

## *De Finetti Theorems for a Boolean Analogue of Easy Quantum Groups*

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**Abstract.** We show an organized form of quantum de Finetti theorem for Boolean independence. We define a Boolean analogue of easy quantum groups for the categories of interval partitions, which is a family of sequences of quantum semigroups.

We construct the Haar states on those quantum semigroups. The proof of our de Finetti theorem is based on the analysis of the Haar states.

### **Introduction**

In the study of distributional symmetries in probability theory, the permutation groups  $S_n$  and the orthogonal groups  $O_n$  play central roles. The de Finetti theorem states that a sequence of real random variables have a joint distribution which is stable under each  $S_n$  action if and only if it is conditionally independent and identically distributed (i.i.d. for short) over its tail  $\sigma$ -algebra. Similarly, the symmetry given by the orthogonal groups  $O_n$  induces conditionally i.i.d. centered Gaussian random variables. See [2] for details.

In noncommutative probability theory, a probability measure space is replaced with a  $W^*$ -probability space  $(M, \varphi)$  which is a pair of a von Neumann algebra and a normal state. A self-adjoint operator in  $M$  has a role as a random variable. Contrary to Kolmogorov probability theory, there are several possible notions of independence in noncommutative probability theory. By [10], there exist only three universal independences; the classical independence, the free independence and the Boolean independence. Free probability theory is one of the most developed noncommutative probability theory [13]. The Boolean independence appeared in [14], [12]. The Boolean

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one occurs only in the non-unital situations. Each universal independence is characterized by a family of multivariate cumulants whose index runs over one of a category of partitions. Free cumulants and Boolean cumulants are determined by noncrossing partitions and interval partitions, respectively. By using Boolean cumulants, it can be proven that the central limit distribution of the Boolean independence is the Bernoulli distribution.

Köstler and Speicher have shown the free de Finetti theorem in [3]. The theorem states that the symmetry given by the free permutation groups  $(C(S_n^+))_{n \in \mathbb{N}}$  induces the conditional free independence. The free permutation group  $C(S_n^+)$  is the liberation, that is, a free analogue, of  $S_n$  (See [1] for the liberation). More precisely, the Hopf algebra  $C(S_n^+)$  is given by eliminating the commuting relations among the generators of the Hopf algebra  $C(S_n)$ . The free permutation group is one of the free quantum groups which appeared in [15], [16].

An easy quantum group is one of Woronowicz's compact matrix quantum groups which is characterized by a tensor category of partitions in the sense of the Tannaka-Krein duality. De Finetti theorems have been proven for easy quantum groups (see [1]) in particular easy groups  $S_n$ ,  $H_n$ ,  $B_n$ ,  $O_n$  and free quantum groups  $C(S_n^+)$ ,  $C(H_n^+)$ ,  $C(B_n^+)$ ,  $C(O_n^+)$ . It is known that every compact quantum group admits a unique Haar state [17], and the Haar states play a main role in the de Finetti theorem.

Liu's work [6] starts the research of the de Finetti theorem for the Boolean independence. He adds a projection  $\mathbf{P}$  to the generators of free quantum groups  $C(S_n^+)$  and defines a quantum semigroup (in the sense of [9])  $\mathcal{B}_s(n)$  and has proven associated Boolean de Finetti theorem. The theorem states that the symmetry given by the family  $(\mathcal{B}_s(n))_{n \in \mathbb{N}}$  characterizes the conditionally Boolean i.i.d. random variables.

## Main Results

To develop the research of the Boolean de Finetti theorem, we are interested in finding the Haar states on Boolean quantum semigroups. By using the Haar state, we can apply the organized strategy for the de Finetti theorems for easy quantum groups [1] a similar way. We define a Boolean analogue of permutation group  $S_n$  in a different form  $Beq_s$ .

We do not prove that  $Beq_s(n)$  and  $\mathcal{B}_s(n)$  are isomorphic, but we prove that  $Beq_s(n)$  and  $\mathcal{B}_s(n)$  admit a same Haar state  $h_s$ . Moreover, we prove that the Boolean quantum semigroups  $Beq_h$  on the category  $I_h$  and the

Boolean quantum semigroups  $Beq_o$  on  $I_o = I_2$  have unique Haar states  $h_h, h_o$ . We do not prove the existence of the Haar state on Boolean quantum semigroups  $Beq_b$  on  $I_b$ , but we prove that of the Haar state on Boolean pr-quantum semigroups  $\mathcal{A}_p[I_b]$ .

We first define the notion of categories of interval partitions which is deeply connected with Boolean independence by Boolean cumulants. By using the categories of interval partitions, we induce the notion of Boolean pre-quantum semigroups  $(\mathcal{A}_p[D; n])_{n \in \mathbb{N}}$  (see Definition 2.3) which is a sequence of unital  $*$ -algebras equipped with coproducts. Taking their  $C^*$ -completion, we define Boolean quantum semigroups  $Beq_x(n)$ .

For a sequence of coalgebras  $(\mathcal{A}(n))_{n \in \mathbb{N}}$ , we say that  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}$ -invariant if its joint distribution is invariant under the coactions of  $(\mathcal{A}(n))_{n \in \mathbb{N}}$ . Then we show the following Boolean de Finetti theorems.

**THEOREM 0.1.** *Let  $(M, \varphi)$  be a pair of a von Neumann algebra and a nondegenerate normal state. Assume  $M$  is  $\sigma$ -weakly generated by self-adjoint elements  $(x_j)_{j \in \mathbb{N}}$ . Let  $M_{\text{nut}}$  be the non-unital tail von Neumann algebra.*

- (s) *The following assertions are equivalent;*
  - (0) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{B}_s$ -invariant.*
  - (alg) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I]$ -invariant.*
  - (beq) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $Beq_s$ -invariant.*
  - (iid) *The elements  $(x_j)_{j \in \mathbb{N}}$  are Boolean i.i.d. over  $M_{\text{nut}}$ .*
- (o) *The following assertions are equivalent;*
  - (alg) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_2]$ -invariant.*
  - (beq) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $Beq_o$ -invariant.*
  - (iid) *The elements  $(x_j)_{j \in \mathbb{N}}$  form a  $M_{\text{nut}}$ -valued Boolean centered Bernoulli family.*
- (h) *The following assertions are equivalent;*
  - (alg) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_h]$ -invariant.*
  - (beq) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $Beq_h$ -invariant.*

- (i) (i) *The elements  $(x_j)_{j \in \mathbb{N}}$  are Boolean independent, and have even and identically distributions, over  $M_{\text{nut}}$ .*
- (b) *The following assertions are equivalent;*
  - (alg) *The sequence  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_b]$ -invariant.*
  - (i) *The elements  $(x_j)_{j \in \mathbb{N}}$  form a  $M_{\text{nut}}$ -valued Boolean shifted Bernoulli family.*

The common difficulty in carrying out the proof is that Boolean independence is a non-unital phenomenon. That is, if  $M$  is a von Neumann algebra and  $\varphi$  is a faithful normal state on  $A$ , and  $(M_1, M_2)$  is a pair of von Neumann subalgebras with  $1_M \in M_1, M_2$  and  $M_1, M_2 \neq \mathbb{C}1_M$ . Then  $(M_1, M_2)$  cannot be Boolean independent in  $(M, \varphi)$ . Hence, we consider non-unital embeddings of von Neumann algebras in the arguments of Boolean independence and conditional Boolean independence.

The main difficulty is to find the Haar states on  $(\text{Beq}_x(n))_{n \in \mathbb{N}}$ . We do that by constructing the GNS-representation of  $\text{Beq}_s(n)$  on the Hilbert space  $L^2(S_n)$  of  $L^2$ -functions on classical permutation group  $S_n$ .

## Related Works

In recent preprints [5] [7], Liu generalizes  $\mathcal{B}_s$  in a different form from  $\text{Beq}_x$  and proves generalized Boolean de Finetti theorems. His strategy does not rely on the Haar states.

## Organization

This paper consists of four sections. Section 1 is devoted to some preliminaries. In Section 2, we introduce the Boolean pre-quantum semigroups  $\mathcal{A}_p[D; n]$  and the Boolean quantum semigroups  $\text{Beq}_x(n)$ . Section 3 provides a detailed exposition of the Haar functionals and the Haar states. In Section 4, our main results, the Boolean de Finetti type results are proved.

## 1. Preliminaries

### 1.1. Partitions

Let us review some notations related to partitions of a set.

*Notation 1.1.*

- (1) A partition of a set  $S$  is a decomposition into mutually disjoint, non-empty subsets. Those subsets are called blocks of the partition. We denote by  $P(S)$  the set of all partitions of  $S$ .
- (2) For a partition  $\pi$  of a set  $S$  and  $r, s \in S$ , we define  $r \underset{\pi}{\sim} s$  if  $r$  and  $s$  belong to the same block of  $\pi$ .
- (3) Let  $S, J$  be any sets and  $\mathbf{j} \in \text{Map}(S, J)$ . We denote by  $\ker \mathbf{j}$  the partition of  $S$  defined as  $r \underset{\ker \mathbf{j}}{\sim} s$  if and only if  $j(r) = j(s)$ .
- (4) For  $\pi, \sigma \in P(S)$ , we write  $\pi \leq \sigma$  if each block of  $\pi$  is a subset of some block of  $\sigma$ . The set  $P(S)$  is a poset under the relation  $\leq$ .
- (5) We set for  $\pi, \sigma \in P(S)$ ,

$$\delta(\pi, \sigma) := \begin{cases} 1, & \text{if } \pi = \sigma, \\ 0, & \text{otherwise,} \end{cases} \quad \zeta(\pi, \sigma) := \begin{cases} 1, & \text{if } \pi \leq \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

We introduce the Möbius function. See [8] for more details.

