling argument as in (2.16) gives

$$\begin{aligned} \left|\frac{\partial u}{\partial t}(x,t)\right| &\leq \frac{C}{(T-t)^{(2p-1)/(4p-2)}} \\ &= \frac{C}{\sqrt{T-t}} \quad \text{for } x \in B_{\delta/2^m}(a) \cap \overline{\Omega},\ \ 0<t<T, \end{aligned}$$

which implies that

$$u(x,t) \leq u(x,0) + \int_0^t u_t(x,\tau)d\tau \leq C \quad \text{for } x \in B_{\delta/2^m}(a) \cap \overline{\Omega},\ \ 0<t<T.$$

The lemma is proved. □

3. Nondegeneracy of blowup limit

We will assume throughout this section that

$$(3.1) \qquad \max_{x\in\overline{\Omega}} u(x,t) \leq \frac{P}{(T-t)^\beta}, \quad \beta = \frac{1}{2(p-1)},$$

for some positive constant P. This estimate is valid if we assume the following (see [15], [16])

$$(3.2) \qquad 1<p<\infty \quad \text{for } n=2, \qquad 1<p<\frac{n}{n-2} \quad \text{for } n\geq 3,$$

$$(3.3) \qquad \partial\Omega \in C^{2+\alpha} \text{ for some } \alpha \in (0,1),$$

$$(3.4) \qquad u_0 \geq 0, \quad \Delta u_0 \geq 0 \quad \text{for } x \in \Omega,$$

$$(3.5) \qquad \frac{\partial u_0}{\partial n} = u_0^p \quad \text{for } x \in \partial\Omega.$$

As in Giga and Kohn [11]–[13], we introduce the scaled solution:

$$\begin{aligned} &w_a(y,s) = (T-t)^\beta u(x,t), \\ &x-a = y\sqrt{T-t}, \quad T-t = e^{-s}, \end{aligned}$$

where a is a fixed point on $\partial\Omega$. If u solves (1.1), then w_a solves

$$\begin{aligned}\frac{\partial}{\partial s}w_a - \Delta w_a + \frac{1}{2}y\cdot\nabla_y w_a + \beta w_a = 0 \quad \text{in } W\\ \frac{\partial w_a}{\partial n} = w_a^p \quad \text{on } \partial_p W,\end{aligned} \tag{3.6}$$

where

$$W = \bigcup_{s>s_0+1}\Omega_a(s),$$

and

$$\Omega_a(s) = \{(y,s);\ e^{-s/2}y + a \in \Omega\}, \quad s_0 = -\ln T.$$

The estimate (3.1) implies the following estimates

$$\left|\frac{\partial}{\partial x_j}u(x,t)\right| \le C(P,p,\Omega)\max_{x\in\overline{\Omega}}u^p(x,t) \le \frac{C(P,p,\Omega)}{(T-t)^{p\beta}},$$

$$\left|\frac{\partial^2}{\partial x_j\partial x_k}u(x,t)\right| \le C(P,p,\Omega)\max_{x\in\overline{\Omega}}u^{2p-1}(x,t) \le \frac{C(P,p,\Omega)}{(T-t)^{(2p-1)\beta}},$$

(see the proof of [16, Theorem 3.1]). As a consequence,

$$0 \le w_a(y,s) \le C(P,p,\Omega) \quad \text{for } (y,s)\in W, \tag{3.7}$$

$$\left|\frac{\partial}{\partial y_j}w_a(y,s)\right| \le C(P,p,\Omega) \quad \text{for } (y,s)\in W, \tag{3.8}$$

$$\left|\frac{\partial^2}{\partial y_j\partial y_k}w_a(y,s)\right| \le C(P,p,\Omega) \quad \text{for } (y,s)\in W, \tag{3.9}$$

where the constants C are independent of the point a. For the following "energy" functional (as in Giga and Kohn [12])

$$E_a(s) = \frac{1}{2}\int_{\Omega_a(s)}\Big(\rho|\nabla_y w_a|^2 + \beta\rho w_a^2\Big)\,dy - \frac{1}{p+1}\int_{\partial\Omega_a(s)}\rho w_a^{p+1}dS, \tag{3.10}$$

$\Big($where $\rho = \exp(-y^2/4)\Big)$, it is established in [16, section 5] that

$$\begin{aligned}\frac{d}{ds}E_a(s) \equiv -\int_{\Omega_a(s)}\rho\left|\frac{\partial w_a}{\partial s}\right|^2 dy + J_a(s),\\ J_a(s) \le C^* e^{-s/4},\\ \frac{d}{ds}\Big[E_a(s) + 4C^* e^{-s/4}\Big] \le 0,\end{aligned} \tag{3.11}$$

and

$$\int_{s_0}^{\infty}\int_{\Omega_a(s)}\rho\left|\frac{\partial w_a}{\partial s}\right|^2 dyds<\infty. \tag{3.12}$$

A careful examination of the proof given in [16] indicates that the constant C^* in inequality (3.11) is independent of the point a; it dependents only on n, p, Ω, and P. Notice that (3.11) claims that the "energy $E_a(s)$" has a limit "$E_a(\infty)$". The energy $E_a(s)$ is the difference of two terms:

$$E_a^1(s)=\int_{\Omega_a(s)}\left(\rho|\nabla_y w_a|^2+\beta\rho w_a^2\right)dy,$$
$$E_a^2(s)=\int_{\partial\Omega(s)}\rho w_a^{p+1}dS.$$

