J. Math. Sci. Univ. Tokyo
23 (2016), 381–403.

Rigidity of Discontinuous Actions on Diamond Homogeneous Spaces

By Imed KÉDIM

Abstract. Let $G = \mathbb{R}^n \ltimes H_{2n+1}$ be the diamond group, H a closed Lie subgroup of G and Γ a discontinuous subgroup for the homogeneous space G/H. We study in this paper some rigidity properties of the discontinuous action of Γ on G/H. Namely, we show that the strong local rigidity fails to hold on the corresponding parameter space. In addition, when particularly H is dilation-invariant, the local rigidity also fails to hold.

1. Introduction

Let G be a Lie group, H a closed subgroup and Γ a finitely generated discontinuous subgroup for G/H. The parameter space of the action of Γ on G/H

$$(1.1)\quad \mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective, } \varphi(\Gamma) \text{ is discrete and} \\ \text{acts properly and fixed point freely} \\ \text{on } G/H \end{array} \right\}$$

is introduced by T. Kobayashi in [11] for the general setting including the non-Riemannian case. The group G acts on the representation space by inner automorphisms, $(g \cdot \varphi)(\gamma) = g\varphi(\gamma)g^{-1}$ for all $g \in G$, $\gamma \in \Gamma$ and $\varphi \in \mathrm{Hom}(\Gamma, G)$. The parameter space is invariant under this action and the deformation space of the discontinuous action of Γ on G/H was introduced in [11] as the quotient space

$$\mathscr{T}(\Gamma, G, H) := \mathscr{R}(\Gamma, G, H)/G \subset \mathrm{Hom}(\Gamma, G)/G.$$

2010 *Mathematics Subject Classification.* Primary 22E25; Secondary 22G15.
Key words: Deformation spaces, diamond group, rigidity, discrete subgroup.
This work was completed with the support of D.G.R.S.R.T through the Laboratory LAMHA (LR11ES52).

The problem of describing explicitly $\mathscr{T}(\Gamma, G, H)$ is posed in [14, Problem C]. The study of geometric and topological features of this space was initiated for general settings as well, and studied by T. Kobayashi and S. Nasrin in the case where G is a certain two step connected simply connected nilpotent Lie group of dimension $2k+1$, $\Gamma \cong \mathbb{Z}^k$ and $G/H \cong \mathbb{R}^{k+1}$, (see [15]). The methods used therein was based on an observation of T. Kobayashi, which allows to use the syndetic hull of Γ (a connected Lie subgroup of G containing Γ co-compactly) instead of Γ. This idea has been generalized in [6] to encompass completely solvable Lie groups and more generally, exponential Lie groups when Γ is abelian. Under this assumption, the subgroup Γ has a unique syndetic hull. Furthermore, the parameter and the deformation spaces are topologically identified respectively to subsets of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})/G$, where $\mathfrak{l}$ is the Lie subalgebra of $\mathfrak{g}$ associated to the syndetic hull of Γ. As an application of this reformulation, by analyzing the equations defining $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ and the proper action condition, some results on local and strongly local rigidity have been obtained in [1, 2, 3, 4, 5, 8, 7]. In this paper, we study some of these questions, when G is the diamond group, which is a non-exponential connected simply connected solvable Lie group, obtained as a semi-direct product $G = \mathbb{R}^n \ltimes H_{2n+1}$, where H_{2n+1} is the $2n+1$-dimensional Heisenberg group. In this context, one can not make use of the arguments developed in the aforementioned references as Γ fails in general to admit a syndetic hull.

Using the fact that the Lie algebra of G is a graded Lie algebra, we prove that the strongly local rigidity property fails to hold. If we restrict to the case $H = U \ltimes H_0$, with U a subgroup of $\mathbb{R}^n$ and H_0 a subgroup of H_{2n+1}, then the local rigidity property also fails to hold for any non-trivial discontinuous subgroup for G/H, in the three following cases: H is connected, H meets the center of H_{2n+1} or H is a dilation-invariant subgroup of G (see section 3.3.2).

The outline of the paper is as follows. The next section is devoted to record some known facts on discontinuous groups, syndetic hulls, the representation space $\mathrm{Hom}(\Gamma, G)$ and the topological features of deformations. In the third section, we introduce the notions of compatible subalgebras, compatible subgroups and invariant subgroups. We also prove some elementary properties of the diamond group. The rigidity results are proved in the fourth section (Theorems 4.1, 4.2 and Corollary 4.3).

2. Backgrounds

This section is devoted to record some basic facts about deformations. The readers could consult the references [9, 10, 11, 12, 13, 14] and some references therein for more details about the subject. Concerning the entire subject, we strongly recommend the papers [10] and [14].

2.1. Proper action

Let X be a locally compact space and L a locally compact topological group. The action of the group L on X is said to be:

(1) Proper if, for each compact subset $S \subset X$ the set $L_S = \{l \in L : l \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or merely free) if, for each $x \in X$, the isotropy group $L_x = \{l \in L : \; lx = x\}$ is trivial.

(3) Discontinuous, if L is discrete and it acts on X properly and fixed point freely. In this case, L is said to be *discontinuous group* for X.

In the case where $X = G/H$ is a homogeneous space and L a subgroup of G, acting on G/H by a left multiplication $l \cdot gH = lgH$, it is well known that the action of L on X is proper if and only if $SHS^{-1} \cap L$ is compact for any compact set S in G. Obviously when H is a compact subgroup, the action of L is proper; and when L is a torsion free and discrete, any proper action is also free. The following Lemma has been obtained in [9].

Lemma 2.1. *Let G be a locally compact group and H, K some subgroups of G. Then the action of K on G/H is proper if and only if the action of H on G/K is proper.*

The following fact is a direct consequence of the definitions of proper and free action,

Lemma 2.2. *Let G be a locally compact group and let H and Γ be subgroups of G. Then the following assertions are equivalent:*

$\imath$) Γ is a discontinuous group for G/H.

$\imath\imath$) Γ is a discontinuous group for G/gHg^{-1}, for any $g \in G$.

In particular, $\mathscr{R}(\Gamma, G, H) \backsimeq \mathscr{R}(\Gamma, G, gHg^{-1})$, for all $g \in G$.

2.2. Syndetic hulls

Let G be a Lie group and Γ a closed subgroup of G. By a *syndetic hull* of Γ, we mean any connected Lie subgroup of G which contains Γ co-compactly. As remarked by T. Kobayashi, there is an equivalence between the proper action of a discrete subgroup Γ and the proper action of its syndetic hull on any locally compact Hausdorff space. The idea of using a syndetic hull was introduced in [9] in the general reductive case and in [15, 16] for the non reductive case. More precisely, one has the following:

Lemma 2.3 ([9, Lemma 2.3]). *Suppose a locally compact group L acts on a Hausdorff, locally compact space X. Let Γ be a co-compact discrete subgroup of L. Then the L-action on X is proper if and only if the Γ-action on X is proper.*

As a direct consequence we can state that,

Lemma 2.4. *Let Γ be a discontinuous subgroup for G/H, assume that H is contained co-compactly in K and Γ is torsion free. Then $\mathscr{R}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, K)$.*

Proof. As Γ is a torsion free, the discontinuous action of Γ on G/H is equivalent to its proper action. Then by definition of the parameter space (1.1), the result comes from lemmas 2.3 and 2.1. $\square$

In general, an arbitrary discrete subgroup of a Lie group does not have a syndetic hull. Nevertheless Saito proved in [19], the following:

Lemma 2.5 ([19]). *Let G be a completely solvable Lie group. Then any closed subgroup of G admits a unique syndetic hull.*

We recall that a completely solvable Lie group is a connected simply connected solvable Lie group such that any endomorphism $\mathrm{ad}_X, X \in \mathfrak{g}$ has real eigenvalues. In particular, a connected simply connected nilpotent Lie group is completely solvable.