DEFINITION 1.2 (The Möbius function). Let  $(P, \leq)$  be a finite poset. The Möbius function  $\mu_P: P^2 \rightarrow \mathbb{C}$  is defined as the inverse of  $\zeta$ , that is, determined by the following relations: for any  $\pi, \sigma \in P$  with  $\pi \not\leq \sigma$ ,  $\mu_P(\pi, \sigma) = 0$ , and for any  $\pi, \sigma \in P$  with  $\pi \leq \sigma$ ,

$$(1.1) \quad \sum_{\substack{\rho \in P \\ \pi \leq \rho \leq \sigma}} \mu_P(\pi, \rho) = \delta(\pi, \sigma), \quad \sum_{\substack{\rho \in P \\ \pi \leq \rho \leq \sigma}} \mu_P(\rho, \sigma) = \delta(\pi, \sigma),$$

The following remark is one of the most important properties of the Möbius function to prove de Finetti theorems.

PROPOSITION 1.3. Let  $Q$  be a subposet of  $P$  which is closed under taking an interval, that is, if  $\pi, \sigma \in Q, \rho \in P$  and  $\pi \leq \rho \leq \sigma$  then  $\rho \in Q$ . Then for any  $\pi, \sigma \in Q$  with  $\pi \leq \sigma$ , we have  $\mu_Q(\pi, \sigma) = \mu_P(\pi, \sigma)$ .

PROOF. The proposition follows from the relations (1.1).  $\square$

We define the notion of categories of interval partitions.

DEFINITION 1.4. A partition  $\pi \in P(k)$  is said to be an *interval partition* of  $[k]$  if each block contains only consecutive elements. We denote by  $I(k)$  the set of all interval partitions of  $[k]$ .

DEFINITION 1.5. The *tensor product*  $\otimes$  of partitions is defined by horizontal concatenation.

DEFINITION 1.6. A *category of interval partitions* is a collection  $D = (D(k))_{k \in \mathbb{N}}$  of subsets  $D(k) \subseteq I(k)$ , subject to the following conditions.

- (1) It is stable under the tensor product  $\otimes$ .
- (2) It contains the pair partition  $\square$ .

For a category of interval partitions  $D$ , let us denote  $L_D := \{k \in \mathbb{N} : \mathbf{1}_k \in D(k)\}$ , where  $\mathbf{1}_k \in P(k)$  is the partition which contains only one block  $\{1, 2, \dots, k\}$ .

Notation 1.7. We denote by  $I_h(k), I_b(k)$ , and  $I_2(k) \subseteq I(k)$  the set of all interval partitions with even block size, with block size  $\leq 2$ , and with block size 2 of  $[k]$ , respectively. Then each  $I_x$  ( $x = h, b, 2$ ) is a category of interval partitions. We also write  $I_s = I$ ,  $I_o = I_2$ . Then we have  $L_{I_s} = \mathbb{N}$ ,  $L_{I_o} = \{2\}$ ,  $L_{I_h} = \{2, 4, 6, \dots\}$  and  $L_{I_b} = \{1, 2\}$ .

Notation 1.8. For  $n \in \mathbb{N}$ , we denote by  $l_n^2$  the standard  $n$ -dimensional Hilbert space. For  $k \in \mathbb{N}$  and  $\pi \in P(k)$ , set a vector in  $l_n^{2 \otimes k}$  by

$$T_\pi^{(n)} := \sum_{\substack{\mathbf{j} \in [n]^k, \\ \pi \leq \ker \mathbf{j}}} e_{\mathbf{j}},$$

where  $(e_i)_{i \in [n]}$  is a fixed complete orthonormal basis of  $l_n^2$  and  $e_{\mathbf{j}} := e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_k}$ . For a category of interval partition  $D$ , let  $H^{D(k)}(n) \in B(l_n^{2 \otimes k})$  be the orthogonal projection onto the subspace  $\text{Span}\{T_\pi^{(n)} \mid \pi \in D(k)\}$ . We omit the index  $(n)$  if there is no confusion. We set

$$H_{\mathbf{i}\mathbf{j}}^{D(k)} := \langle e_{\mathbf{i}}, H^{D(k)} e_{\mathbf{j}} \rangle.$$

DEFINITION 1.9 (The Weingarten function). For  $\pi, \sigma \in P(k)$ , set the Gram matrix  $G_{k,n}$  by  $G_{k,n}(\pi, \sigma) := \langle T_\pi^{(n)}, T_\sigma^{(n)} \rangle = n^{|\pi \vee \sigma|}$ . Let  $D$  be a category of interval partitions. Since the family  $(T_\pi^{(n)})_{\pi \in D(k)}$  is linearly independent for large  $n$ ,  $G_{k,n}$  is invertible for sufficiently large  $n$ . We define the Weingarten function  $W_{k,n}^D$  to be its inverse.

PROPOSITION 1.10. Let  $D$  be a category of interval partitions. For any  $\mathbf{i}, \mathbf{j} \in [n]^k$  and sufficiently large  $n$ , we have

$$H_{\mathbf{i}, \mathbf{j}}^{D(k)} = \sum_{\substack{\pi, \sigma \in D(k) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{k,n}^D(\pi, \sigma).$$

PROOF. This is a special case of a well-known result, see [1] for more details.  $\square$

DEFINITION 1.11. A category  $D$  of interval partitions is said to be closed under taking an interval if for any  $k \in \mathbb{N}$  and  $\rho, \sigma \in D(k)$ , we have

$$\{\pi \in I(k) \mid \rho \leq \pi \leq \sigma\} = \{\pi \in D(k) \mid \rho \leq \pi \leq \sigma\}.$$

PROPOSITION 1.12 (The Weingarten estimate). Assume  $D$  is closed under taking an interval. For any  $\pi, \sigma \in D(k)$ ,

$$n^{|\pi|} W_{k,n}^D(\pi, \sigma) = \mu_{I(k)}(\pi, \sigma) + O\left(\frac{1}{n}\right) \text{ (as } n \rightarrow \infty),$$

PROOF. By [1, Prop.3.4], it holds that  $n^{|\pi|} W_{k,n}^D(\pi, \sigma) = \mu_{D(k)}(\pi, \sigma) + O(1/n)$ , as  $n \rightarrow \infty$ . Since the subposet  $D(k) \subseteq I(k)$  is closed under taking an interval, we have  $\mu_{I(k)} = \mu_{D(k)}$ , which proves the proposition.  $\square$

REMARK 1.13. We call a category of interval partition  $D$  is *join-stable* or  *$\vee$ -stable* if  $\sigma \vee \rho \in D(k)$  for any  $\sigma, \rho \in D(k), k \in \mathbb{N}$ . We see that each category of interval partitions  $I_s, I_o, I_h$  is  $\vee$ -stable. Therefore, for  $x = s, o, h$ , there exists an interval partition  $\max_{I_x(k)} W \in I_x(k)$  such that for any nonempty subset  $W \subseteq I_x(k)$  with  $W \vee W \subseteq W$ .

However, the category  $I_b$  is not  $\vee$ -stable. For example,

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \vee \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \notin I_b(3).$$

*Notation 1.14.* Let the index  $x$  be one of  $s, o, h$ . For any  $k \in \mathbb{N}$  and  $\sigma \in P(k)$ , we write

$$\inf_{I_x} \sigma := \max\{\pi \in I_x(k) \mid \pi \leq \sigma\}.$$

**1.2. Nonunital tail von Neumann algebras**

Let us define non-unital tail von Neumann algebras. In this paper, we do not assume that an embedding of  $*$ -algebras,  $C^*$ -algebras or von Neumann algebras is unital.

DEFINITION 1.15.

- (1) For  $n \in \mathbb{N}$ , denote by  $\mathcal{P}_n^o$  (resp.  $\mathcal{P}_\infty^o$ ) the  $*$ -algebra of all polynomials without constant terms in noncommutative  $n$ -variables  $X_1, \dots, X_n$  (resp. countably infinite many variables  $(X_j)_{j \in \mathbb{N}}$ ).
- (2) Let  $M$  be a von Neumann algebra. Let  $(x_j)_{j \in \mathbb{N}}$  be a sequence of self-adjoint elements in  $M$ . Denote by  $\text{ev}_x: \mathcal{P}_\infty^o \rightarrow M$  the evaluation map  $\text{ev}_x(X_j) = x_j$ . Let us denote by  $M_{\text{nut}}$  the non-unital tail von Neumann algebra, that is,

$$M_{\text{nut}} := \bigcap_{n=1}^{\infty} \overline{\text{ev}_x(\mathcal{P}_{\geq n}^o)}^{\sigma w},$$

where  $\mathcal{P}_{\geq n}^o := \{f \in \mathcal{P}_\infty^o \mid f \text{ is a polynomial in variables } X_j (j \geq n)\}$ .

We define the notion of conditional expectations for non-unital embeddings.

DEFINITION 1.16. Let  $\eta: B \hookrightarrow A$  be an embedding of  $*$ -algebras. A linear map  $E: A \rightarrow B$  is said to be a conditional expectation with respect to  $\eta$  if it satisfies the following conditions:

- (1)  $E(x^*x) \geq 0$  for all  $x \in A$ ,
- (2)  $E \circ \eta = \text{id}_B$ ,



(3)  $E(\eta(b)x) = bE(x), E(x\eta(b)) = E(x)b$  for all  $b \in B, x \in A$ .

DEFINITION 1.17. Let  $A, B, \eta$  and  $E$  be the same as in Definition 1.16. Let  $(a_j)_{j \in J}$  be self-adjoint elements in  $A$ . We say  $(a_j)_{j \in J}$  are identically distributed over  $(E, B)$  if  $E[a_i b_1 a_i b_2 \cdots a_i b_k] = E[a_i b_1 a_i b_2 \cdots a_i b_k]$  holds for any  $i, j \in J, k \in \mathbb{N}$ , and  $b_1, \dots, b_k \in B \cup \{1_A\}$ .

Let us introduce the notion of conditional Boolean independence.