It is proved in [16] that the limits of both terms exist, and

$$\lim_{s\to\infty}E_a^1(s)=\frac{2(p+1)}{p-1}E_a(\infty),\quad \lim_{s\to\infty}E_a^2(s)=\frac{2(p+1)}{p-1}E_a(\infty). \tag{3.13}$$

Lemma 3.1. *Let the assumption* (3.1) *be in force. If for some* $K>0$

$$\liminf_{t\to T}(T-t)^\beta\inf_{|y|\le K}u(a+y\sqrt{T-t},t)=0, \tag{3.14}$$

then,

$$\limsup_{t\to T}(T-t)^\beta\sup_{|y|\le K}u(a+y\sqrt{T-t},t)=0. \tag{3.15}$$

Proof. Without loss of generality we may assume that the exterior normal direction at the point a is $(-1,0,\cdots,0)$. By (3.14), there exists (y_j,s_j) such that

$$\lim_{j\to\infty}y_j=\widetilde{y}\in\overline{R_+^n},\quad \lim_{j\to\infty}s_j=+\infty,\quad\text{and }\lim_{j\to\infty}w_a(y_j,s_j)=0, \tag{3.16}$$

where $R_+^n=\{(y_1,\cdots,y_n);\ y_1>0\}$. From (3.7)–(3.9), there exists a further subsequence of s_j's, still denoted by s_j, such that $w_a(y,s+s_j)$ converges uniformly on any compact set to a function $w_a^\infty(y,s)$. The estimate (3.12) implies that w_a^∞ is independent of s. Hence

$$-\Delta_y w_a^\infty+\frac{1}{2}y\cdot\nabla_y w_a^\infty+\beta w_a^\infty=0\quad\text{in } R_+^n, \tag{3.17}$$

$$\frac{\partial w_a^\infty}{\partial n}=(w_a^\infty)^p\quad\text{on } y_1=0. \tag{3.18}$$

It is clear that $w_a^\infty(y) \geq 0$ and, by (3.16), $w_a^\infty(\tilde{y}) = 0$. If $\tilde{y} \in R_+^n$, then $w_a^\infty(y) \equiv 0$, by strong maximum principle. If $\tilde{y} \in \partial R_+^n$, then by (3.18), $w_a^\infty(\tilde{y}) = \frac{\partial}{\partial n} w_a^\infty(\tilde{y}) = 0$. Thus $w_a^\infty(y) \equiv 0$, by Hopf's lemma. It follows that $E_a(\infty) = E_a[w_a^\infty] = 0$. Thus by (3.13),

$$\lim_{s\to\infty} E_a^1(s) = \lim_{s\to\infty} \int_{\Omega_a(s)} \Big(\rho|\nabla_y w_a|^2 + \beta\rho w_a^2\Big)\, dy = 0. \tag{3.19}$$

The estimates (3.7)–(3.9) implies that, for any given sequence $s_j \to \infty$, there is a further subsequence $\{s_{j_k}\}$ such that $w_a(y, s_{j_k})$ converges uniformly on any compact set to a limit function as $s_j \to \infty$. (3.19) implies that this limit function has to be identically 0. Thus $w_a(y, s_{j_k})$ converges to 0 uniformly on any compact set. Since the limit function $w_a^\infty \equiv 0$ is independent of the choices of the sequences $\{s_j\}$, the function $w_a(y, s)$ has to converge to 0 uniformly on any compact set, as $s \to \infty$ (not just on subsequences). The lemma is proved. □

We next prove

LEMMA 3.2. *Let the assumption* (3.1) *be in force. If*

$$u(x,t) \leq \frac{\varepsilon}{(T-t)^\beta} \quad \textit{for } (x,t) \in \Big(B_\delta(a) \cap \partial\Omega\Big) \times (T-\delta^2, T), \tag{3.20}$$

then there exists $c = c(\varepsilon, \delta, P, p, n) > 0$ *such that*

$$u(x,t) \leq \frac{3\varepsilon}{(T-t)^\beta} \quad \textit{for } x \in \overline{\Omega}, |x-a| \leq \frac{\sqrt{\varepsilon}\delta}{\sqrt{P}},\ T-c < t < T. \tag{3.21}$$

PROOF. It is easy to check that (assuming that $a = 0$)

$$\frac{\varepsilon}{(T-t)^\beta}\left(1 + \frac{P|x|^2}{\varepsilon\delta^2}\right) + L \quad \Big(L = \frac{P}{(\varepsilon\beta\delta^2/2nP)^\beta} \geq \max_{\{\tau = T-\varepsilon\beta\delta^2/2nP\}} u(x,\tau)\Big)$$

is a supersolution for $T - \frac{\varepsilon\beta\delta^2}{2nP} < t < T$, $|x| < \delta$. Restricting x to $|x| \leq \frac{\sqrt{\varepsilon}\delta}{\sqrt{P}}$ and further restricting t to $t \geq T - (\varepsilon/L)^{2(p-1)}$, we obtain (3.21). □

We next derive some inequalities. Multiplying the equation (3.6) with $w_a\rho$ and integrating over $\Omega_a(s)$ with respect to y, we obtain

$$E_a^1(s) - E_a^2(s) = -\int_{\Omega_a(s)} \rho w_a \frac{\partial w_a}{\partial s} dy. \tag{3.22}$$