2.3. The homomorphisms set $\mathrm{Hom}(\Gamma, G)$

We recall first that any discrete subgroup of a connected simply connected solvable Lie group is finitely generated, see [18] for details. Assume then for the rest of this paper that Γ is generated by $\{\gamma_1, \dots, \gamma_k\}$. Let $F(k)$ be the non-abelian free group with a k generators $x_1, \dots, x_k$ and φ a group homomorphism from $F(k)$ to G, then φ is completely determined by the image of the generators. As $F(k)$ is a group without any non-trivial relation, the map

$$\mathrm{Hom}(F(k), G) \to G^k, \quad \varphi \mapsto (\varphi(x_1), \dots, \varphi(x_k)),$$

is a bijection and when $\mathrm{Hom}(F(k), G)$ is endowed with the point wise convergence topology and G^k with the product topology, this map becomes a homeomorphism. Now the natural map

$$\pi_k \, : \, F(k) \to \Gamma, \quad x_{i_1}^{n_1} \cdots x_{i_r}^{n_r} \mapsto \gamma_{i_1}^{n_1} \cdots \gamma_{i_r}^{n_r}.$$

is a group homomorphism. This allows to conclude that $\Gamma \cong F(k)/\ker \pi_k$ and

$$\mathrm{Hom}(\Gamma, G) = \{\varphi \in \mathrm{Hom}(F(k), G), \, \ker \varphi \supset \ker \pi_k\}.$$

If we consider the identification of $\mathrm{Hom}(F(k), G)$ with G^k, then we can write:

$$\mathrm{Hom}(\Gamma, G) = \{(g_1, \dots, g_k) \in G^k, \, g_{i_1} \cdots g_{i_r} = e \text{ for all } x_{i_1} \cdots x_{i_r} \in \ker \pi_k\},$$

which is clearly closed subset. As G is a real analytic manifold and the multiplication map is also analytic, it comes out that $\mathrm{Hom}(\Gamma, G)$ is an analytic subset of G^k whenever Γ is finitely presented, i.e. the kernel of π_k is finitely generated, see [14, Section 5.2].

2.4. The concept of rigidity

We keep the same notation and assumptions. Generalizing Weil's notion of local rigidity of discontinuous groups for Riemannian symmetric spaces, T. Kobayashi introduced the notion of local rigidity and rigidity of discontinuous for non-Riemannian homogeneous spaces (see page 135 of [11]). Notably, he proved in [13] that for the reductive case, the local rigidity may fail even for irreducible symmetric space of high dimensions. We briefly recall here some details. For a comprehensible information, we refer the readers to the references [6, 9, 10, 11, 14]. For $\varphi \in \mathscr{R}(\Gamma, G, H)$, the discontinuous group $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (resp. *strongly locally rigid*) ([11]) as a discontinuous group for G/H

if the orbit of φ under the inner conjugation is open in $\mathscr{R}(\Gamma, G, H)$ (resp. in $\operatorname{Hom}(\Gamma, G)$). This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. So, the homomorphisms which are locally rigid are those which correspond to isolated points in the deformation space $\mathscr{T}(\Gamma, G, H)$. When every point in $\mathscr{R}(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and the manifold $\Gamma\backslash G/H$ does not admit *continuous deformations*. Otherwise, it said to be continuously deformable.

3. The Diamond Group and First Preliminary Results

3.1. Co-exponential bases

Let G be a real connected simply connected solvable Lie group, $\mathfrak{g}$ its Lie algebra and

$$\exp : \mathfrak{g} \longrightarrow G$$

the exponential map. When G is an exponential Lie group, that is the exponential map is a diffeomorphism, the structure of the Lie algebra allows to construct the group G, the multiplication law is given by the formula,

$$\exp(X)\exp(Y) = \exp(C(X,Y)) \tag{3.1}$$

where $C(X,Y)$ is the Campbell–Backer–Hausdorff product of X and Y. This consideration leads us to the identification of G with the topological space $\mathbb{R}^n$, $n = \dim \mathfrak{g}$, where the exponential map is nothing but

$$\exp : \mathfrak{g} \to \mathbb{R}^n, \quad \sum_{i=1}^{n} \alpha_i X_i \mapsto (\alpha_1, \dots, \alpha_n),$$

and the multiplication law is given by (3.1). Here $\{X_1, \dots, X_n\}$ is any basis of $\mathfrak{g}$. Now if G is not exponential, the exponential map is not a diffeomorphism. Nevertheless, using the concept of the co-exponential basis we can also construct a diffeomorphism between G and $\mathbb{R}^n$.

Definition 3.1 ([17]). Let G be a Lie group, H a connected Lie subgroup of G and $\mathfrak{h}$ its Lie subalgebra. A free family $\{b_1, \dots, b_d\}$ in $\mathfrak{g}$, $d = \dim(\mathfrak{g}/\mathfrak{h})$ is said to be a co-exponential basis to $\mathfrak{h}$ in $\mathfrak{g}$ if the map

$$\mathbb{R}^d \times H \longrightarrow G, \quad (x_1, \dots, x_d, h) \longmapsto \exp(x_1 b_1) \cdots \exp(x_d b_d) h,$$

is a diffeomorphism. In particular if $\mathfrak{h}$ is trivial, a co-exponential basis to $\{0\}$ in $\mathfrak{g}$ gives a diffeomorphism between $\mathbb{R}^d$ and G.

It is well known that any Lie subalgebra of a solvable Lie algebra admits a co-exponential basis. A constructive proof of the existence of such a basis is given in [17] and based on the three following assertions :

($\imath$) If $\mathfrak{h}$ is one co-dimensional ideal of $\mathfrak{g}$, then any vector in $\mathfrak{g} \setminus \mathfrak{h}$ is a co-exponential basis.

($\imath\imath$) If $\mathfrak{g} \supset \mathfrak{h}' \supset \mathfrak{h}$ and $\{b_1, \dots, b_d\}$ respectively $\{c_1, \dots, c_r\}$ is a co-exponential basis for $\mathfrak{h}'$ in $\mathfrak{g}$ respectively for $\mathfrak{h}$ in $\mathfrak{h}'$, then $\{b_1, \dots, b_d, c_1, \dots, c_r\}$ is a co-exponential basis for $\mathfrak{h}$ in $\mathfrak{g}$.

($\imath\imath\imath$) If $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$, which is not an ideal of $\mathfrak{g}$, then any co-exponential basis for $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ in $[\mathfrak{g}, \mathfrak{g}]$ is also a co-exponential basis for $\mathfrak{h}$ in $\mathfrak{g}$.

As a direct consequence, we get the following lemmas, (see [17]).