DEFINITION 1.18. Let  $\eta: B \hookrightarrow A$  be a non-unital embedding of unital  $*$ -algebras  $A, B$  with a conditional expectation  $E: A \rightarrow B$ . Let  $1_A$  be a unit of  $A$ . Let  $(x_j)_{j \in J}$  be a family of self-adjoint elements of  $A$ . Write

$$B\langle x_j \rangle^o := \text{Span} \bigcup_{n=1}^{\infty} \{b_0 x_{j_1} b_1 x_{j_2} \dots b_{n-1} x_{j_n} b_n \mid b_0, \dots, b_n \in B \cup \{1_A\}\}.$$

The elements  $(x_j)_{j \in J}$  are said to be *Boolean independent over  $(E, B)$*  if

$$E[y_1 \cdots y_k] = E[y_1] \cdots E[y_k],$$

whenever  $k \in \mathbb{N}, j_1, \dots, j_k \in J, j_1 \neq j_2 \neq \dots \neq j_k$ , and  $y_l \in B\langle x_{j_l} \rangle^o, l = 1, \dots, k$ .

Notation 1.19.

- (1) Let  $(S, \leq)$  be a finite totally ordered set and we write  $S = \{s_1 < s_2 < \dots < s_n\}$ . For a family  $(a_s)_{s \in S}$  of elements in  $M$ , we denote by  $\prod_{s \in S}^{\rightarrow} a_s$  the ordered product  $\prod_{s \in S}^{\rightarrow} a_s = a_{s_1} \cdots a_{s_n}$ .
- (2) For an interval partition  $\pi$  and blocks  $V, W \in \pi$ , we write  $V \leq W$  if  $k \leq l$  for any  $k \in V$  and  $l \in W$ . The set  $\pi$  is a totally ordered set under the relation  $\leq$ .

LEMMA 1.20. *The elements  $(x_j)_{j \in \mathbb{N}}$  are Boolean independent and identically distributed over  $(E, B)$  if and only if the following holds: for any  $j_1, \dots, j_k \in \mathbb{N}$  and  $b_0, b_1, \dots, b_k \in B \cup \{1_A\}$ ,*

$$E[b_0 x_{j_1} b_1 x_{j_2} b_2 \cdots x_{j_k} b_k] = b_0 \cdot \prod_{V \in \text{inf}_I \ker \mathbf{j}}^{\rightarrow} E[\prod_{l \in V}^{\rightarrow} x_{j_l} b_l].$$

PROOF. For  $r, s \in [n], r \sim^{\text{inf}_I \ker \mathbf{j}} s$  if and only if  $r$  and  $s$  are consecutive elements and  $j_r = j_s$ . By the linearity of  $E$ , we have the claim.  $\square$

**1.3. Boolean cumulants**

In operator-valued free probability, operator-valued cumulants characterize the conditional free independence (see [8] [11]). We introduce some properties of the operator-valued Boolean cumulants. They combinatorially characterize conditional Boolean independence. Single variate Boolean cumulants are defined in [12]. As far as the author knows, multivariate Boolean cumulants first appeared in [4].

Throughout this section, we suppose  $B \subseteq A$  is an embedding of  $*$ -algebras (not necessarily unital) with a normal conditional expectation  $E$ .

DEFINITION 1.21. Let us define  $B$ -valued multilinear functions  $K_\pi^E : A^n \rightarrow B$  ( $\pi \in I(k), k \in \mathbb{N}$ ) inductively by the following three relations:

- (1) For  $k \in \mathbb{N}$  and  $y_1, \dots, y_k \in M$ ,  $E[y_1 \cdots y_k] = \sum_{\pi \in I(k)} K_\pi^E[y_1, \dots, y_k]$ .
- (2) For  $k \in \mathbb{N}$  and  $\pi \in I(k)$ ,  $K_\pi^E[y_1, \dots, y_k] = \prod_{V \in \pi} K_{(V)}^E[y_1, \dots, y_k]$ .
- (3) For  $\pi \in I(k)$  and  $V \in \pi$ ,  $K_{(V)}^E[y_1, \dots, y_k] := K_{\mathbf{1}_m}^E[y_{j_1}, \dots, y_{j_m}]$  where  $V = \{j_1 < j_2 < \cdots < j_m\}$ .

We call them *Boolean cumulants* with respect to  $E$ . We write  $K_n^E = K_{\mathbf{1}_n}^E$  for  $n \in \mathbb{N}$ .

PROPOSITION 1.22. For  $\pi \in I(k)$ ,  $y_1, \dots, y_k$  and  $k \in \mathbb{N}$ , set  $E^\pi[y_1, \dots, y_k] := \prod_{V \in \pi} E[\prod_{j \in V} y_j]$ . Then for  $\pi \in P(k)$ ,  $y_1, \dots, y_k \in M$  and  $k \in \mathbb{N}$ ,

$$E^\pi[y_1, \dots, y_k] = \sum_{\substack{\sigma \in I(k) \\ \sigma \leq \pi}} K_\sigma^E[y_1, \dots, y_k].$$

Hence we have  $K_\pi^E[y_1, \dots, y_k] = \sum_{\substack{\sigma \in I(k) \\ \sigma \leq \pi}} E^\sigma[y_1, \dots, y_k] \mu_{I(k)}(\sigma, \pi)$ .

PROOF. The proof is a straight induction on  $|\pi|$ .  $\square$

The conditional Boolean independence can be characterized by vanishing of mixed cumulants.

**THEOREM 1.23.** *Let  $(x_j)_{j \in J}$  be a family of self-adjoint elements in  $A$ . Then  $(x_j)_{j \in J}$  are Boolean independent identically distributed with respect to  $E$  if and only if*

$$E[b_0 x_{j_1} b_1 x_{j_2} b_2 \cdots x_{j_k} b_k] = \sum_{\substack{\pi \in I(k) \\ \pi \leq \ker \mathbf{j}}} K_\pi^E [b_0 x_1 b_1, x_1 b_2, \dots, x_1 b_k]$$

for any  $b_1, \dots, b_k \in B \cup \{1_A\}$ ,  $\mathbf{j} \in J^k$ ,  $k \in \mathbb{N}$ .

**PROOF.** We have

$$\begin{aligned} b_0 \cdot \prod_{V \in \inf_I \ker \mathbf{j}}^{\rightarrow} E[\prod_{l \in V}^{\rightarrow} x_1 b_l] &= b_0 E^{\inf_I \ker \mathbf{j}} [x_1 b_1, x_2 b_2, \dots, x_1 b_k] \\ &= \sum_{\substack{\pi \in I(k) \\ \pi \leq \inf_I \ker \mathbf{j}}} K_\pi^E [b_0 x_1 b_1, x_1 b_2, \dots, x_1 b_k]. \end{aligned}$$

We see that  $\{\pi \in I(k) \mid \pi \leq \inf_I \ker \mathbf{j}\} = \{\pi \in I(k) \mid \pi \leq \ker \mathbf{j}\}$ . Lemma 1.20 completes the proof.  $\square$

**DEFINITION 1.24.** Let  $x$  be a self-adjoint element in  $(M, E)$ .

- (1) The element  $x$  is said to *have centered Bernoulli distribution* if for any  $b_1, \dots, b_{k-1} \in N \cup \{1_M\}$  and  $k \in \mathbb{N}$ ,

$$E[x b_1 x b_2 \cdots b_{k-1} x] = \sum_{\pi \in I_2(k)} K_\pi^E [x b_1, x b_2, \dots, x].$$

We see immediately that if  $N = \mathbb{C}1_M$ ,  $x$  has centered Bernoulli distribution if and only if that is  $(\delta_\sigma + \delta_{-\sigma})/2$  where  $\sigma := \sqrt{K_2^E[x, x]}$ .

- (2) The element  $x$  is said to *have shifted Bernoulli distribution* if for any  $b_1, \dots, b_{k-1} \in N \cup \{1_M\}$  and  $k \in \mathbb{N}$ ,

$$E[x b_1 x b_2 \cdots b_{k-1} x] = \sum_{\pi \in I_b(k)} K_\pi^E [x b_1, x b_2, \dots, x].$$

We check easily that if  $N = \mathbb{C}1_N$ ,  $x$  has shifted Bernoulli distribution with  $K_1^E[x] = \mu$  and  $K_2^E[x, x] = \sigma^2$  if and only if its distribution is

$$\text{Ber}(\mu, \sigma^2) := \frac{\alpha \delta_\alpha + \beta \delta_{-\beta}}{\alpha + \beta},$$

where  $\alpha, \beta > 0$ , and  $(\alpha, -\beta)$  is the pair of distinct solutions of the quadratic equation  $Z^2 - \mu Z - \sigma^2 = 0$  in the variable  $Z$ . Its  $n$ -th moment is given by  $E[x^n] = (\alpha^{n+1} - (-\beta)^{n+1}) / (\alpha + \beta)$ .

A Bernoulli distribution is the central limit distribution of Boolean i.i.d. self-adjoint elements (see [12]). Hence the Bernoulli distribution is the Boolean analogue of Gaussian distribution.

## 2. Boolean Analogues of Easy Quantum Groups

### 2.1. Boolean quantum semigroups

In this section we introduce the notions of Boolean quantum semigroups on categories of interval partitions.

DEFINITION 2.1. For a category  $D$  of interval partitions, consider the following three conditions.

(D1) It is *block-stable*, which means that for any  $k \in \mathbb{N}$ ,

$$D(k) = \{\pi \in D(k) \mid \{V\} \in D(|V|), V \in \pi\}.$$

(D2) It is closed under taking an interval.

(D3) It *has enough partitions*, which means that for  $l \in \mathbb{N}$ , it holds that  $D(l) \neq \emptyset$  if there is  $k \in L_D$  with  $D(k+l) \neq \emptyset$ .

We say that  $D$  is *blockwise* if it satisfies (D1)–(D3).

*Example 2.2.* Categories  $I_s, I_o, I_h, I_b$  of interval partitions are blockwise.