It follows that

$$
\begin{aligned}
\frac{p-1}{2}E_a^1(s) &= (p+1)E_a(s) - \Big[E_a^1(s) - E_a^2(s)\Big] \\
&= (p+1)E_a(s) + \int_{\Omega_a(s)} \rho w_a \frac{\partial w_a}{\partial s} dy \qquad (3.23)\\
&\le (p+1)E_a(s) + \frac{1}{8}\int_{\Omega_a(s)} \rho w_a^2 dy \\
&\quad + 2\int_{\Omega_a(s)} \rho \left|\frac{\partial w_a}{\partial s}\right|^2 dy,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{p-1}{2}\int_{\Omega_a(s)} \rho |\nabla_y w_a|^2 dy + \frac{1}{8}\int_{\Omega_a(s)} \rho w_a^2 dy \\
\le\ & (p+1)E_a(s) + 2\int_{\Omega_a(s)} \rho \left|\frac{\partial w_a}{\partial s}\right|^2 dy \\
=\ & (p+1)E_a(s) + 2\Big(J_a(s) - \frac{d}{ds}E_a(s)\Big).
\end{aligned}
$$

Integrating this equation from s to $s+1$, we get

$$
\begin{aligned}
&\frac{p-1}{2}\int_s^{s+1}\int_{\Omega_a(s)} \rho |\nabla_y w_a|^2 dyds + \frac{1}{8}\int_s^{s+1}\int_{\Omega_a(s)} \rho w_a^2 dyds \qquad (3.24)\\
&\qquad \le (p+5)\max_{s\le\sigma\le s+1} E_a(\sigma) + 2C^* e^{-s/4}.
\end{aligned}
$$

The estimates (3.7)–(3.9), together with the equation (3.6), imply

$$
\left|\frac{\partial w_a}{\partial s}(y,s)\right| \le C(P,p,\Omega) \quad \text{for } y \in \overline{\Omega_a(s)},\ |y|\le 1,\ s > s_0+1. \qquad (3.25)
$$

The boundary $\partial\Omega_a(s)$ is C^2 uniformly as $s\to\infty$; it is certainly uniformly Lipschitz. Using the elliptic version of the interpolation theorem (the proof is similar to those in [14]), (3.8) and (3.25), viewing t as another variable, we obtain

$$
\begin{aligned}
\|w_a\|_{C(B)} &\le C\Big[\|w_a\|_{C(B)} + \Big\|\Big(\frac{\partial}{\partial s}, \nabla_y\Big) w_a\Big\|_{C(B)}\Big]^{\frac{n+1}{n+3}} \\
&\quad \cdot \Big[\iint_B w_a^2 dy d\sigma\Big]^{\frac{1}{n+3}}
\end{aligned}
$$

$$\tag{3.26} \begin{aligned} &\leq C(P,p,\Omega)\Big[\iint_B \rho w_a^2 dy d\sigma\Big]^{\frac{1}{n+3}} \\ &\leq C(P,p,\Omega)\Big[(p+5)\max_{s\leq\sigma\leq s+1} E_a(\sigma) + 2C^* e^{-s/4}\Big]^{\frac{1}{n+3}}, \end{aligned}$$

where $B = \{(y,\sigma);\ y \in \Omega_a(s),\ s < \sigma < s+1\}$. (3.22)–(3.26) are valid for all $a \in \partial\Omega$, with constants independent of a.

Our main result of this section is

PROPOSITION 3.3. *Let the assumption* (3.1) *be in force. If for some* $K > 0$

$$\tag{3.27} \liminf_{t\to T}(T-t)^\beta \inf_{|y|\leq K} u(a + y\sqrt{T-t}, t) = 0,$$

then a *is not a blowup point.*

PROOF. By Lemma 3.1,

$$\lim_{s\to\infty} E_a(s) = 0.$$

Thus for any $\eta > 0$, there exists s^* large enough such that

$$\tag{3.28} 4C^* e^{-s^*/4} \leq \eta, \qquad E_a(s^*) \leq \eta.$$

For this fixed s^*, it is clear that $E_b(s^*)$ is a continuous function in the variable $b \in \partial\Omega$. Therefore there exists a neighborhood N of a such that

$$\tag{3.29} E_b(s^*) \leq 2\eta \quad \text{for } b \in \partial\Omega \cap N.$$

Now by (3.11), (3.28) and (3.29),

$$E_b(s) \leq E_b(s^*) + 4C^* e^{-s^*/4} \leq 3\eta \quad \text{for any } s \geq s^*,\ b \in \partial\Omega \cap N.$$

Substituting this inequality into (3.26), we obtain

$$\tag{3.30} \begin{aligned} w_b(0,s) \leq C(P,p,\Omega)\Big[(3p+16)\eta\Big]^{\frac{1}{n+3}} \\ \text{for any } s \geq s^*,\ b \in \partial\Omega \cap N. \end{aligned}$$

We now take η to be sufficiently small so that the right-hand-side of (3.30) is less than $\varepsilon_0/3$, where ε_0 is determined in Proposition 2.1. Since ε_0 is independent of the neighborhood N, the conclusion now follows from Lemma 3.2 and Proposition 2.1. □

4. Single-point-blowup

In this section, we restrict our attention to the case of space dimension $n = 2$ and the domain is radially symmetric, say, $\Omega = B_1(0) = \{(x_1, x_2);\ x_1^2 + x_2^2 < 1\}$. If the initial data is radially symmetric, then the solution is also radially symmetric; in this case, the problem is essentially one space dimensional and the solution will blow up everywhere on the boundary $|x| = 1$.