Lemma 3.2. *Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ and V a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus V$. Then any basis of V is a co-exponential basis to $\mathfrak{h}$ in $\mathfrak{g}$.*

Lemma 3.3. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Then there exists a co-exponential basis $\{u_1, \dots, u_r\}$ to $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] = W \oplus \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, where W is the linear span of $u_1, \dots, u_r$.*

3.2. The diamond Algebra and its compatible subalgebras

Recall that the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is a two step real nilpotent Lie algebra of dimension $2n+1$ admitting a basis $Z, X_1, Y_1, \dots, X_n, Y_n$ such that the non-vanishing brackets are

$$[X_l, Y_l] = Z, \qquad \text{for all } l = 1, \dots, n. \tag{3.2}$$

The *diamond algebra* $\mathfrak{g}$ is defined as the direct sum of $\mathfrak{h}_{2n+1}$ and an n dimensional abelian Lie algebra $\mathfrak{a} = \oplus_{l=1}^n \mathbb{R}A_i$ with the additional non-trivial brackets

$$[A_l, X_l] = Y_l \text{ and } [A_l, Y_l] = -X_l, \qquad \text{for } l = 1, \dots, n. \tag{3.3}$$

Recall that a Lie algebra $\mathfrak{l}$ is called a graded Lie algebra if there is a decomposition
$$\mathfrak{l} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{l}_d,$$
where $\mathfrak{l}_d, d \in \mathbb{Z}$ are a subspaces of $\mathfrak{l}$ such that:
$$[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_{i+j}, \quad i, j \in \mathbb{Z}.$$
From the brackets relations (3.3) and (3.2), the diamond algebra $\mathfrak{g}$ is a graded Lie algebra with the decomposition
$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$
where
$$\mathfrak{g}_0 = \mathfrak{a}, \quad \mathfrak{g}_1 = \bigoplus_{i=1}^{n} (\mathbb{R}X_i \oplus \mathbb{R}Y_i), \quad \mathfrak{g}_2 = \mathbb{R}Z \text{ is the center.}$$
As a consequence, for all $t \in \mathbb{R}_+^* := \{t \in \mathbb{R}, t > 0\}$, the dilation $\mu_t : \mathfrak{g} \to \mathfrak{g}$ defined for $v \in \mathfrak{g}_d$ by $\mu_t(v) = t^d v$ is an automorphism of $\mathfrak{g}$.

DEFINITION 3.4. A subalgebra $\mathfrak{h}$ of a graded Lie algebra $\mathfrak{l}$ is called *subgraded* subalgebra if
$$\mathfrak{h} = \bigoplus_{d \in \mathbb{Z}} (\mathfrak{h} \cap \mathfrak{l}_d).$$

The subgraded subalgebras of $\mathfrak{g}$ are the subalgebras of $\mathfrak{g}$ which are stable by the one parameter family of dilations μ_t, $t \in \mathbb{R}_+^*$.

3.2.1 Compatible subalgebras

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *compatible with a given Levi-decomposition* if it is a direct sum $\mathfrak{u} \oplus \mathfrak{h}_0$, where $\mathfrak{u} = \mathfrak{a} \cap \mathfrak{h}$ and $\mathfrak{h}_0 = \mathfrak{h}_{2n+1} \cap \mathfrak{h}$. Any subgraded subalgebra is compatible with a given Levi-decomposition, and conversely a subalgebra which is compatible with a given Levi decomposition and contains the center of $\mathfrak{g}$ is subgraded subalgebra. In general, a subalgebra which is compatible with a given Levi decomposition is not subgraded. The subalgebra generated by $X_1 + Z$ is compatible with a given Levi-decomposition but not subgraded subalgebra of $\mathfrak{g}$. Nevertheless, we have the following statement :

PROPOSITION 3.5. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ which is compatible with a given Levi-decomposition. Then there exists $g \in G$ such that $\mathrm{Ad}_g(\mathfrak{h})$ is a subgraded subalgebra.*

Assume for the rest of this paragraph that $\mathfrak{h} = \mathfrak{u} \oplus \mathfrak{h}_0$, with $\mathfrak{u}$ and $\mathfrak{h}_0$ as before. To prove Proposition 3.5 we need some preliminary remarks. Let $\underline{A}$ be an element of $\mathfrak{u}$, we define the *support of* $\underline{A}$ as the unique subset $I_{\underline{A}} \subseteq \{1, \dots, n\}$ such that :

$$\underline{A} = \sum_{i \in I_{\underline{A}}} \alpha_i A_i \text{ with } \alpha_i \neq 0, \ \forall\, i \in I_{\underline{A}}. \tag{3.4}$$

We define also the *support of* $\mathfrak{h}$ as the set $I := \cup_{\underline{A} \in \mathfrak{u}} I_{\underline{A}}$. Obviously there exists $\underline{A} \in \mathfrak{u}$ such that $I = I_{\underline{A}}$, such a vector is called *generic element of* $\mathfrak{u}$. Denote by $\mathfrak{U}_I$ the linear subspace of $\mathfrak{g}$ generated by X_i, Y_i, $i \in I$ and let $\mathfrak{h}_I = \mathbb{R}Z \oplus \mathfrak{U}_I$. Note that $\mathfrak{h}_{2n+1} = \mathfrak{U}_I \oplus \mathfrak{h}_{I^C}$ and $[\mathfrak{h}_I, \mathfrak{h}_{I^C}] = [\mathfrak{u}, \mathfrak{h}_{I^C}] = \{0\}$.

LEMMA 3.6. *Let I be the support of $\mathfrak{h}$. Then*

$$\mathfrak{h} = \mathfrak{u} \oplus (\mathfrak{h}_0 \cap \mathfrak{U}_I) \oplus (\mathfrak{h}_0 \cap \mathfrak{h}_{I^C}),$$

as a direct sum of linear subspaces.

PROOF. Let $v \in \mathfrak{h}_0$ such that $v = v_I + v_{I^C}$, where $v_I \in \mathfrak{U}_I$ and $v_{I^C} \in \mathfrak{h}_{I^C}$. To conclude, we have to prove that $v_I \in \mathfrak{h}$. Let $\underline{A}$ be a generic element of $\mathfrak{u}$. Then we have $\mathrm{ad}(\underline{A})^m v = \mathrm{ad}(\underline{A})^m v_I$ and $\mathrm{ad}(\underline{A})^m v_I \in \mathfrak{h}$ for all $m > 0$, where $\mathrm{ad}(\underline{A})^m v_I = [\underline{A}, \mathrm{ad}(\underline{A})^{m-1} v_I]$. Let $\underline{A}$ be written as in (3.4) and

$$v_I = \sum_{i \in I} v_i, \quad \text{with } v_i \in \mathfrak{U}_{\{i\}}.$$

Consider the set of strictly positive integers $\{\beta_1, < \cdots, < \beta_q\} := \{|\alpha_i|, i \in I\}$, then from the expressions

$$\underline{A} = \sum_{i=1}^{q} \beta_i \sum_{|\alpha_j| = \beta_i} \frac{\alpha_j}{|\alpha_j|} A_j \quad \text{and} \quad v_I = \sum_{i=1}^{q} \sum_{|\alpha_j| = \beta_i} v_j,$$

we deduce that

$$\mathrm{ad}(\underline{A})^m v_I = \sum_{i=1}^{q} \beta_i^m \sum_{|\alpha_j| = \beta_i} \big(\frac{\alpha_j}{|\alpha_j|}\big)^m \mathrm{ad}(A_j)^m v_j.$$