DEFINITION 2.3. Let  $D$  be a blockwise category of partitions. Denote by  $\mathcal{A}[D; n]$  the non-unital  $*$ -algebra generated by self-adjoint elements  $u_{ij}^{(n)}$  ( $1 \leq i, j \leq n$ ) and an orthogonal projection  $p^{(n)}$  with the following

relations: for any  $k \in L_D$  and  $\mathbf{i}, \mathbf{j} \in [n]^k$ ,

$$\sum_{i=1}^n u_{ij_1}^{(n)} \cdots u_{ij_k}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & j_1 = \cdots = j_k, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{j=1}^n u_{i_1j}^{(n)} \cdots u_{i_kj}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & i_1 = \cdots = i_k, \\ 0, & \text{otherwise.} \end{cases}$$

If there is no confusion, we omit the index  $(n)$  and simply write  $u_{i,j}$  and  $p$ . There is a linear map  $\Delta: A[D; n] \rightarrow A[D; n] \otimes A[D; n]$  with

$$\Delta(p) := p \otimes p, \quad \Delta(u_{ij}) := \sum_{k=1}^n u_{ik} \otimes u_{kj} \quad (i, j = 1, 2, \dots, n).$$

It is easy to check that  $\Delta$  is a coproduct, that is, the following holds:

$$(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta.$$

Set a linear map  $\varepsilon: A[D; n] \rightarrow \mathbb{C}$  by  $\varepsilon(u_{ij}) = \delta_{ij}$ ,  $\varepsilon(p) = 1$ . We have  $(\text{id} \otimes \varepsilon)\Delta = \text{id} = (\varepsilon \otimes \text{id})\Delta$ . Hence  $A[D; n]$  is a coalgebra with the coproduct  $\Delta$  and the counit  $\varepsilon$ . We define a sequence of unital  $*$ -algebra equipped with a coproduct by

$$\mathcal{A}_p[D; n] := pA[D; n]p$$

We call  $(\mathcal{A}_p[D; n])_{n \in \mathbb{N}}$  the *Boolean pre-quantum semigroups on  $D$* .

DEFINITION 2.4. We call the sequences of pairs  $((\text{Beq}_x(n), \Delta))_{n \in \mathbb{N}}$  defined by the following *Boolean quantum semigroups on  $D$*  for  $I_x$  ( $x = s, o, h, b$ ).

- (1) For  $a \in \mathcal{A}[I_x; n]$ , we set

$$\|a\| := \sup\{\|\pi(a)\| \mid \pi \text{ is a } * \text{-representation of } \mathcal{A}[I_x; n], \\ \pi(p) = 1, \|(\pi(u_{ij}))_{i,j}\|_n \leq 1\},$$

where  $\|\cdot\|_n$  is the operator norm on  $B(H) \otimes M_n(\mathbb{C})$  for each  $*$ -representation  $(\pi, H)$ . Since there is the  $*$ -representation  $\pi: A[I; n] \rightarrow C(S_n) \subseteq B(H)$  defined by  $\pi(u_{ij})(\sigma) = \delta(\sigma(i), j)$  ( $\sigma \in S_n$ ), we obtain  $0 \leq \|a\| < \infty$ . Hence  $\|\cdot\|$  is a  $C^*$ -seminorm on  $\mathcal{A}[I_x; n]$ .

- (2) Let  $B$  be the  $C^*$ -completion of  $\mathcal{A}[I_x; n]/\langle \|\cdot\| = 0 \rangle$ . We define the Boolean quantum semigroup on  $I_x$  of  $n$  by

$$Beq_x(n) := pBp.$$

- (3) We denote by  $\iota_n$  the unital  $*$ -hom  $\mathcal{A}_p[I_x; n] \rightarrow Beq_x(n)$  which is the restriction of the  $*$ -hom  $\mathcal{A}[I_x; n] \rightarrow Beq_x(n)$  determined by  $\iota_n(u_{ij}) = [u_{ij}]$  ( $i, j \in [n]$ ),  $\iota_n(p) = [p]$ . By abuse of notation, we use the same symbols  $u_{ij}, p$  for the generators  $[u_{ij}], [p]$  of  $Beq_x(n)$ .
- (4) For any  $*$ -representation  $\pi$  of  $\mathcal{A}[I_x; n]$  with  $\|(\pi(u_{ij}))_{ij}\|_n \leq 1$ , we obtain  $\|(\pi(\Delta u_{ij}))_{ij}\|_n \leq 1$ . Hence we can extend the domain of  $\Delta$ , that is, there is a unique bounded  $*$ -hom  $\bar{\Delta}: Beq_x(n) \rightarrow Beq_x(n) \otimes_{\min} Beq_s(n)$  with  $\bar{\Delta}(u_{ij}) = \sum_{s \in [n]} u_{is} \otimes u_{sj}$  and  $\bar{\Delta}(p) = p \otimes p$ . We simply denote by  $\Delta$  the bounded  $*$ -hom  $\bar{\Delta}$  if there is no confusion. It is easy to check that  $\Delta$  is a coproduct of  $Beq_x(n)$ .

LEMMA 2.5. *Let the index  $x$  be one of  $s, o, h, b$ . Then for any  $k, n \in \mathbb{N}$  and  $\pi \in I_x(k)$ , we have*

$$\sum_{\substack{\mathbf{i} \in [n]^k \\ \pi \leq \ker \mathbf{i}}} u_{i_1 j_1}^{(n)} \cdots u_{i_k j_k}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & \pi \leq \ker \mathbf{j}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} u_{i_1 j_1}^{(n)} \cdots u_{i_k j_k}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & \pi \leq \ker \mathbf{i}, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. The proof is induction on  $|\pi|$ .  $\square$

REMARK 2.6. We denote by  $P_{ij} \in B(L^2(S_n))$  ( $i, j \leq n$ ) the generators of  $C(S_n)$ , where  $P_{ij}(\sigma) = \delta_{i, \sigma(j)}$  ( $\sigma \in S_n$ ). We see at once that there is a  $*$ -representation  $Beq_s \rightarrow B(L^2(S_n))$  which maps  $u_{ij}$  to  $P_{ij}$  and  $P$  to 1.

In Section 3.2, we prove that there is another  $*$ -representation on  $L^2(S_n)$  (see Notation 3.8, Proposition 3.9). In the construction, we use  $P_{ij}$  in a different way. Let  $\hat{P}_{ij}$  (resp.  $\hat{1}$ ) be the image of  $P_{ij}$  (resp. 1) with respect to the standard inclusion  $C(S_n) \hookrightarrow L^2(S_n)$ . The  $*$ -representation maps  $u_{ij}$

(resp.  $p$ ) to the one dimensional projection onto the closed subspace  $\mathbb{C}\hat{P}_{ij}$  (resp.  $\mathbb{C}\hat{1}$ )  $\subseteq L^2(S_n)$ . Furthermore, we show that  $Beq_s$  admits the unique Haar state and that this  $*$ -representation is the GNS-representation of the Haar state (see Theorem 3.14).

Next we consider coactions on the  $*$ -algebra of noncommutative polynomials without constant terms.

DEFINITION 2.7. Let  $\mathcal{A}$  be a unital  $*$ -algebra equipped with a coproduct  $\Delta$ . For any  $*$ -algebra  $\mathcal{P}$ , a  $*$ -preserving linear map  $T: \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{A}$  is said to be a *linear coaction on  $\mathcal{P}$*  if we have

$$(T \otimes \text{id}) \circ T = (\text{id} \otimes \Delta) \circ T.$$

Notation 2.8. Let  $D$  be a blockwise category of interval partitions.

- (1) For  $m, n \in \mathbb{N}$  with  $m \geq n$ , we define a  $*$ -hom  $r_{nm}: \mathcal{A}[D; m] \rightarrow \mathcal{A}[D; n]$  by

$$r_{nm}(u_{ij}^{(m)}) := \begin{cases} u_{ij}^{(n)}, & i, j \leq n, \\ \delta_{ij} 1_{\mathcal{A}[D; n]}, & \text{otherwise,} \end{cases} \quad r_{nm}(p^{(m)}) := p^{(n)}.$$

- (2) Define a linear map  $\Lambda_n: \mathcal{P}_n^o \rightarrow \mathcal{P}_n^o \otimes \mathcal{A}_p[D; n]$  by

$$\Lambda_n(X_{j_1} \cdots X_{j_k}) := \sum_{\mathbf{i} \in [n]^k} X_{i_1} \cdots X_{i_k} \otimes pu_{i_1 j_1} \cdots u_{i_k j_k} p.$$

We define a linear map  $\Psi_n: \mathcal{P}_\infty^o \rightarrow \mathcal{P}_\infty^o \otimes \mathcal{A}_p[D; n]$  by

$$\Psi_n(f) := (\text{id} \otimes r_{nm}) \circ \Lambda_m(f),$$

for  $f \in \mathcal{P}_m^o \subseteq \mathcal{P}_\infty^o$ . Then by a direct calculation, each  $\Psi_n$  is a linear coaction of  $\mathcal{A}_p[D; n]$  on  $\mathcal{P}_\infty^o$ .

- (3) We define a coaction  $\Phi_n$  of  $Beq_x(n)$  on  $\mathcal{P}_\infty^o$  by

$$\Phi_n := (\text{id} \otimes \iota_n) \circ \Psi_n.$$

DEFINITION 2.9. Let  $(M, \varphi)$  be a pair of a von Neumann algebra and a state. For any sequence  $(x_j)_{j \in \mathbb{N}}$  of self-adjoint elements in  $M$ , we say that its joint distribution is  $\mathcal{A}_p[D]$ -invariant if it is invariant under the coactions of  $(\mathcal{A}[D; n])_{n \in \mathbb{N}}$ , that is, for any  $n \in \mathbb{N}$ ,

$$(\varphi \circ \text{ev}_x \otimes \text{id}) \circ \Psi_n = \varphi \circ \text{ev}_x \otimes p.$$

We also say that it is  $Beq_x$ -invariant if for any  $n \in \mathbb{N}$ ,

$$(\varphi \circ \text{ev}_x \otimes \text{id}) \circ \Phi_n = \varphi \circ \text{ev}_x \otimes p.$$

It is clear that  $\mathcal{A}[I_x]$ -invariance implies  $Beq_x$ -invariance.