Here, we shall consider those initial data which are not radially symmetric. We shall establish the single-point-blowup for those *nice* initial data with only one *hill* on the boundary.

We first rewrite the equations (1.1)–(1.3) in polar coordinates

$$
\begin{aligned}
&(4.1) & \mathcal{L}[u] &= 0 \quad \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi, \\
&(4.2) & \frac{\partial u}{\partial r} &= u^p \quad \text{for } r = 1,\ 0 \le \theta < 2\pi, \\
&(4.3) & u(r,\theta,0) &= u_0(r,\theta) \quad \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi, \qquad (u_0 \ge 0),
\end{aligned}
$$

where $1 < p < \infty$ and

$$
(4.4) \qquad \mathcal{L} = \frac{\partial}{\partial t} - \Delta_x = \frac{\partial}{\partial t} - \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta} \Big).
$$

We assume that $u_0(r,\theta) = \widetilde{u_0}(x)$ $(x_1 = r\cos\theta,\ x_2 = r\sin\theta)$ is C^2 and satisfies:

$$
\begin{aligned}
&(4.5) & & u_0 \ge 0, \quad \Delta_x \widetilde{u_0} \ge 0 \quad \text{for } x \in B_1(0), \\
&(4.6) & & \frac{\partial u_0}{\partial r} = u_0^p \quad \text{for } |x| = 1, \\
&(4.7) & & \frac{\partial u_0}{\partial \theta}(r,\theta) < 0 \quad \text{for } 0 < r < 1,\ 0 < \theta < \pi, \\
&(4.8) & & u_0(r,\theta) = u_0(r, 2\pi - \theta) \quad \text{for } 0 \le r \le 1,\ 0 \le \theta \le \pi.
\end{aligned}
$$

The main result of this section is

Theorem 4.1. *Under the assumption* (4.5)–(4.8), *the point* $(r,\theta) = (1,0)$ *is the only blowup point.*

Proof. First, by uniqueness of the system and (4.8), we easily obtain

$$
(4.9) \quad u(r,\theta,t) = u(r, 2\pi - \theta, t) \quad \text{for } 0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 < t < T,
$$

where T is the blowup time. This implies that

$$\frac{\partial u}{\partial \theta} = 0 \quad \text{on } \theta = 0 \text{ and } \theta = \pi. \tag{4.10}$$

Using (4.7), (4.10) and applying the maximum principle to $\frac{\partial u}{\partial \theta}$ (notice that the operator $\mathcal{L}$ may have a singularity at $r = 0$, however, there is no problem since we can apply the maximum principle to $\frac{\partial u}{\partial \theta} = -x_2 \widetilde{u}_{x_1} + x_1 \widetilde{u}_{x_2}$ in rectangular coordinates), we obtain

$$\frac{\partial u}{\partial \theta} < 0 \quad \text{for } 0 < r < 1,\ 0 < \theta < \pi. \tag{4.11}$$

The blowup occurs only at the boundary ([16, Corollary 4.2]). Therefore, if the conclusion is not true, then there exists $\theta_0 \in (0, \pi]$ such that $(r, \theta) = (1, \theta_0)$ is a blowup point. But then (4.11) implies that

$$(r, \theta) = (1, \theta) \quad (0 \le \theta \le \theta_0) \text{ are all blowup points.} \tag{4.12}$$

Thus by Proposition 3.3 and (4.11),

$$\liminf_{t \to T-0} \inf_{\{0 \le \theta \le \theta_0\}} (T-t)^{\beta} u(1, \theta, t) \ge \liminf_{t \to T-0} (T-t)^{\beta} u(1, \theta_0, t) > 0.$$

It turns out that the above estimate is not enough for our proof, for technical reasons. We need more accurate estimate (see (4.19)–(4.20) below) for the solution u. We need the following facts:

Claim 1.

$$\liminf_{t \to T-0} \frac{u(1, \theta_1, t)}{u(1, \theta_2, t)} = 1 \qquad \text{for any } 0 < \theta_2 < \theta_1 < \theta_0. \tag{4.13}$$

If (4.13) is not true (notice that $u(1, \theta_1, t) < u(1, \theta_2, t)$), then there exist $\varepsilon > 0$ and $t_j \to T - 0$ such that

$$\frac{u(1, \theta_1, t_j)}{u(1, \theta_2, t_j)} \le 1 - \varepsilon \quad \text{for } j = 1, 2, 3, \cdots. \tag{4.14}$$

We now take θ^* such that $\theta_1 < \theta^* < \theta_0$. We now go back to the rectangular coordinate for the solution $\widetilde{u}(x_1, x_2, t) = u(r, \theta, t)$. Set $x^* = (x_1^*, x_2^*) =$