Now observe that $\mathrm{ad}(A_j)^4 v_j = v_j$, then

$$\mathrm{ad}(\underline{A})^{4m} v_I = \sum_{i=1}^{q} \beta_i^{4m} \sum_{|\alpha_j|=\beta_i} v_j.$$

For $u_i = \sum_{|\alpha_j|=\beta_i} v_j$, the sequence $w_m = \sum_{i=1}^{q} \beta_i^{4m} u_i = \mathrm{ad}(\underline{A})^{4m} v_I$ is contained in the subspace L generated by the free family $\{u_1, \dots, u_q\}$. The matrix associated to the coordinates of $w_1, \dots, w_q$ via the basis $\{u_1, \dots, u_q\}$, is the Vandermonde matrix

$$\begin{pmatrix} \beta_1^4 & \cdots & \beta_1^{4q} \\ \vdots & & \vdots \\ \beta_q^4 & \cdots & \beta_q^{4q} \end{pmatrix},$$

which is non-singular, because $\beta_i^4 \neq \beta_j^4$ for all $i \neq j$. This means that the vectors $w_1, \dots, w_q$ form a basis of L. As $w_m \in \mathfrak{h}$ for all $m > 0$, the subspace $L \subset \mathfrak{h}$, in particular the vector $v_I = \sum_{i=1}^{q} u_i \in \mathfrak{h}$. $\square$

The following Lemma has been obtained in [7, Proposition 3.1.].

LEMMA 3.7. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{h}_{2n+1}$ such that $\mathfrak{z}(\mathfrak{h}_{2n+1}) \not\subset \mathfrak{h}$. Then $\dim \mathfrak{h} \leq n$ and there exists a basis $\mathcal{B}_{\mathfrak{h}} = \{Z, X'_1 \dots, X'_n, Y'_1, \dots, Y'_n\}$ of $\mathfrak{h}_{2n+1}$ with the Lie commutation relations*

$$[X'_i, Y'_j] = \delta_{i,j} Z,\ i, j = 1, ..., n \tag{3.5}$$

such that $\mathfrak{h}$ is generated by $X'_1, \dots, X'_p$, where $p = \dim \mathfrak{h}$. The symbol $\delta_{i,j}$ designates here the Kronecker index.

PROOF OF THE PROPOSITION 3.5. If $\mathfrak{h}$ contains the center, then obviously $\mathfrak{h}$ is subgraded. Assume that $\mathfrak{h}$ does not contain the center. Using Lemma 3.7, where we replace $\mathfrak{h}_{2n+1}$ by $\mathfrak{h}_{IC}$ and $\mathfrak{h}$ by $\mathfrak{h} \cap \mathfrak{h}_{IC}$, we deduce the existence of a basis $\{Z, X'_1 \dots, X'_m, Y'_1, \dots, Y'_m\}$ of $\mathfrak{h}_{IC}$ satisfying the relation (3.5) and such that $X'_1 \dots, X'_p$ is a basis of $\mathfrak{h} \cap \mathfrak{h}_{IC}$, where $p = \dim(\mathfrak{h} \cap \mathfrak{h}_{IC})$ and $2m+1 = \dim \mathfrak{h}_{IC}$. For $X'_i = X''_i + \alpha_i Z$, with $X''_i \in \mathfrak{g}_1$, let $X = \sum_{i=1}^{p} \alpha_i Y'_i$. Then a direct calculation shows that $\mathrm{Ad}_{\exp(X)}(\mathfrak{h} \cap \mathfrak{h}_{IC})$ is the linear span of $X''_1, \dots, X''_p$, in particular $\mathrm{Ad}_{\exp(X)}(\mathfrak{h} \cap \mathfrak{h}_{IC}) \subset \mathfrak{g}_1$. Observe that for all $X \in \mathfrak{h}_{IC}$, we have $[X, \mathfrak{u}] = 0$ and $[X, \mathfrak{U}_I] = 0$. Thus

$\mathrm{Ad}_{\exp(X)}(\mathfrak{u}) = \mathfrak{u}$ and $\mathrm{Ad}_{\exp(X)}(\mathfrak{U}_I \cap \mathfrak{h}) = \mathfrak{U}_I \cap \mathfrak{h}$. Using Lemma 3.6, we conclude that

$$\mathrm{Ad}_{\exp(X)}(\mathfrak{h}) = \mathfrak{u} \oplus (\mathfrak{U}_I \cap \mathfrak{h}) \oplus \mathrm{Ad}_{\exp(X)}(\mathfrak{h} \cap \mathfrak{h}_{I^C}),$$

which is graded since $\mathfrak{U}_I \cap \mathfrak{h} \oplus \mathrm{Ad}_{\exp(X)}(\mathfrak{h} \cap \mathfrak{h}_{I^C}) \subseteq \mathfrak{g}_1$. □

3.3. The diamond group

The Heisenberg group H_{2n+1} is the connected simply connected Lie group of dimension $2n+1$ associated to $\mathfrak{h}_{2n+1}$. As $\mathfrak{h}_{2n+1}$ is nilpotent, H_{2n+1} is an exponential Lie group. We identify H_{2n+1} with the affine space $\mathbb{R}^{2n+1} = (\mathbb{R}^2)^n \times \mathbb{R}$. Any element x of H_{2n+1} can be written as

$$x = (x_1, \ldots, x_n, z), \text{ where } x_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \in \mathbb{R}^2.$$

Thanks to the Campbell–Backer–Hausdorff formula for two step nilpotent Lie groups:

$$C(X,Y) = X + Y + \frac{1}{2}[X,Y],$$

the multiplication is obtained by the formula

$$xx' = (x_1 + x'_1, \ldots, x_n + x'_n, z + z' + \frac{1}{2}\sum_{l=1}^{n} b(x_l, x'_l)),$$

where b is the standard non-degenerate skew-symmetric bilinear form on $\mathbb{R}^2$, given by $b(x_l, x'_l) = det(x_l,\, x'_l)$.

As $[\mathfrak{g},\mathfrak{g}] = \mathfrak{h}_{2n+1}$, the family $\{A_1, \ldots, A_n\}$ is co-exponential basis to $\mathfrak{h}_{2n+1}$ in $\mathfrak{g}$, the *diamond group* is identified to the product space $\mathbb{R}^n \times H_{2n+1}$ and the map

$$\mathfrak{a} \times \mathfrak{h}_{2n+1} \to G, \quad (A, X) \mapsto \exp(A)\exp(X)$$

is a diffeomorphism.