### 2.2. Relations with Liu’s Boolean quantum semigroups

We introduce Liu’s boolean permutation quantum semigroup defined in [6]. Let  $B_s(n)$  be the universal unital  $C^*$ -algebra generated by projections  $\mathbf{P}, U_{i,j}(i, j = 1, \dots, n)$  and relations such that

$$\begin{aligned} \sum_{i=1}^n U_{ij} \mathbf{P} &= \mathbf{P}, \quad j = 1, \dots, n, \\ U_{i_1 j} U_{i_2 j} &= 0, \quad \text{if } i_1 \neq i_2, \quad \text{for any } j = 1, \dots, n, \\ U_{i j_1} U_{i j_2} &= 0, \quad \text{if } j_1 \neq j_2, \quad \text{for any } i = 1, \dots, n. \end{aligned}$$

By [6, Lemma 3.3], we have

$$\sum_{j=1}^n U_{ij} \mathbf{P} = \mathbf{P}, \quad i = 1, \dots, n.$$

We see that  $B_s(n)$  admits a coproduct  $\Delta$  determined by  $\Delta(\mathbf{P}) := \mathbf{P} \otimes \mathbf{P}, \Delta(U_{ij}) := \sum_{k=1}^n U_{ik} \otimes U_{kj} (i, j = 1, 2, \dots, n)$ . Then let us introduce Liu’s boolean permutation quantum semigroup.

DEFINITION 2.10. We set  $\mathcal{B}_s(n) = \mathbf{P}B_s(n)\mathbf{P}$ , and we call  $(\mathcal{B}_s(n), \Delta)$  the boolean permutation quantum semigroup of  $n$ .

We can check that each  $\mathcal{B}_s(n)$  is a quantum semigroup in the sense of Soltan [9].



LEMMA 2.11. *There is a  $*$ -hom  $\alpha: \text{Beq}_s(n) \rightarrow \mathcal{B}_s(n)$  with  $\alpha(u_{ij}) = U_{ij}$  ( $i, j \leq n$ ) and  $\alpha(p) = \mathbf{P}$ .*

PROOF. We see that for any  $k \in \mathbb{N}$  and  $\mathbf{i}, \mathbf{j} \in [n]^k$ ,

$$\sum_{i=1}^n U_{i j_1} \cdots U_{i j_k} \mathbf{P} = \begin{cases} \mathbf{P}, & j_1 = \cdots = j_k, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{j=1}^n U_{i_1 j} \cdots U_{i_k j} \mathbf{P} = \begin{cases} \mathbf{P}, & i_1 = \cdots = i_k, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof.  $\square$

Notation 2.12. We set a linear map  $L_n: \mathcal{P}_n^o \rightarrow \mathcal{P}_n^o \otimes \mathcal{B}_s(n)$  by  $L_n(X_{j_1} \cdots X_{j_k}) := \sum_{\mathbf{i} \in [n]^k} X_{i_1} \cdots X_{i_k} \otimes \mathbf{P} U_{i_1 j_1} \cdots U_{i_k j_k} \mathbf{P}$ . We set a linear map  $\mathbf{L}_n: \mathcal{P}_\infty^o \rightarrow \mathcal{P}_\infty^o \otimes \mathcal{B}_s(n)$  by  $\Psi_n(f) := (\text{id} \otimes r_{nm}) \circ \Lambda_m(f)$ , for  $f \in \mathcal{P}_m^o \subseteq \mathcal{P}_\infty^o$ . Then by a direct calculation, each  $\mathbf{L}_n$  is a linear coaction of  $\mathcal{B}_s(n)$  on  $\mathcal{P}_\infty^o$ .

Let  $(M, \varphi)$  be a von Neumann algebra and a nondegenerate normal state and  $(x_j)_{j \in \mathbb{N}}$  be a sequence of self-adjoint elements in  $M$ . We may assume  $M \subseteq B(H)$ , and  $\varphi$  is implemented by  $\Omega \in H$ , which is a cyclic vector for  $M$ . We suppose that  $\text{ev}_x(\mathcal{P}_\infty^o)$  is  $\sigma$ -weakly dense in  $M$ , where  $\text{ev}_x$  is the evaluation map.

Notation 2.13. We say that  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{B}_s$ -invariant if for any  $n \in \mathbb{N}$ ,  $(\varphi \circ \text{ev}_x \otimes \text{id}) \circ \mathbf{L}_n = \varphi \circ \text{ev}_x \otimes \mathbf{P}$ .

LEMMA 2.14. *Assume  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_x]$ -invariant or  $\text{Beq}_x$ -invariant for one of  $x = s, o, h, b$ . Then it is  $\mathcal{B}_s$ -invariant.*

PROOF. This follows immediately from Lemma 2.11.  $\square$

We review that  $\mathcal{B}_s$ -invariance implies the existence of the normal conditional expectation onto the non-unital tail von Neumann algebra. Assume that  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{B}_s$ -invariant. Then by [6, Lemma 6.4], for  $a \in \text{ev}_x(\mathcal{P}_\infty^o)$ ,  $E[a] := \sigma\text{-}\lim_{n \rightarrow \infty} \text{sh}^n(a)$  is well-defined,  $E[a] \in M_{\text{nut}}$  and  $E$

is state-preserving. By [6, Lemma 6.7], we have for any  $a, b, c \in \text{ev}_x(\mathcal{P}_\infty^o)$ ,  $\langle E[a]b\Omega, c\Omega \rangle = \langle aE_{\text{nut}}[b]\Omega, E[c]\Omega \rangle$ . By [6, Lemma 6.8], we can define  $E_{\text{nut}} : M \rightarrow M_{\text{nut}}$  by

$$(2.1) \quad E_{\text{nut}}[y] := \sigma w\text{-}\lim_{n \rightarrow \infty} E[y_n],$$

where  $(y_n)$  is a bounded sequence in  $\text{ev}_x(\mathcal{P}_\infty^o)$  with  $\sigma w\text{-}\lim_{n \rightarrow \infty} y_n = y$ . By [6, Lemma 6.9],  $E_{\text{nut}}$  is normal. By [6, Lemma 6.10],  $E[b] = b$  for any  $b \in M_{\text{nut}}$ . By [6, Lemma 6.11] and since  $E$  is normal, it holds that for any  $y, z_1, z_2 \in M$ ,

$$(2.2) \quad \langle E_{\text{nut}}[y]z_1\Omega, z_2\Omega \rangle = \langle yE_{\text{nut}}[z_1]\Omega, E_{\text{nut}}[z_2]\Omega \rangle.$$

In particular,  $\varphi \circ E_{\text{nut}} = \varphi$ . By [6, Lemma 6.12],  $E[by] = bE[y]$ ,  $E[yb] = E[y]b$  for any  $b \in M_{\text{nut}}$  and  $y \in M$ . Hence  $E_{\text{nut}}$  is a normal conditional expectation onto  $M_{\text{nut}}$ , which is state-preserving.

**PROPOSITION 2.15.** *Assume that  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{B}_s$ -invariant. Let  $E_{\text{nut}} : M \rightarrow M_{\text{nut}}$  be the conditional expectation defined by (2.1). Set  $e_{\text{nut}} \in B(H)$  be the orthogonal projection onto the closed subspace  $\overline{M_{\text{nut}}\Omega}$ . Then it holds that*

$$E_{\text{nut}}[y] = e_{\text{nut}} y e_{\text{nut}} \quad (y \in M).$$

In particular,  $M_{\text{nut}} = e_{\text{nut}} M e_{\text{nut}}$ .

**PROOF.** Let  $b \in M_{\text{nut}}$ ,  $y \in M$ . As  $E_{\text{nut}}[b^*y] = b^*E_{\text{nut}}[y]$ ,  $\langle b\Omega, (y - E_{\text{nut}}[y])\Omega \rangle = \langle \Omega, (b^*y - E_{\text{nut}}[b^*y])\Omega \rangle = 0$ . Hence  $(b^*y - E_{\text{nut}}[b^*y])\Omega \in \overline{M_{\text{nut}}\Omega}$ ,  $e_{\text{nut}}y\Omega = E_{\text{nut}}[y]\Omega$ . By (2.2), for any  $y \in M$ ,  $a, b \in \text{ev}_x(\mathcal{P}_\infty^o)$ ,

$$\begin{aligned} \langle E_{\text{nut}}[y]a\Omega, b\Omega \rangle &= \langle yE_{\text{nut}}[a]\Omega, E_{\text{nut}}[b]\Omega \rangle = \langle ye_{\text{nut}}a\Omega, e_{\text{nut}}b\Omega \rangle \\ &= \langle e_{\text{nut}}ye_{\text{nut}}a\Omega, b\Omega \rangle. \end{aligned}$$

Since the subspace  $\text{ev}_x(\mathcal{P}_\infty^o)\Omega$  is dense in  $H$ , it holds that  $E_{\text{nut}}[y] = e_{\text{nut}}ye_{\text{nut}}$ . As  $E_{\text{nut}}[M] = M_{\text{nut}}$ , it holds that  $M_{\text{nut}} = e_{\text{nut}} M e_{\text{nut}}$ .  $\square$

**COROLLARY 2.16.** *Assume  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_x]$ -invariant or  $\text{Beq}_x$ -invariant for one of  $x = s, o, h, b$ . Then  $E_{\text{nut}}[y] := e_{\text{nut}} y e_{\text{nut}} \quad (y \in M)$  is a nondegenerate normal conditional expectation onto  $M_{\text{nut}}$  with respect to the embedding  $M_{\text{nut}} \subseteq M$ .*

**PROOF.** This follows from Lemma 2.14 and Proposition 2.15.  $\square$

### 3. Haar Functionals and Haar States

#### 3.1. Haar functionals on $\mathcal{A}[D; n]$

At first, we construct a linear functional with an invariance property on  $\mathcal{A}[D; n]$  instead of a Haar state.

DEFINITION 3.1. Let  $\mathcal{A}$  be a unital  $*$ -algebra. Assume  $\mathcal{A}$  is equipped with a coproduct  $\Delta$ . A linear functional  $h$  (resp. a state) on  $\mathcal{A}$  is called a Haar functional (resp. a Haar state) if it satisfies the following Haar invariance property:

$$(3.1) \quad (\text{id} \otimes h)\Delta = h(\cdot)1_{\mathcal{A}} = (h \otimes \text{id})\Delta.$$

PROPOSITION 3.2. Under the assumption of  $\mathcal{A}$  in Definition 3.1, the unital Haar functional on  $\mathcal{A}$  is unique if it exists.