$(\cos\theta^*, \sin\theta^*)$. Let $w_1(y,s)$ and $w_2(y,s)$ be the solution in similarity variable at the point $(x_1,x_2)=(1,0)$ and the point $(x_1,x_2)=(x_1^*,x_2^*)$, respectively. We assume that a rotation (in y) has been made so that $(-1,0)$ is the exterior normal direction at the point $(y_1,y_2)=(0,0)$. More precisely,

$$w_1(y,s)=(T-t)^{\beta}\widetilde{u}(x,t), \quad \frac{x-(1,0)}{\sqrt{T-t}}=\mathcal{R}_1 y, \quad T-t=e^{-s}, \tag{4.15}$$

$$w_2(y,s)=(T-t)^{\beta}\widetilde{u}(x,t), \quad \frac{x-x^*}{\sqrt{T-t}}=\mathcal{R}_2 y, \quad T-t=e^{-s}, \tag{4.16}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are rotation operators. Clearly, (3.7)–(3.9) and (3.12) are valid for both w_1 and w_2. Now let $s_j=-\log(T-t_j)$. There exists a subsequence of s_j's, still denoted by s_j, such that $w_1(y,s+s_j)$ and $w_2(y,s+s_j)$ converge uniformly on any compact set. (3.12) implies that the limits, denoted by w_1^{∞} and w_2^{∞} respectively, are independent of the variable s. It is also clear that w_1^{∞} and w_2^{∞} are $C^2(\overline{R^2_+})$ functions satisfying the equations (3.17)–(3.18).

Now by (4.9), (4.11) and (4.14),

$$\begin{aligned}(T-t_j)^{\beta}u(1,\theta^*+\theta,t_j) &\le (T-t_j)^{\beta}u(1,\theta_1,t_j)\\ &\le (1-\varepsilon)(T-t_j)^{\beta}u(1,\theta_2,t_j)\\ &\le (1-\varepsilon)(T-t_j)^{\beta}u(1,\theta,t_j),\end{aligned}$$

for any θ such that $|\theta|\le\min[\theta_2,\theta^*-\theta_1]$. Therefore by letting $s_j\to\infty$, we easily conclude that

$$w_2^{\infty}\le(1-\varepsilon)w_1^{\infty} \quad \text{on } \{y_1=0\}.$$

Thus by the boundary condition (3.18),

$$\frac{\partial w_2^{\infty}}{\partial n}=(w_2^{\infty})^p\le(1-\varepsilon)^p(w_1^{\infty})^p=(1-\varepsilon)^p\frac{\partial w_1^{\infty}}{\partial n}.$$

By the comparison principle (see Lemma 4.2 below),

$$w_2^{\infty}\le(1-\varepsilon)^p w_1^{\infty} \quad \text{on } \overline{R^2_+}.$$

Continue this iteration process, we obtain $w_2^{\infty}\le(1-\varepsilon)^{p^m}w_1^{\infty}$ for any positive integer m, and hence $w_2^{\infty}\equiv 0$. This implies that $w_2(y,s+s_j)$ converges to 0 uniformly on any compact set; especially,

$$\lim_{j\to\infty} w_2(0,s_j)=0,$$

i.e.,

$$\lim_{j\to\infty}(T-t_j)^{\beta}u(1,\theta^*,t_j)=0.$$

Thus Proposition 3.2 implies that $(r,\theta)=(1,\theta^*)$ is not a blowup point, which is a contradiction. This proves the Claim 1.

Claim 2. Let $\psi(y_1)$ be a positive function of the variable y_1 only satisfying (3.17)–(3.18) (the one dimensional, strictly positive, bounded solution of the system (3.17)–(3.18) is unique, see [7, Lemma 3.1]). Let w_{θ^*} be the solution in similarity variable in (4.16) at the point $(r,\theta)=(1,\theta^*)$, then

$$\liminf_{s\to\infty} w_{\theta^*}(y,s)=\limsup_{s\to\infty} w_{\theta^*}(y,s)=\psi(y_1)\quad \text{for any } 0<\theta^*<\theta_0, \tag{4.17}$$

where the limit in the above equality is taken uniformly on any compact set.

In fact, since the estimates (3.7)–(3.9) and (3.12) are valid for w_{θ^*}, we can always take subsequence s_j such that the sequence of the solution $w_{\theta^*}(y,s+s_j)$ converges uniformly on any compact set to a function $w_{\theta^*}^{\infty}$. As before, (3.12) implies that $w_{\theta^*}^{\infty}$ is independent of the variable s.

Let $t_j=T-\exp(-s_j)$. We take θ_1 and θ_2 such that $0<\theta_2<\theta^*<\theta_1<\theta_0$. Then by the monotonicity (4.11) and Claim 1,

$$\begin{aligned}(T-t_j)^{\beta}u(1,\theta^*+\overline{\theta},t_j) &\le (T-t_j)^{\beta}u(1,\theta_2,t_j)\\ &\le (1+\varepsilon(t_j))(T-t_j)^{\beta}u(1,\theta_1,t_j)\\ &\le (1+\varepsilon(t_j))(T-t_j)^{\beta}u(1,\theta^*+\widehat{\theta},t_j)\\ &\qquad (\lim_{t_j\to T-0}\varepsilon(t_j)=0\quad \text{by (4.13)}),\end{aligned}$$