The abelian group $\exp(\mathfrak{a}) = \mathbb{R}^n$ acts on $\mathfrak{h}_{2n+1}$ via the adjoint representation

$$\mathrm{Ad}_{\exp(t_i A_i)}(\alpha_j X_j + \beta_j Y_j) = \begin{cases} \alpha_j X_j + \beta_j Y_j & \text{if} \quad i \neq j \\ \alpha'_j X_j + \beta'_j Y_j & \text{if} \quad i = j, \end{cases}$$

where $\begin{pmatrix} \alpha'_j \\ \beta'_j \end{pmatrix} = r(t_j) \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$ and $r(t_i)$ is the rotation transformation

$$r(t_i) = \begin{pmatrix} \cos(t_i) & -\sin(t_i) \\ \sin(t_i) & \cos(t_i) \end{pmatrix},$$

through the basis $\{X_i, Y_i\}$. Thus the diamond group is the semi-direct product $G = \mathbb{R}^n \ltimes H_{2n+1}$, where $\mathbb{R}^n$ acts on H_{2n+1} as follows:

$$(t_1, \dots, t_n) \cdot x = (r(t_1)x_1, \dots, r(t_n)x_n, z).$$

More precisely, let us consider the projections:

$$\xi_l : G \to \mathbb{R}^2, \quad \tau_l : G \to \mathbb{R} \quad \text{and} \quad \zeta : G \to \mathbb{R}$$

defined by $\xi_l(g) = x_l$, $\tau_l(g) = t_l$ for $l = 1, \dots, n$ and $\zeta(g) = z$, where

$$g = (t_1, \dots, t_n, x_1, \dots, x_n, z).$$

The product of two elements $g, g' \in G$ is determined by the formulas

$$\xi_l(gg') = r_l(g')\xi_l(g) + \xi_l(g'), \ \tau_l(gg') = \tau_l(g) + \tau_l(g'), \tag{3.6}$$

and

$$\zeta(gg') = \zeta(g) + \zeta(g') + \frac{1}{2}\sum_{l=1}^{n} b\big(r_l(g')\xi_l(g), \xi_l(g')\big), \tag{3.7}$$

where $r_l(g) = r(\tau_l(g))^{-1}$. As a consequence, the action of G on itself by conjugation is given by

$$\begin{aligned} &\delta_l\big(gg'g^{-1}\big) = \delta_l\big(g'\big), \\ &\xi_l\big(gg'g^{-1}\big) = r_l(g)^{-1}\big((r_l(g') - I)\xi_l(g) + \xi_l(g')\big), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \zeta(gg'g^{-1}) = \zeta(g') &+ \frac{1}{2}\sum_{l=1}^{n} b\big(r_l(g')\xi_l(g), \xi_l(g')\big) \\ &- \frac{1}{2}\sum_{l=1}^{n} b\big(r_l(g')\xi_l(g), \xi_l(g)\big) + b\big(\xi_l(g'), \xi_l(g)\big) \end{aligned} \tag{3.9}$$

3.3.1 Compatible subgroups

Let H_0 be a subgroup of H_{2n+1} and U a subgroup of $\exp(\mathfrak{a})$ such that U normalizes H_0. Then the semi-direct product $U \ltimes H_0$ is a well defined subgroup of G.

Definition 3.8. A closed subgroup H of G is said to be compatible with a given Levi decomposition if it is a semi-direct product $U \ltimes H_0$, where $U = \exp(\mathfrak{a}) \cap H$ and $H_0 = H \cap H_{2n+1}$.

Proposition 3.9. *Let $H = U \ltimes H_0$ be a subgroup of G compatible with a given Levi decomposition and $L(H_0)$ the syndetic hull of H_0. Then U normalizes $L(H_0)$ and the subgroup $U \ltimes L(H_0)$ contains $U \ltimes H_0$ co-compactly. In particular if U is connected and $U \ltimes L(H_0)$ is a syndetic hull of H.*

Note that since a compatible subgroup with a given Levi-decomposition is closed, H_0 is closed subgroup of H_{2n+1}. Thus the syndetic hull of H_0 exists and it is unique. Similarly, any closed subgroup of the abelian group $\exp(\mathfrak{a})$, has a unique syndetic hull. More precisely we can state that:

Lemma 3.10. *Let K be a closed subgroup of a completely solvable Lie group. Then the syndetic hull of K is the connected Lie subgroup K' generated by the one parameter subgroups $\{\exp(tX) \,|\, t \in \mathbb{R}\}$, for all $\exp(X) \in K$.*

Proof. The existence of the syndetic hull is given by Lemma 2.5. If G is a completely solvable by Lemma 2.5, K admits a unique syndetic hull, denote it by $L(K)$. Clearly K' is connected subgroup containing K and $K' \subset L(K)$. The homogeneous space $L(K)/K'$ is homeomorphic to $\mathbb{R}^d$, where $d = \dim L(K) - \dim K'$ and the map

$$L(K)/K \to L(K)/K', \quad xK \mapsto xK'$$

is a surjective and continuous. As $L(K)/K$ is compact, $d = 0$ and $L(K) = K'$. $\square$

Proof of Proposition 3.9. Using Lemma 3.10, it is sufficient to show that $U \ltimes H_0'$ is a subgroup of G. Or equivalently, U normalizes H_0'. Let $\exp(tX)$ be a generator of H_0' and $\exp(A)$ an element of U then,

$$\exp(A)\exp(tX)\exp(-A) = \exp\big(t\,\mathrm{Ad}_{\exp(A)}(X)\big) \text{ and } \mathrm{Ad}_{\exp(A)}(X) \in H_0.$$

Thus, the one parameter subgroup $\exp\big(t\,\mathrm{Ad}_{\exp(A)}(X)\big)$ is also a generator of H_0'.

For the second statement, let S be a compact set in $L(H_0)$ such that $L(H_0) = H_0 S$. Then $U \ltimes L(H_0) = U \ltimes H_0 L = (U \ltimes H_0)(0, S)$ and $(0, S)$ is a compact set in $U \ltimes L(H_0)$. $\square$

Lemma 3.11. *Let H be a subgroup of G, which is compatible with a given Levi-decomposi- tion. Assume that H_{2n+1} is the syndetic hull of H_0. If $L(U)$ is the syndetic hull of U then the connected subgroup $L(U) \ltimes H_{2n+1}$ is a syndetic hull of H.*

Proof. If $L(U) = SU$ for a compact set S in $\exp(\mathfrak{a})$ and $H_{2n+1} = H_0 S'$ for S' a compact set of H_{2n+1}. Then

$$\begin{aligned} L(U) \ltimes H_{2n+1} &= (S,0)(U \ltimes H_0)(0,S') \\ &= (S,0)(0,S'^{-1})(U \ltimes H_0) \quad \square \\ &= (S,S'^{-1})H. \end{aligned}$$

3.3.2 Subgraded and dilation-invariant subgroups

Recall that, if L is a connected simply connected Lie groups, then any automorphism μ of the Lie algebra $\mathfrak{l}$ of L lifts to an automorphism $\tilde{\mu}$ of G, by the formula

$$\tilde{\mu}\big(\exp(X)\big) = \exp\big(\mu(X)\big), \qquad \text{for all } X \in \mathfrak{l}.$$

In particular, if $\mathfrak{l}$ is graded Lie algebra, for any $t \in \mathbb{R}_+^*$ the dilation automorphism μ_t induces an automorphism of L. This lead to define an action of $\mathbb{R}_+^*$ on L.

Definition 3.12. Assume that L is a connected simply connected Lie group with a graded Lie algebra $\mathfrak{l}$. A closed subgroup H of L is said to be :

1) dilation-invariant, if H is stable under the action of $\mathbb{R}_+^*$ on L.

2) subgraded, if H is connected and its Lie subalgebra is a subgraded subalgebra of $\mathfrak{l}$.