PROOF. Assume that  $g, h$  are unital Haar linear functionals on  $\mathcal{A}$ . Combining invariant properties, for any  $a \in \mathcal{A}$  we obtain  $(h \otimes g)\Delta(a) = (h \otimes \text{id})(\text{id} \otimes g)\Delta(a) = (h \otimes \text{id})(1_{\mathcal{A}} \otimes g(a)) = g(a)$ . Similarly,  $(h \otimes g)\Delta(a) = (\text{id} \otimes g)(h \otimes \text{id})\Delta(a) = (\text{id} \otimes g)(h(a) \otimes 1_{\mathcal{A}}) = h(a)$ . This completes the proof.  $\square$

Notation 3.3. Let  $D$  be a category of interval partitions.

(1) Set

$$V_n^D := \text{Span}(\{p\} \cup \{pu_{i_1 j_1} \cdots u_{i_k j_k} p \mid \mathbf{i}, \mathbf{j} \in [n]^k, k \in \mathbb{N}\}) \subseteq \mathcal{A}_p[D; n].$$

We see at once that  $\Delta(V_n^D) \subseteq V_n^D \otimes V_n^D$ .

(2) We write  $u_{\mathbf{ij}} = u_{i_1 j_1} \cdots u_{i_k j_k}$  for  $\mathbf{i}, \mathbf{j} \in [n]^k, k \in \mathbb{N}$ . Fix a complete orthonormal basis  $\{e_i\}_{i \in [n]}$  of the standard  $n$  dimensional Hilbert space  $l_n^2$ . Set  $e_{\mathbf{i}} := e_{i_1} \otimes \cdots \otimes e_{i_k}$  for  $\mathbf{i} \in [n]^k$ .

(3) We denote by  $\Lambda_n^k$  the linear map  $l_n^{2 \otimes k} \rightarrow l_n^{2 \otimes k} \otimes V_n^D$  defined by

$$\Lambda_n^k(e_{\mathbf{j}}) := \sum_{\mathbf{i} \in [n]^k} e_{\mathbf{i}} \otimes pu_{\mathbf{ij}}p.$$

By a direct calculation,  $\Lambda_n^k$  is a linear coaction of  $V_n^D$ , that is,

$$(\text{id} \otimes \Delta)\Lambda_n^k = (\Lambda_n^k \otimes \text{id})\Lambda_n^k.$$

(4) Let  $\text{Fix}(\Lambda_n^k)$  denote the invariant subspace of the coaction  $\Lambda_n^k$ , that is,

$$\text{Fix}(\Lambda_n^k) := \{\xi \in l_n^{2 \otimes k} \mid \Lambda_n^k(\xi) = \xi \otimes p\}.$$

LEMMA 3.4. *Let  $g$  be a functional on  $\mathcal{A}_p[I; n]$ . Assume  $g|_{V_n^D}$  satisfies the Haar invariance property and  $g(apb) = g(a)g(b)$  for any  $a, b \in V_n^D$ . Then  $g$  is a Haar functional.*

PROOF. For any  $k, l \in [n]$  and for any multi-indices  $\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \dots, \mathbf{i}^{(l)}, \mathbf{j}^{(1)}, \mathbf{j}^{(2)}, \dots, \mathbf{j}^{(l)} \in [n]^k$ ,

$$\begin{aligned} & (\text{id} \otimes h)\Delta(pu_{\mathbf{i}^{(1)}\mathbf{j}^{(1)}}pu_{\mathbf{i}^{(2)}\mathbf{j}^{(2)}}p \cdots pu_{\mathbf{i}^{(l)}\mathbf{j}^{(l)}}p) \\ &= \sum_{\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(l)} \in [n]^k} pu_{\mathbf{i}^{(1)}\mathbf{s}^{(1)}}pu_{\mathbf{i}^{(2)}\mathbf{s}^{(2)}}p \cdots pu_{\mathbf{i}^{(l)}\mathbf{s}^{(l)}}p \\ & \quad \cdot h(pu_{\mathbf{s}^{(1)}\mathbf{j}^{(1)}}p)h(pu_{\mathbf{s}^{(2)}\mathbf{j}^{(2)}}p) \cdots h(pu_{\mathbf{s}^{(l)}\mathbf{j}^{(l)}}p) \\ &= (\text{id} \otimes h)\Delta(pu_{\mathbf{s}^{(1)}\mathbf{j}^{(1)}}p) \cdot (\text{id} \otimes h)\Delta(pu_{\mathbf{s}^{(2)}\mathbf{j}^{(2)}}p) \cdots (\text{id} \otimes h)\Delta(pu_{\mathbf{s}^{(l)}\mathbf{j}^{(l)}}p). \end{aligned}$$

This finishes the proof by using the Haar invariance on  $V_n^D$ .  $\square$

LEMMA 3.5. *For any  $k, n \in \mathbb{N}$ ,  $\pi \in D(k)$ , and  $\mathbf{i} \in [n]^l$ ,*

$$\Lambda_n^{k+l}(T_\pi \otimes e_{\mathbf{i}}) = T_\pi \otimes \Lambda_n^l(e_{\mathbf{i}}), \quad \Lambda_n^{l+k}(e_{\mathbf{i}} \otimes T_\pi) = \Lambda_n^l(e_{\mathbf{i}}) \otimes T_\pi.$$

PROOF.

$$\begin{aligned} \Lambda_n^{k+l}(T_\pi \otimes e_{\mathbf{i}}) &= \sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} \Lambda_n^{k+l}(e_{\mathbf{j}} \otimes e_{\mathbf{i}}) \\ &= \sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} \sum_{\mathbf{s} \in [n]^k, \mathbf{r} \in [n]^l} e_{\mathbf{s}} \otimes e_{\mathbf{r}} \otimes pu_{s_1 j_1} \cdots u_{s_k j_k} u_{r_1 i_1} \cdots u_{r_l i_l} p \\ &= \sum_{\mathbf{s} \in [n]^k, \mathbf{r} \in [n]^l} e_{\mathbf{s}} \otimes e_{\mathbf{r}} \otimes \sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} (pu_{s_1 j_1} \cdots u_{s_k j_k}) u_{r_1 i_1} \cdots u_{r_l i_l} p. \end{aligned}$$

By Lemma 2.5, we have  $\sum_{\mathbf{j} \in [n]^k, \pi \leq \ker \mathbf{j}} p u_{s_1 j_1} \cdots u_{s_k j_k} = \zeta(\pi, \ker \mathbf{s})p$ . Hence

$$\Lambda_n^{k+l}(T_\pi \otimes e_{\mathbf{i}}) = \sum_{\substack{\mathbf{s} \in [n]^k \\ \pi \leq \ker \mathbf{s}}} \sum_{\mathbf{r} \in [n]^l} e_{\mathbf{s}} \otimes e_{\mathbf{r}} \otimes p u_{r_1 i_1} \cdots u_{r_l i_l} p = T_\pi \otimes \Lambda_n^l(e_{\mathbf{i}}).$$

The proof for the second equation is similar to that of the first one.  $\square$

LEMMA 3.6. *Let  $D$  be a category of interval partitions with  $D(l) \neq \emptyset$  for a fixed index  $l \in \mathbb{N}$ . For any  $k \in L_D$  and  $\mathbf{j} \in [n]^l$ , we have*

$$H^{D(k+l)}(T_{\mathbf{1}_k} \otimes e_{\mathbf{j}}) = T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}}.$$

PROOF. Since  $D$  is  $\otimes$ -stable, we have  $H^{D(k+l)} \geq H^{D(k)} \otimes H^{D(l)}$ . As  $D(k), D(l) \neq \emptyset$ , it holds that  $D(k+l) \neq \emptyset$ . We have  $H^{D(k+l)}(T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}}) = T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}}$ . We only need to show that

$$(3.2) \quad \langle T_\pi, T_{\mathbf{1}_k} \otimes e_{\mathbf{j}} \rangle = \langle T_\pi, T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}} \rangle, \text{ for any } \pi \in D(k+l).$$

As  $D(l) \neq \emptyset$  by the assumption, there are scalars  $(\alpha_\sigma)_{\sigma \in D(l)}$  with  $H^{D(l)}e_{\mathbf{j}} = \sum_{\sigma \in D(l)} \alpha_\sigma T_\sigma$ . Then for any  $\rho \in D(l)$ ,

$$(3.3) \quad \langle T_\rho, e_{\mathbf{j}} \rangle = \sum_{\sigma \in D(k)} \alpha_\sigma \langle T_\rho, T_\sigma \rangle = \sum_{\sigma \in D(l)} \alpha_\sigma n^{|\rho \vee \sigma|}.$$

For any  $\pi \in D(k+l)$ ,

$$(3.4) \quad \langle T_\pi, T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}} \rangle = \sum_{\sigma \in D(l)} \alpha_\sigma \langle T_\pi, T_{\mathbf{1}_k} \otimes T_\sigma \rangle = \sum_{\sigma \in D(l)} \alpha_\sigma n^{|\pi \vee (\mathbf{1}_k \otimes \sigma)|}.$$

Consider the case  $k \sim^\pi k+1$ . Set  $\pi' := \pi|_{[k+1, k+l]}$ . We have  $\pi \vee (\mathbf{1}_k \otimes \sigma) = (\uparrow^{\otimes(k-1)} \otimes \square \otimes \uparrow^{\otimes(l-1)}) \vee (\mathbf{1}_k \otimes (\pi' \vee \sigma))$ . Hence  $|\pi \vee (\mathbf{1}_k \otimes \sigma)| = |\pi' \vee \sigma|$ . By (3.3), (3.4),

$$\langle T_\pi, T_{\mathbf{1}_k} \otimes H^{D(l)}e_{\mathbf{j}} \rangle = \sum_{\sigma \in D(l)} \alpha_\sigma n^{|\pi' \vee \sigma|} = \langle T_{\pi'}, e_{\mathbf{j}} \rangle.$$

As  $k \sim^\pi k+1$ , we have  $\langle T_\pi, T_{\mathbf{1}_k} \otimes e_{\mathbf{j}} \rangle = \langle T_\pi, e_{j_1}^{\otimes k} \otimes e_{\mathbf{j}} \rangle = \langle T_{\pi'}, e_{\mathbf{j}} \rangle$ . Hence in this case we have shown (3.2).