for any $\overline{\theta}$ and $\widehat{\theta}$ such that $|\overline{\theta}|,\ |\widehat{\theta}|\le \min(\theta^*-\theta_2,\theta_1-\theta^*)$. Letting $s_j=-\log(T-t_j)\to\infty$, we easily obtain that the function $w_{\theta^*}^{\infty}$ is independent of the variable y_2 on the boundary $\{y_1=0\}$. Thus $w_{\theta^*}^{\infty}\in C^{\infty}(\overline{R_+^2})$ and $\dfrac{\partial w_{\theta^*}^{\infty}}{\partial y_2}$ is bounded on R_+^2. Differentiating the equation for $w_{\theta^*}^{\infty}$ with respect to y_2 and apply the maximum principle (see Lemma 4.2 below), we obtain that $\dfrac{\partial w_{\theta^*}^{\infty}}{\partial y_2}\equiv 0$ on R_+^2. Thus $w_{\theta^*}^{\infty}$ is a function of the variable y_1 only. Since the one dimensional strictly positive bounded solution ψ of the system (3.17)–(3.18) is unique, we conclude that $w_{\theta^*}^{\infty}\equiv\psi(y_1)$ on R_+^2.

Now for any sequence $s_j \to \infty$, there is a further subsequence s_{j_k} such that $w_{\theta^*}(y, s+s_{j_k})$ converges uniformly on any compact set. We just proved that the limit function (on this subsequence s_{j_k}) has to be the one dimensional solution $\psi(y_1)$, which is unique. Thus the limit $\lim_{s\to\infty} w_{\theta^*}(y,s)$ exists (not just on subsequences), and the limit equals $\psi(y_1)$. Claim 2 is proved.

Now let

$$D = \psi(0). \tag{4.18}$$

Claim 2 implies that

$$\begin{aligned}\frac{D-\varepsilon(t)}{(T-t)^\beta} &\le u(1, \frac{3\theta_0}{4}, t) \le u(1,\theta,t)\\ &\le u(1, \frac{\theta_0}{4}, t) \le \frac{D+\varepsilon(t)}{(T-t)^\beta} \quad \text{for } \frac{\theta_0}{4} \le \theta \le \frac{3\theta_0}{4},\end{aligned} \tag{4.19}$$

where

$$\lim_{t\to T-0} \varepsilon(t) = 0. \tag{4.20}$$

Construction of a comparison function: We now construct an auxiliary function with the help of the one dimensional solution $\psi(y_1)$ of the system (3.17)–(3.18).

Recall that ([7, Lemma 3.1])

$$\begin{aligned}\psi(\xi) &= d_0 U\Big(\beta, \frac{1}{2}, \frac{\xi^2}{4}\Big) \quad \text{for } \xi \ge 0,\\ U(a,b,\mu) &= \frac{1}{\Gamma(a)}\int_0^\infty e^{-\mu t} t^{a-1}(1+t)^{b-a-1}dt,\\ d_0 &= \frac{1}{\sqrt{\pi}}\Big[\beta\frac{\Gamma^p(\beta+\frac{1}{2})}{\Gamma(\beta+1)}\Big]^{2\beta},\\ \psi(\xi) &= K\xi^{-2\beta}[1+O(\xi^{-2})] \quad (K>0).\end{aligned} \tag{4.21}$$

A direct calculation shows that (notice that $b-a-1 = \frac{1}{2}-\beta-1 < 0$)

$$\frac{\dfrac{\partial U}{\partial \mu}(a,b,\mu)}{U(a,b,\mu)} = \frac{-\displaystyle\int_0^\infty e^{-\mu t} t^{a}(1+t)^{b-a-1}dt}{\displaystyle\int_0^\infty e^{-\mu t} t^{a-1}(1+t)^{b-a-1}dt}$$

$$
\begin{aligned}
&= -\frac{1}{\mu}\frac{\displaystyle\int_0^\infty e^{-t}\, t^{a}(1+t/\mu)^{b-a-1}dt}{\displaystyle\int_0^\infty e^{-t}\, t^{a-1}(1+t/\mu)^{b-a-1}dt} \\
&\leq -\frac{c_1}{\mu} \quad \text{for } \mu \geq 1 \\
&\quad (c_1 \text{ is a positive constant depending only on } a \text{ and } b),
\end{aligned}
$$

which implies that

$$
\frac{\dfrac{d\psi}{d\xi}}{\psi(\xi)} = \frac{\xi\dfrac{\partial U}{\partial\mu}\Big(\beta,\dfrac{1}{2},\dfrac{\xi^2}{4}\Big)}{2U\Big(\beta,\dfrac{1}{2},\dfrac{\xi^2}{4}\Big)} \leq -\frac{2c_1}{\xi} \quad \text{for } \xi \geq 2. \tag{4.22}
$$

Notice that $\psi'(\xi) < 0$ for all $\xi \geq 0$. Therefore (4.22) implies that

$$
\frac{\psi'(\xi)}{\psi(\xi)} \leq \frac{-c_2}{1+\xi} \quad \text{for all } \xi \geq 0, \tag{4.23}
$$

where c_2 is a positive constant depending only on p and β.