If H is subgraded, then obviously H is a dilation-invariant subgroup. Conversely, a dilation-invariant subgroup is not necessarily connected; and a closed subgroup with a subgraded Lie subalgebra may or not be dilation-invariant. As an example, in our situation, for $g \in G$ the dilation action is given by

$$\tau_l(\tilde{\mu}_t(g)) = \tau_l(g),\ \xi_l(\tilde{\mu}_t(g)) = t\xi_l(g) \text{ and } \zeta(\tilde{\mu}_t(g)) = t^2\zeta(g). \tag{3.10}$$

The center of G

$$Z(G) = \{g \in G,\ \xi_l(g) = 0 \text{ and } \tau_l(g) \in 2\pi\mathbb{Z} \text{ for all } l = 1, \dots, n\}$$

and the subgroup $H = \exp(\mathbb{Z}X)\exp(\mathbb{R}Z)$ for $X \in \mathfrak{g}_1$ are both non-connected and have a same Lie subalgebra the center $\mathfrak{g}_2$, which is subgraded. The center $Z(G)$ is dilation-invariant but H is not. The following Lemma is a characterization of dilation-invariant subgroups in G.

Lemma 3.13. *A closed subgroup H of G is a dilation-invariant subgroup if and only if H is compatible with a given Levi decomposition such that $H \cap H_{2n+1}$ is a subgraded subgroup.*

Proof. Suppose that H is dilation-invariant and let U and H_0 be as in definition 3.8. As H_{2n+1} is a dilation-invariant subgroup, then so is H_0. Clearly, U normalizes H_0 and $U \ltimes H_0$ is a closed subgroup of H. Let $h = \exp(A)\exp(X) \in H$. As H is closed

$$\lim_{t \to 0} \tilde{\mu}_t\big(\exp(A)\exp(X)\big) = \lim_{t \to 0} \exp(A)\exp\big(\mu_t(X)\big) = \exp(A) \in U,$$

thus $\exp(X) \in H_0$ and $H = U \ltimes H_0$. As $\lim_{t\to 0}\exp\big(\mu_t(X)\big) = e$ for all $\exp(X)$ in H_0, the dilation invariance of H_0 leads to conclude that H_0 is connected and the Lie subalgebra of H_0 is $\mathfrak{h}_0 = \{X, \exp(X) \in H_0\}$. Let $X \in \mathfrak{h}_0$, write $X = X' + X''$ with $X' \in \mathfrak{g}_1$ and $X'' \in \mathfrak{g}_2$. For $t \neq 0$, we have $X - \frac{1}{t}\mu_t(X) = tX'' \in \mathfrak{h}_0$. Then $X', X'' \in \mathfrak{h}_0$ and $\mathfrak{h}_0 = \mathfrak{h}_0 \cap \mathfrak{g}_1 \oplus \mathfrak{h}_0 \cap \mathfrak{g}_2$.

Conversely, if H_0 is subgraded then H_0 is dilation-invariant and we have $\tilde{\mu}_t\big(U \ltimes H_0\big) = U \ltimes \tilde{\mu}_t(H_0) = U \ltimes H_0$. $\square$

4. The Rigidity Problem

Our main results in this section are the following :

Theorem 4.1. *Let G be the diamond group and Γ a non-trivial finitely generated subgroup of G (not necessarily discrete). Then there is no open*

G-orbit in $\operatorname{Hom}(\Gamma, G)$. *In particular if Γ is a discontinuous group for a homogeneous space G/H, then strong local rigidity property fails to hold for every element in* $\mathscr{R}(\Gamma, G, H)$.

Theorem 4.2. *Assume that H is a dilation-invariant subgroup of G and Γ is a non-trivial discontinuous group for G/H. Then for every $\varphi \in \mathscr{R}(\Gamma, G, H)$, local rigidity property fails to hold.*

Corollary 4.3. *Let $H = U \ltimes H_0$ be a subgroup of G compatible with a given Levi-decomposition. Assume that one of the following statements is satisfied:*

ı) The subgroup $H_0 \cap Z(H_{2n+1})$ is non-trivial.

ıı) The subgroup U is connected, which is the case when H is connected. Then for any non-trivial discontinuous group for G/H, local rigidity property fails to hold everywhere.

To prove these results, we need some preliminary results. The following observation is an important tool. The action of $\mathbb{R}_+^*$ on G defined by (3.10), induces a natural action of $\mathbb{R}_+^*$ on $\operatorname{Hom}(\Gamma, G)$ by the formula $t \cdot \varphi = \tilde{\mu}_t \circ \varphi$.

Lemma 4.4. *The map $\mathbb{R}_+^* \times \operatorname{Hom}(\Gamma, G)/G \to \operatorname{Hom}(\Gamma, G)/G$, $(t, [\varphi]) \mapsto [t \cdot \varphi]$ gives a well defined continuous action of $\mathbb{R}_+^*$ on $\operatorname{Hom}(\Gamma, G)/G$. In particular for all $\varphi \in \operatorname{Hom}(\Gamma, G)$, the map $\mathbb{R}_+^* \to \operatorname{Hom}(\Gamma, G)/G$, $t \mapsto t \cdot [\varphi]$ is continuous.*

Proof. The fact that the map is a well-defined action is derived from the equality

$$t \cdot (g \cdot \varphi) = (t \cdot g) \cdot (t \cdot \varphi).$$

Since the quotient map $\operatorname{Hom}(\Gamma, G) \to \operatorname{Hom}(\Gamma, G)/G$ is open and the action of $\mathbb{R}_+^*$ on $\operatorname{Hom}(\Gamma, G)$ is continuous, the action of $\mathbb{R}_+^*$ on $\operatorname{Hom}(\Gamma, G)/G$ is also continuous. □

4.1. Strong local rigidity

The aim of this subsection is to prove Theorem 4.1. Let $\gamma_1, \dots, \gamma_k$ be the generators of Γ and identify $\operatorname{Hom}(\Gamma, G)$ to a subspace of G^k via the injection $\varphi \mapsto (\varphi(\gamma_1), \dots, \varphi(\gamma_k))$. To simplify the notation we write φ_i instead of $\varphi(\gamma_i)$. Let $\operatorname{Fix}(\Gamma, G)$ be the set of fixed points in $\operatorname{Hom}(\Gamma, G)/G$

under the action of $\mathbb{R}_+^*$. Then as a direct consequence of Lemma 4.4, we obtain,

Lemma 4.5. *Let* $\varphi \in \mathrm{Hom}(\Gamma, G)$. *If the* G*-orbit of* φ *is open in* $\mathrm{Hom}(\Gamma, G)$, *then* $[\varphi] \in \mathrm{Fix}(\Gamma, G)$.

Proof. The action of $\mathbb{R}_+^*$ is continuous. If $[\varphi]$ is an open point in $\mathrm{Hom}(\Gamma, G)/G$, then the stabilizer of $[\varphi]$ is an open subgroup of $\mathbb{R}_+^*$. □

We consider the subset,

$$F'(\Gamma, G) := \{(\varphi_1, \ldots, \varphi_k) \in \mathrm{Hom}(\Gamma, G),\ \xi_l(\varphi_i) = 0 \text{ for all } i \text{ and } l\} \tag{4.1}$$

and $F''(\Gamma, G)$ its image by the quotient map: $\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G)/G$.