Consider the case  $k \not\sim^\pi k + 1$ . Since  $D$  is block-stable, there are  $\pi_1 \in D(k)$  and  $\pi_2 \in D(l)$  with  $\pi = \pi_1 \otimes \pi_2$ . Then  $\pi \vee (\mathbf{1}_k \otimes \sigma) = \mathbf{1}_k \otimes (\pi_2 \vee \sigma)$ , and  $|\pi \vee (\mathbf{1}_k \otimes \sigma)| = 1 + |\pi_2 \vee \sigma|$ . By (3.3), (3.4),

$$\begin{aligned} \langle T_\pi, T_{\mathbf{1}_k} \otimes H^{D(l)} e_j \rangle &= \sum_{\sigma \in D(l)} \alpha_\sigma n^{1+|\pi_2 \vee \sigma|} = n \langle T_{\pi_2}, e_j \rangle \\ &= \langle T_{\pi_1}, T_{\mathbf{1}_k} \rangle \langle T_{\pi_2}, e_j \rangle = \langle T_\pi, T_{\mathbf{1}_k} \otimes e_j \rangle. \end{aligned}$$

Hence we have shown (3.2). Then we have proven the lemma.  $\square$

**THEOREM 3.7 (The Haar Functionals).** *Assume  $D$  is blockwise. Then for any  $k \in \mathbb{N}$ ,*

$$(3.5) \quad \text{Fix}(\Lambda_n^k) = \text{Span}\{T_\pi : \pi \in D(k)\}.$$

Moreover, for any  $n \in \mathbb{N}$  there exists a unique unital Haar functional  $h_D$  on  $\mathcal{A}_p[D; n]$  with

- (1)  $h_D(p) = 1$ ,
- (2)  $h_D(pu_{\mathbf{i}\mathbf{j}}p) = H_{\mathbf{i}\mathbf{j}}^{D(k)}$  for  $\mathbf{i}, \mathbf{j} \in [n]^k$ , and  $k \in \mathbb{N}$ ,
- (3)  $h_D(a_1 \cdots a_l) = h_D(a_1) \cdots h_D(a_l)$  for any  $l \in \mathbb{N}$ ,  $a_1, \dots, a_l \in V_n^D$ .

**PROOF.** By a direct calculation,  $\text{Fix}(\Lambda_n^k) \supseteq \text{Span}\{T_\pi : \pi \in D(k)\}$ . We prove the opposite inclusion. We have  $\sum_{\mathbf{s} \in [n]^k, \pi \leq \ker \mathbf{s}} H_{\mathbf{r}\mathbf{s}}^{D(k)} = \langle e_{\mathbf{r}}, H^{D(k)} T_\pi \rangle = \langle e_{\mathbf{r}}, T_\pi \rangle = \zeta(\pi, \ker \mathbf{r})$ . Similarly we have  $\sum_{\mathbf{r} \in [n]^k, \pi \leq \ker \mathbf{r}} H_{\mathbf{r}\mathbf{s}}^{D(k)} = \zeta(\pi, \ker \mathbf{s})$ .

Assume  $k \in L_D$ . We prove that for any  $l \in \mathbb{N}$  and  $\mathbf{i}, \mathbf{j}, \mathbf{r} \in [n]^{k+l}$ ,

$$(3.6) \quad \sum_{\substack{\mathbf{s} \in [n]^k \\ 1_k \leq \ker \mathbf{s}}} H_{\mathbf{r}\Pi\mathbf{i}, \mathbf{s}\Pi\mathbf{j}}^{D(k+l)} = \zeta(\pi, \ker \mathbf{r}) H_{\mathbf{i}\mathbf{j}}^{D(l)}.$$

In the case  $H^{D(k+l)} = 0$  it holds that  $H^{D(l)} = 0$  as  $D$  is  $D(k) \neq \emptyset$  and (D1). Assume that  $H^{D(k+l)} \neq 0$ . By condition (D3),  $D(l) \neq \emptyset$ . Thus by Lemma 3.6, we have

$$\sum_{\substack{\mathbf{s} \in [n]^k \\ 1_k \leq \ker \mathbf{s}}} H_{\mathbf{r}\Pi\mathbf{i}, \mathbf{s}\Pi\mathbf{j}}^{D(k+l)} = \langle e_{\mathbf{r}} \otimes e_{\mathbf{i}}, H^{D(k+l)}(T_{1_k} \otimes e_{\mathbf{j}}) \rangle = \langle e_{\mathbf{r}} \otimes e_{\mathbf{i}}, T_{1_k} \otimes H^{D(l)} e_{\mathbf{j}} \rangle$$

This proves the claim (3.6). Similarly we have

$$\begin{aligned} \sum_{\substack{\mathbf{r} \in [n]^k \\ \pi \leq \ker \mathbf{r}}} H_{\mathbf{r} \parallel \mathbf{i}, \mathbf{s} \parallel \mathbf{j}}^{D(k+l)} &= \zeta(\pi, \ker \mathbf{s}) H_{\mathbf{ij}}^{D(l)}, \text{ for any } \mathbf{s} \in [n]^k, \\ \sum_{\substack{\mathbf{s} \in [n]^k \\ \pi \leq \ker \mathbf{s}}} H_{\mathbf{i} \parallel \mathbf{r}, \mathbf{j} \parallel \mathbf{s}}^{D(k+l)} &= \zeta(\pi, \ker \mathbf{r}) H_{\mathbf{ij}}^{D(l)}, \text{ for any } \mathbf{r} \in [n]^k, \\ \sum_{\substack{\mathbf{r} \in [n]^k \\ \pi \leq \ker \mathbf{r}}} H_{\mathbf{i} \parallel \mathbf{r}, \mathbf{j} \parallel \mathbf{s}}^{D(k+l)} &= \zeta(\pi, \ker \mathbf{s}) H_{\mathbf{ij}}^{D(l)}, \text{ for any } \mathbf{s} \in [n]^k. \end{aligned}$$

Therefore, there is a functional  $h_D$  on  $V_n^D$  with (1) and (2).

For any  $\xi \in \text{Fix}(\Lambda_n^k)$ ,  $(id \otimes h_D)\Lambda_n^k(\xi) = (id \otimes h_D)(\xi \otimes p) = \xi \otimes 1$ . On the other hand, we have

$$(id \otimes h_D)\Lambda_n^k(\xi) = \sum_{\mathbf{i}, \mathbf{j} \in [n]^k} \xi_{\mathbf{i}} e_{\mathbf{j}} \otimes H_{\mathbf{ij}}^{D(k)} p = H^{D(k)} \xi \otimes p.$$

Thus we have  $H^{D(k)} \xi = \xi$ , which proves  $\text{Fix}(\Lambda_n^k) = \text{Span}\{T_\pi : \pi \in D(k)\}$ . As  $\Lambda_n^k(H^{D(k)} e_{\mathbf{j}}) = H^{D(k)} e_{\mathbf{j}}$  it holds that  $\sum_{\mathbf{s} \in [n]^k} p u_{\mathbf{is}} p H_{\mathbf{sj}}^{D(k)} = H_{\mathbf{ij}}^{D(k)} p$ . Hence

$$\begin{aligned} (id \otimes h_D)\Delta(pu_{i_1 j_1} \cdots u_{i_k j_k} p) &= \sum_{\mathbf{s} \in [n]^k} pu_{s_1 j_1} \cdots u_{s_k j_k} p H_{\mathbf{sj}}^{D(k)} \\ &= H_{\mathbf{ij}}^{D(k)} p = h_D(pu_{i_1 j_1} \cdots u_{i_k j_k} p). \end{aligned}$$

Therefore, we have  $(id \otimes h_D)\Delta = h_D(\cdot)p$ . The other invariance property follows from a similar proof. By Lemma 3.4, we can extend  $h_D$  to  $\mathcal{A}_p[D; n]$  by (3) with the Haar invariance.  $\square$

### 3.2. Haar states on $\text{Beq}_x$

In this section, we construct a  $*$ -representation of  $\mathcal{A}[I; n]$  on  $L^2(S_n)$ , which induces the GNS-representation of  $(\mathcal{A}_p[I; n], h_I)$ . In particular, we see that  $h_I$  is a state and that  $\text{Beq}_s$  has unique Haar states. By a similar discussion, we show that  $\text{Beq}_h, \text{Beq}_o$  have the unique Haar state.