Define

$$
z_1(r,t) = z_1(r,\theta,t) = \frac{1}{(T-t)^\beta}\psi\Big(\frac{1-r}{\sqrt{T-t}}\Big), \qquad \Big(\beta = \frac{1}{2(p-1)}\Big). \tag{4.24}
$$

By using the equation for ψ, we get

$$
\begin{aligned}
\mathcal{L}[z_1] &= \frac{1}{r}\,\frac{1}{(T-t)^\beta}\,\frac{1}{\sqrt{T-t}}\psi'\Big(\frac{1-r}{\sqrt{T-t}}\Big) \\
&\leq \frac{-\,c_2\psi}{r\,(T-t)^\beta[(1-r)+\sqrt{T-t}]} \\
&= \frac{-\,c_2}{r\,[(1-r)+\sqrt{T-t}]}z_1 \quad \text{for } 0<r<1,\ 0<t<T,
\end{aligned} \tag{4.25}
$$

where the operator $\mathcal{L}$ is defined in (4.4). On the boundary,

$$
\frac{\partial z_1}{\partial r} = z_1^p \quad \text{for } r=1,\ 0<t<T. \tag{4.26}
$$

Now take a fixed η such that $1<\eta<p$. Then

$$\begin{aligned}\mathcal{L}[z_1^\eta] &\le \eta z_1^{\eta-1}\mathcal{L}[z_1]\\ &\le \frac{-c_2\eta}{r\,[(1-r)+\sqrt{T-t}]}z_1^\eta \quad \text{for } 0<r<1,\ 0<t<T,\end{aligned}\tag{4.27}$$

$$\frac{\partial z_1^\eta}{\partial r}=\eta z_1^{p-1}z_1^\eta=\frac{\eta D^{p-1}}{\sqrt{T-t}}z_1^\eta \quad \text{for } r=1,\ 0<t<T,\tag{4.28}$$

where the constant D is defined in (4.18). Define

$$z_2(r,\theta,t)=z_1^\eta(r,t)\sin\Big[\frac{2\pi}{\theta_0}\Big(\theta-\frac{\theta_0}{4}\Big)\Big] \quad \text{for } \frac{\theta_0}{4}\le\theta\le\frac{3\theta_0}{4}.\tag{4.29}$$

Then by (4.27),

$$\begin{aligned}\mathcal{L}[z_2] &= \Big\{\mathcal{L}[z_1^\eta]+\frac{z_1^\eta(r,t)}{r^2}\Big(\frac{2\pi}{\theta_0}\Big)^2\Big\}\sin\Big[\frac{2\pi}{\theta_0}\Big(\theta-\frac{\theta_0}{4}\Big)\Big]\\ &\le \Big\{\frac{-c_2\eta}{[(1-r)+\sqrt{T-t}]}+\frac{1}{r}\Big(\frac{2\pi}{\theta_0}\Big)^2\Big\}\frac{z_1^\eta}{r}\sin\Big[\frac{2\pi}{\theta_0}\Big(\theta-\frac{\theta_0}{4}\Big)\Big]\\ &< 0 \quad \text{for } T-\delta_0^2<t<T,\ 1-\delta_0<r<1,\ \frac{\theta_0}{4}<\theta_0<\frac{3\theta_0}{4},\end{aligned}\tag{4.30}$$

provided δ_0 is small enough (depending only on p and θ_0). It is also clear that

$$z_2=0 \quad \text{for } \theta=\frac{\theta_0}{4} \text{ and } \theta=\frac{3\theta_0}{4},\ 0<t<T,\ 0<r<1,\tag{4.31}$$

$$\frac{\partial z_2}{\partial r}=\frac{\eta D^{p-1}}{\sqrt{T-t}}z_2 \quad \text{for } r=1,\ \frac{\theta_0}{4}<\theta<\frac{3\theta_0}{4},\ 0<t<T.\tag{4.32}$$

Completing the proof: The function $\dfrac{\partial u}{\partial\theta}$ satisfies the equations

$$\begin{aligned}\mathcal{L}\Big[\frac{\partial u}{\partial\theta}\Big] &= 0 \quad \text{for } 0\le r<1,\ 0\le\theta<2\pi, 0<t<T,\\ \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial\theta}\Big) &= pu^{p-1}\frac{\partial u}{\partial\theta} \quad \text{for } r=1,\ 0\le\theta<2\pi,\ 0<t<T.\end{aligned}$$

By (4.20), we can take δ ($0<\delta\le\delta_0$, with the δ_0 given in (4.30)) to be small enough such that

$$p\Big(D-\varepsilon(t)\Big)^{p-1}>\eta D^{p-1} \quad \text{for } T-\delta^2\le t<T\tag{4.33}$$

(this is possible since $\eta < p$); we now fix such an δ. Using (4.19) and (4.33), we find that $-\dfrac{\partial u}{\partial \theta}$ satisfies

$$\begin{aligned}
\mathcal{L}\Big[-\frac{\partial u}{\partial \theta}\Big] &= 0 \quad \text{for } 1-\delta < r < 1,\ \frac{\theta_0}{4} < \theta < \frac{3\theta_0}{4},\ T-\delta^2 < t < T,\\
\frac{\partial}{\partial r}\Big(-\frac{\partial u}{\partial \theta}\Big) &> \frac{\eta D^{p-1}}{\sqrt{T-t}}\Big(-\frac{\partial u}{\partial \theta}\Big)\\
&\qquad\qquad \text{for } r=1,\ \frac{\theta_0}{4} < \theta < \frac{3\theta_0}{4},\ T-\delta^2 \le t < T.
\end{aligned}$$