Lemma 4.6. $\mathrm{Fix}(\Gamma, G) \subset F''(\Gamma, G)$.

Proof. Let $[\varphi] \in \mathrm{Fix}(\Gamma, G)$, then for all $t \in \mathbb{R}_+^*$ there exists an element $g_t \in G$ such that $t \cdot \varphi = g_t \cdot \varphi$. Thus for all $l = 1, \ldots, n$ and $i = 1, \ldots, k$, we have

$$t\xi_l(\varphi_i) = \xi_l(g_t \varphi_i g_t^{-1}),$$

which is equivalent by (3.8) to

$$t\xi_l(\varphi_i) = r_l(g_t)^{-1} r_l(\varphi_i) \xi_l(g_t) + r_l(g_t)^{-1} \xi_l(\varphi_i) - r_l(g_t)^{-1} \xi_l(g_t). \tag{4.2}$$

If $\tau_l(\varphi_i) \in 2\pi\mathbb{Z}$, then $t\xi_l(\varphi_i) = r_l(g_t)^{-1}\xi_l(\varphi_i)$ for all t, which implies that $\xi_l(\varphi_i) = 0$. For $\tau_l(\varphi_i) \notin 2\pi\mathbb{Z}$, let

$$x_{\varphi,l,i} := (I - r_l(\varphi_i))^{-1} \xi_l(\varphi_i). \tag{4.3}$$

Then from (4.2) we get

$$\xi_l(g_t) = (I - t r_l(g_t)) x_{\varphi,l,i}.$$

In particular for all i, j such that $\tau_l(\varphi_i)$, $\tau_l(\varphi_i) \notin 2\pi\mathbb{Z}$, we have the relation $x_{\varphi,l,i} = x_{\varphi,l,j}$. Let $g \in G$ be an element of G satisfying

$$\xi_l(g) = \begin{cases} x_{\varphi,l,i} & \text{if there exist } i \text{ such that } \tau_l(\varphi_i) \notin 2\pi\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

By a direct calculation, using (3.8), we show that $\xi_l(g\varphi_i g^{-1}) = 0$ for all l and i. Therefore $g \cdot \varphi \in F'(\Gamma, G)$ and $[g \cdot \varphi] = [\varphi]$. $\square$

LEMMA 4.7. *The restriction of the quotient map* $\pi : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G)/G$ *to* $F'(\Gamma, G)$ *is a continuous bijection from* $F'(\Gamma, G)$ *to* $F''(\Gamma, G)$.

PROOF. The restriction is continuous and by definition of $F''(\Gamma, G)$ is surjective. To see that the restriction is injective, suppose that there exits $\varphi \in F'(\Gamma, G)$ and $g \in G$ such that $g \cdot \varphi \in F'(\Gamma, G)$, we have to prove that $\varphi = g \cdot \varphi$. By definition of $F'(\Gamma, G)$,

$$\xi_l(\varphi_i) = \xi_l(g\varphi_i g^{-1}) = 0 \text{ for all } l \text{ and } i. \tag{4.4}$$

If there exists i such that $\tau_l(\varphi_i) \notin 2\pi\mathbb{Z}$, the last equality implies that $\xi_l(g) = 0$. Then from equality (3.9), we conclude that $\zeta(g\varphi_i g^{-1}) = \zeta(\varphi_i)$, therefore $\varphi = g \cdot \varphi$. $\square$

Observe that for all $\varphi \in F'(\Gamma, G)$, the image $\varphi(\Gamma)$ is an abelian subgroup of the abelian subgroup

$$\mathbb{R}^n \times Z(H_{2n+1}) = \{g \in G,\ \xi_l(g) = 0,\ l = 1, \dots, n\} \cong \mathbb{R}^{n+1}.$$

Then φ factors through the quotient map $\pi :, \Gamma \to \Gamma' = \Gamma/[\Gamma, \Gamma]$, where $[\Gamma, \Gamma]$ is the derivative group. We consider the map

$$\begin{array}{rccc} \xi : & F'(\Gamma, G) & \longrightarrow & \mathrm{Hom}(\Gamma', \mathbb{R}^{n+1}), \\ & \varphi & \longmapsto & \tilde{\varphi} \end{array}$$

where $\tilde{\varphi}(\gamma[\Gamma, \Gamma]) = \varphi(\gamma)$.

LEMMA 4.8. *The map* ξ *is a homeomorphism.*

PROOF. The inverse of ξ is the map $\tilde{\varphi} \mapsto \tilde{\varphi} \circ \pi$, the bi-continuity is clear. $\square$

The abelian group Γ' is finitely generated. Let $T(\Gamma')$ be the subgroup of the torsion elements in Γ'. Then the quotient group $\Gamma'' = \Gamma'/T(\Gamma')$ is an abelian torsion free finitely generated group. Thus Γ'' is a free abelian group isomorphic to $\mathbb{Z}^l$, where $l = \mathrm{rk}(\Gamma'')$.

Lemma 4.9. *The topological spaces* $\mathrm{Hom}(\Gamma', \mathbb{R}^{n+1})$ *and* $\mathrm{Hom}(\Gamma'', \mathbb{R}^{n+1})$ *are homeomorphic to each other.*

Proof. Note that $T(\Gamma') \subset \ker(\psi)$ for all $\psi \in \mathrm{Hom}(\Gamma', \mathbb{R}^{n+1})$. Then as before, the map

$$\begin{array}{rcl} \xi' : \mathrm{Hom}(\Gamma', \mathbb{R}^{n+1}) & \longrightarrow & \mathrm{Hom}(\Gamma'', \mathbb{R}^{n+1}), \\ \psi & \longmapsto & \tilde{\psi} \end{array}$$

defined by $\tilde{\psi}(\gamma T(\Gamma')) = \psi(\gamma)$, is a continuous bijection, its inverse is $\tilde{\psi} \mapsto \tilde{\psi} \circ \pi'$, where π' is the quotient map $\Gamma' \to \Gamma''$. □

Proof of the Theorem 4.1. Suppose that $[\varphi]$ is open in $\mathrm{Hom}(\Gamma, G)/G$, then by the lemmas 4.5 and 4.6, $[\varphi]$ is open point in $F''(\Gamma, G)$. Using Lemma 4.7, we see that the inverse image of $[\varphi]$ in $F'(\Gamma, G)$ is an isolated point. As a consequence of lemmas 4.8 and 4.9, the space $\mathrm{Hom}(\Gamma'', R^{n+1})$ contains an isolated point. If the rank l of Γ'' is non zero, then $\mathrm{Hom}(\Gamma'', R^{n+1})$ is homeomorphic to $M_{n,l}(\mathbb{R})$. Thus $l = 0$ and the set $\mathrm{Hom}(\Gamma'', R^{n+1})$ is reduced to the trivial homomorphism. Using again Lemmas 4.9 and 4.8 we see that $F'(\Gamma, G) = \{0\}$. In particular φ is the trivial homomorphism. In this case φ is a fixed point by the action of G on $\mathrm{Hom}(\Gamma, G)$ and thus φ is an isolated point in $\mathrm{Hom}(\Gamma, G)$. But $\mathrm{Hom}(\Gamma, G)$ is closed, then for all $\psi \in \mathrm{Hom}(\Gamma, G)$, $\lim_{t \to 0} t \cdot \psi \in F'(\Gamma, G) = \{\varphi\}$. It comes out that φ is an isolated point if only if $\mathrm{Hom}(\Gamma, G)$ is reduced to the trivial homomorphism, which is impossible because the natural injection of Γ in G is not trivial. □

4.2. Local rigidity

The aim of this section is to prove Theorem 4.2 and corollary 4.3.