By Hopf's lemma and strong maximum principle,

$$\sigma_1 = \inf_{\{\theta_0/4 \le \theta \le 3\theta_0/4,\ 1-\delta \le r \le 1\}} \Big[-\frac{\partial u}{\partial \theta}(r,\theta,T-\delta^2)\Big] > 0. \tag{4.34}$$

Since the solution u is uniformly bounded in the region $\{r \le 1-\delta/2,\ T-\delta^2 \le t < T\}$, the strong maximum principle also implies that

$$\sigma_2 = \inf_{\{\theta_0/4 \le \theta \le 3\theta_0/4,\ T-\delta^2 \le t < T\}} \Big[-\frac{\partial u}{\partial \theta}(1-\delta,\theta,t)\Big] > 0. \tag{4.35}$$

Using the fourth equation in (4.21), we know that z_2 is uniformly bounded on $\{r = 1-\delta,\ T-\delta^2 \le t < T\}$. Therefore by comparison principle,

$$-\frac{\partial u}{\partial \theta} \ge \gamma z_2 \quad \text{for } \frac{\theta_0}{4} \le \theta \le \frac{3\theta_0}{4},\ 1-\delta \le r \le 1,\ T-\delta^2 \le t < T, \tag{4.36}$$

provided γ is small enough such that

$$\begin{aligned}
&\sigma_1 > \gamma z_2(r,\theta,T-\delta^2) \quad \text{for } 1-\delta \le r \le 1, \frac{\theta_0}{4} \le \theta \le \frac{3\theta_0}{4},\\
&\sigma_2 > \gamma z_2(1-\delta,\theta,t) \quad \text{for } T-\delta^2 \le t < T,\ \frac{\theta_0}{4} \le \theta \le \frac{3\theta_0}{4}.
\end{aligned}$$

If we further restrict θ to $[3\theta_0/8, 5\theta_0/8]$, (4.36) implies that

$$\begin{aligned}
-\frac{\partial u}{\partial \theta} &\ge \frac{\gamma\sqrt{2}}{2}\frac{D^\eta}{(T-t)^{\beta\eta}}\\
&\qquad \text{for } r=1,\ \frac{3\theta_0}{8} \le \theta \le \frac{5\theta_0}{8},\ T-\delta^2 \le t < T.
\end{aligned} \tag{4.37}$$

Now by (4.37) and (3.1),

$$\begin{aligned} 0 \le u\Big(1, \frac{5\theta_0}{8}, t\Big) &= u\Big(1, \frac{3\theta_0}{8}, t\Big) + \int_{3\theta_0/8}^{5\theta_0/8} \frac{\partial u}{\partial \theta}(1, \theta, t) d\theta \\ &\le \frac{P}{(T-t)^{\beta}} - \frac{\theta_0}{4} \frac{\gamma\sqrt{2}}{2} \frac{D^{\eta}}{(T-t)^{\beta\eta}}. \end{aligned}$$

This is a contradiction if $T - t$ is small enough. The Theorem is proved. □

To complete this section, we next state the following comparison lemma used in the proof.

LEMMA 4.2. *Suppose that* $w_j(y)$ $(j = 1, 2)$ *are two* $C^1(\overline{R^n_+}) \cap C^2(R^n_+)$ *functions satisfying*

$$\begin{aligned} &\limsup_{|y|\to\infty} \frac{|w_j(y)|}{\exp(\alpha y^2)} = 0 \quad (j = 1, 2) \quad \textit{for some } 0 < \alpha < \min\Big(\frac{\beta}{2n}, \frac{1}{4}\Big), \\ &-\Delta_y w_1 + \frac{1}{2} y \cdot \nabla_y w_1 + \beta w_1 \ge -\Delta_y w_2 + \frac{1}{2} y \cdot \nabla_y w_2 + \beta w_2 \quad \textit{in } R^n_+, \\ &a(y)\frac{\partial w_1}{\partial n} + b(y) w_1 \ge a(y) \frac{\partial w_2}{\partial n} + b(y) w_2 \quad \textit{on } y_1 = 0, \quad \Big(\frac{\partial}{\partial n} = -\frac{\partial}{\partial y_1}\Big), \end{aligned}$$

where $R^n_+ = \{(y_1, \cdots, y_n);\ y_1 > 0\}$, $\beta > 0$, $a(y) \ge 0$, $b(y) \ge 0$ *and* $a(y) + b(y) > 0$. *Then*

$$w_1(y) \ge w_2(y) \quad \textit{on } R^n_+.$$

PROOF. Since

$$-2\alpha n + \beta > 0, \qquad -4\alpha^2 + \alpha > 0,$$

the function $h(y) = \exp(\alpha y^2)$ satisfies

$$\begin{aligned} &-\Delta_y h + \frac{1}{2} y \cdot \nabla_y h + \beta h > 0 \quad \text{in } R^n_+, \\ &\frac{\partial h}{\partial n} = 0 \quad \text{on } y_1 = 0, \qquad h(y) > 0 \quad \text{on } \overline{R^n_+}. \end{aligned}$$

Using maximum principle, we obtain,

$$w_1(y) + \varepsilon h(y) \ge w_2(y) \quad \text{on } R^n_+,$$

for any $\varepsilon > 0$ (we use maximum principle on a finite domain since $h(y)$ dominate when $|y|$ is large). Now the lemma follows by letting $\varepsilon \to 0+$. □

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