Lemma 4.10. *If H is a dilation-invariant subgroup, then $\mathscr{R}(\Gamma, G, H)$ is an $\mathbb{R}_+^*$-stable subspace of* $\mathrm{Hom}(\Gamma, G)$. *In particular $\mathbb{R}_+^*$ acts continuously on $\mathscr{T}(\Gamma, G, H)$.*

Proof. As $\mathbb{R}_+^*$ acts by a continuous automorphism, φ is injective and $\varphi(\Gamma)$ is discrete if and only if $(t \cdot \varphi)$ is also injective and $(t \cdot \varphi)(\Gamma)$ is discrete. Recall that G is simply connected, then the subgroup Γ is torsion free. Thus the action of Γ on G/H is proper only if it is free. By definition of the parameter space (1.1), we have to prove that the action of Γ on G/H via φ is proper if and only if the action of Γ on G/H via $t\cdot\varphi$ is proper. Let S be a compact set in G. Then

$$\begin{aligned}(t\cdot\varphi)(\Gamma)_S &= \{\gamma\in(t\cdot\varphi)(\Gamma),\ \gamma SH\cap SH\neq\emptyset\}\\ &= t\{\gamma\in\varphi(\Gamma),\ (t\cdot\gamma)SH\cap SH\neq\emptyset\}\\ &= t\{\gamma\in\varphi(\Gamma),\ t^{-1}\big((t\cdot\gamma)SH\cap SH\big)\neq\emptyset\}\\ &= t\{\gamma\in\varphi(\Gamma),\ \gamma(t\cdot S)H\cap(t\cdot S)H\neq\emptyset\}\\ &= t\varphi(\Gamma)_{t\cdot S}.\end{aligned}$$

Therefore the sets $(t\cdot\varphi)(\Gamma)_S$ and $\varphi(\Gamma)_{t\cdot S}$ are homeomorphic. □

LEMMA 4.11. *Let $\varphi\in\mathscr{R}(\Gamma,G,H)$. If φ is a locally rigid, then $G\cdot\varphi\cap F'(\Gamma,G)\neq\emptyset$. In particular Γ is an abelian subgroup.*

PROOF. Assume that φ is locally rigid, then $[\varphi]$ is an open point in $\mathscr{T}(\Gamma,G,H)$. By continuity of the action of $\mathbb{R}_+^*$, we deduce that $[\varphi]\in\mathrm{Fix}(\Gamma,G)$ and we conclude by Lemma 4.6 that $G\cdot\varphi\cap F'(\Gamma,G)\neq\emptyset$. Let φ' be an element of the intersection, then $\varphi'(\Gamma)$ is a subgroup of the abelian subgroup $\mathbb{R}^n\times Z(H_{2n+1})$. As φ' is injective, Γ is abelian. □

LEMMA 4.12. *Let $\varphi\in F'(\Gamma,G)\cap\mathscr{R}(\Gamma,G,H)$. Then for any real number $t\neq 0$, the element $\varphi_t\in F'(\Gamma,G)$ defined by*

$$\zeta\big((\varphi_t)_i\big)=t\zeta(\varphi_i)\ \text{ and }\ \tau_l\big((\varphi_t)_i\big)=t\tau_l(\varphi_i)\ \text{ for all } l=1,\dots,n,$$

is also an element of $\mathscr{R}(\Gamma,G,H)$.

PROOF. The subgroup $\varphi(\Gamma)$ is a discrete subgroup of the connected simply connected Lie subgroup $\mathbb{R}^n\times Z(H_{2n+1})$. Thus $\varphi(\Gamma)$ has a syndetic hull L_φ, which is the linear span of the generators $\varphi_1,\dots,\varphi_k$. Similarly if $t\neq 0$ the subgroup $\varphi_t(\Gamma)$ is a discrete subgroup of $\mathbb{R}^n\times Z(H_{2n+1})$ and its syndetic hull is equal to L_φ. By Lemma 2.3 the action of $\varphi(\Gamma)$ is proper if and only if the action of $\varphi_t(\Gamma)$ is proper. □

Let $\mathbb{R}^*:=\mathbb{R}\setminus\{0\}$. From Lemma 4.12, for $\varphi\in F'(\Gamma,G)\cap\mathscr{R}(\Gamma,G,H)$, the map

$$\begin{aligned}c_\varphi:\mathbb{R}^* &\longrightarrow \mathscr{R}(\Gamma,G,H)\\ t &\longmapsto \varphi_t\end{aligned}$$

is a well defined and continuous injection. As a consequence we can state the following :

LEMMA 4.13. *The map* $\overline{c_\varphi} : \mathbb{R}^* \to \mathscr{T}(\Gamma, G, H)$, $t \mapsto [\varphi_t]$, *is a continuous injection.*

PROOF. The map $\overline{c_\varphi}$ is the composition of two continuous injective maps, c_φ and the restriction of the quotient map to $F'(\Gamma, G) \cap \mathscr{R}(\Gamma, G, H)$. □

PROOF OF THE THEOREM 4.2. Suppose that φ is locally rigid. By Lemma 4.11, we can assume that $\varphi \in F'(\Gamma, G)$. Then the map c_φ is well defined and using Lemma 4.13, we conclude that $\overline{c_\varphi}^{-1}\big([\varphi]\big)$ is an open point in $\mathbb{R}^*$, which is impossible. □

PROOF OF COROLLARY 4.3. Assume that U is connected. By Proposition 3.9, the subgroup $K = U \ltimes L(H_0)$ is a syndetic hull of H, where $L(H_0)$ is the syndetic hull of H_0. As $\varphi(\Gamma)$ is a torsion free, $\mathscr{R}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, L(H))$ by Lemma 2.4. Proposition 3.5 induces the existence of $g \in G$ such that $\mathrm{Ad}_g(\mathfrak{k})$ is a subgraded subalgebra, where $\mathfrak{k}$ is the Lie subalgebra of K. Then gKg^{-1} is a dilation-invariant subgroup and by Lemma 2.2 we conclude that $\mathscr{R}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, gKg^{-1})$.

Assume now $H \cap Z(H_{2n+1})$ is non-trivial. As before, we have $\mathscr{R}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, K)$. By hypothesis $L(H_0) \cap Z(H_{2n+1})$ is a non-trivial connected subgroup of $Z(H_{2n+1})$ which is one dimensional, thus $Z(H_{2n+1}) \subset L(H_0)$. Then the Lie subalgebra of $L(H_0)$ is graded and from Lemma 3.13 we conclude that K is dilation-invariant. □

Acknowledgments. The author is immensely thankful to the referee for his extremely careful reading of the previous version of this article. His valuable suggestions have been followed in revising the manuscript.

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(Received July 16, 2013)
(Revised October 15, 2014; February 12, 2015)

University of Carthage
Department of Mathematics
Faculty of Sciences of Bizerte
Tunisia
E-mail: imed.kedim@gmail.com