*Notation 3.8.* Let  $(L^2(S_n))_{n \in \mathbb{N}}$  be the sequence of the Hilbert spaces of all  $L^2$ -functions on permutation groups  $S_n$  with respect to the uniform

probability measure on  $S_n$ , which is precisely the normalized counting measure. In particular, the inner product of elements  $\xi, \eta \in L^2(S_n)$  is given by

$$\langle \xi, \eta \rangle = \frac{\sum_{\sigma \in S_n} \bar{\xi}(\sigma)\eta(\sigma)}{\#S_n}.$$

We write  $\|\xi\| := \sqrt{\langle \xi, \xi \rangle}$ . Let  $C(S_n)$  be the  $C^*$ -algebra of all continuous functions on  $S_n$ . Let  $\tau_n$  be the tracial state on  $C(S_n)$  given by the normalized counting measure on  $S_n$ . The  $C^*$ -algebra  $C(S_n)$  acts on  $L^2(S_n)$  by the left multiplication. Set  $\hat{\cdot} : C(S_n) \rightarrow L^2(S_n)$  be the inclusion map then  $(L^2(S_n), \hat{\cdot})$  is the GNS-representation of  $(C(S_n), \tau_n)$ . We set the projections  $P_{ij} \in C(S_n)$  ( $i, j \leq n$ ) by

$$P_{ij}(\sigma) := \delta_{i, \sigma(j)} \quad (\sigma \in S_n).$$

We have  $\|\hat{P}_{ij}\|^2 = \tau_n(P_{ij}^*P_{ij}) = \tau_n(P_{ij}) = 1/n$  and  $\|\hat{1}\|^2 = \tau_n(1) = 1$ . For  $\xi \in L^2(S_n)$ , let us denote by  $Q(\xi)$  the orthogonal projection onto the one dimensional subspace  $\mathbb{C}\xi \subseteq L^2(S_n)$ . For  $\eta, \xi \in L^2(S_n)$ , we denote by  $|\xi\rangle\langle\eta|$  the bounded linear operator  $\zeta \in L^2(S_n) \mapsto \langle\eta, \zeta\rangle\xi \in L^2(S_n)$ . We have

$$Q(\hat{P}_{ij}) = \frac{1}{\|\hat{P}_{ij}\|^2} |\hat{P}_{ij}\rangle\langle\hat{P}_{ij}|.$$

We denote by  $\omega$  the vector state on  $B(L^2(S_n))$  induced by the unit vector  $\hat{1}$ .

We obtain that if  $n \geq 2$ ,  $i_1, i_2, j_1, j_2 \in [n]$  with  $i_1 \neq i_2$  and  $j_1 \neq j_2$  then

$$Q(\hat{P}_{i_1j_1})Q(\hat{P}_{i_2j_2}) = \frac{\langle \hat{P}_{i_1j_1}, \hat{P}_{i_2j_2} \rangle}{\|\hat{P}_{i_1j_1}\|^2\|\hat{P}_{i_2j_2}\|^2} |\hat{P}_{i_1j_1}\rangle\langle\hat{P}_{i_2j_2}| = \frac{\#S_{n-2}}{n^2} |\hat{P}_{i_1j_1}\rangle\langle\hat{P}_{i_2j_2}|.$$

Hence  $Q(\hat{P}_{i_1j_1})Q(\hat{P}_{i_2j_2}) \neq Q(\hat{P}_{i_2j_2})Q(\hat{P}_{i_1j_1})$ , and  $(Q(\hat{P}_{ij}))_{ij}$  contains mutually noncommutative pairs.

We prove that the  $*$ -representation of  $\mathcal{A}[I; n]$  determined by  $u_{ij} \mapsto Q(\hat{P}_{ij})$  and  $p \mapsto Q(\hat{1})$  is well-defined. At first, we show that the operators  $Q(\hat{P}_{ij})$  and  $Q(\hat{1})$  satisfy the relations which appear in the definition of Liu's Boolean quantum permutation group.



PROPOSITION 3.9. Let  $u_{ij}(i, j \leq n)$ ,  $p$  be the generators of  $\mathcal{A}[I; n]$ . Then we have

$$\begin{aligned} Q(\hat{P}_{ij_1})Q(\hat{P}_{ij_2}) &= \delta_{j_1, j_2} Q(\hat{P}_{ij_1}), \text{ for any } i \in [n], j_1, j_2 \in [n], \\ Q(\hat{P}_{i_1j})Q(\hat{P}_{i_2j}) &= \delta_{i_1, i_2} Q(\hat{P}_{i_1j}), \text{ for any } j \in [n], i_1, i_2 \in [n], \\ Q(\hat{P}_{ij})Q(\hat{1}) &= |\hat{P}_{ij}\rangle\langle\hat{1}|, \text{ for any } i, j \in [n]. \end{aligned}$$

PROOF. For any indices  $i \leq n$  and  $j_1, j_2 \leq n$ , it holds that

$$\begin{aligned} \langle \hat{P}_{ij_1}, \hat{P}_{ij_2} \rangle &= \frac{\sum_{\sigma \in S_n} \delta_{\sigma(i), j_1} \delta_{\sigma(i), j_2}}{\#S_n} = \frac{\delta_{j_1, j_2} \#S_{n-1}}{\#S_n}, \\ Q(\hat{P}_{ij_1})Q(\hat{P}_{ij_2}) &= \frac{\langle \hat{P}_{ij_1}, \hat{P}_{ij_2} \rangle}{\langle \hat{P}_{ij_2}, \hat{P}_{ij_2} \rangle} Q(\hat{P}_{ij_1}) = \delta_{j_1, j_2} Q(\hat{P}_{ij_1}). \end{aligned}$$

Similarly,  $Q(\hat{P}_{i_1j})Q(\hat{P}_{i_2j}) = \delta_{i_1, i_2} Q(\hat{P}_{i_1j})$  for any indices  $i_1, i_2 \leq n$  and  $j \leq n$ . Then

$$Q(\hat{P}_{ij})Q(\hat{1}) = \frac{\langle \hat{P}_{ij}, \hat{1} \rangle}{\langle \hat{P}_{ij}, \hat{P}_{ij} \rangle \langle \hat{1}, \hat{1} \rangle} |\hat{P}_{ij}\rangle\langle\hat{1}| = |\hat{P}_{ij}\rangle\langle\hat{1}|. \quad \square$$

COROLLARY 3.10. There is the unique  $*$ -representation  $\pi_s : \mathcal{A}_p[I; n] \rightarrow B(L^2(S_n))$  with

$$(3.7) \quad \pi_s(u_{ij}) = Q(\hat{P}_{ij}) \ (i, j \leq n), \quad \pi_s(p) = Q(\hat{1}).$$

Moreover, there are  $*$ -representations  $\bar{\pi}_s : \text{Beq}_s(n) \rightarrow B(L^2(S_n))$  and  $\Pi_s : \mathcal{B}_s(n) \rightarrow B(L^2(S_n))$  with  $\bar{\pi}_s([u_{ij}]) = \pi_s(u_{ij}) = \Pi_s(U_{ij})(i, j \leq n)$  and  $\bar{\pi}_s([p]) = \pi_s(p) = \Pi_s(\mathbf{P})$ .

PROOF. Since  $\sum_{i=1}^n \hat{P}_{ij} = \hat{1} \ (j \leq n)$  and  $\sum_{j=1}^n \hat{P}_{ij} = \hat{1} \ (i \leq n)$ , we have for any  $k \in \mathbb{N}$  and  $\mathbf{i}, \mathbf{j} \in [n]^k$ ,

$$\begin{aligned} \sum_{i=1}^n Q(\hat{P}_{ij_1}) \dots Q(\hat{P}_{ij_k}) Q(\hat{1}) &= \begin{cases} Q(\hat{1}), & j_1 = \dots = j_k, \\ 0, & \text{otherwise,} \end{cases} \\ \sum_{j=1}^n Q(\hat{P}_{i_1j}) \dots Q(\hat{P}_{i_kj}) Q(\hat{1}) &= \begin{cases} Q(\hat{1}), & i_1 = \dots = i_k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the  $*$ -representation  $\pi_s$  (3.7) is well-defined. The existence of  $\Pi_s$  directory follows from Proposition 3.9. Since  $\|\pi_s(u_{ij})\|_n \leq 1$  and  $\pi_s(p) = 1$ , we have  $\bar{\pi}_s$  is well-defined.  $\square$

LEMMA 3.11. *For any  $\mathbf{i}, \mathbf{j} \in [n]^k$  and  $k \in \mathbb{N}$ , we have*

$$(3.8) \quad Q(\hat{\mathbf{1}})Q(P_{i_1 j_1}) \cdots Q(P_{i_k j_k})Q(\hat{\mathbf{1}}) = \frac{\delta(\inf_I \ker \mathbf{i}, \inf_I \ker \mathbf{j})}{n(n-1)^{|\inf_I \ker \mathbf{i}|-1}} Q(\hat{\mathbf{1}}).$$

PROOF. In the case  $|\inf_I \ker \mathbf{i}| = 1$  it holds that  $\ker \mathbf{i} = \mathbf{1}_k$ . Then the left hand side of the equation (3.8) is equal to

$$\delta(\mathbf{1}_k, \ker \mathbf{j})Q(\hat{\mathbf{1}})Q(P_{i_1 j_1})Q(\hat{\mathbf{1}}) = \frac{\delta(\mathbf{1}_k, \ker \mathbf{j})}{n} Q(\hat{\mathbf{1}}).$$

This proves (3.8).

Let  $m \in \mathbb{N}$  and assume that the equation holds if  $|\inf_I \ker \mathbf{i}| \leq m$ . Let  $\inf_I \ker \mathbf{i} = \{V_1 < V_2 < \cdots < V_b\}$ , where  $b = |\inf_I \ker \mathbf{i}|$ , and  $s_\nu := \min V_\nu$  for  $\nu \in [b]$ . Then the left hand side of the equation (3.8) is equal to

$$\zeta(\inf_I \ker \mathbf{i}, \ker \mathbf{j})Q(\hat{\mathbf{1}})Q(P_{i(s_1)j(s_1)}) \cdots Q(P_{i(s_b)j(s_b)})Q(\hat{\mathbf{1}}).$$

Since  $i(s_\nu) \neq i(s_{\nu+1})$  for any  $\nu < b$ ,  $Q(P_{i(s_\nu)j(s_\nu)})Q(P_{i(s_{\nu+1})j(s_{\nu+1})}) = 0$ , whenever  $j(s_\nu) = j(s_{\nu+1})$ . Now

$$\zeta(\inf_I \ker \mathbf{i}, \ker \mathbf{j}) \prod_{\nu=1}^b 1(j(s_\nu) \neq j(s_{\nu+1})) = \delta(\inf_I \ker \mathbf{i}, \inf_I \ker \mathbf{j}).$$

Assume indices satisfy  $i_1 \neq i_2$  and  $j_1 \neq j_2$ . Then

$$\langle P_{i_1 j_1}, P_{i_2 j_2} \rangle = \frac{\sum_{\sigma \in S_n} \delta_{\sigma(i_1), j_1} \delta_{\sigma(i_2), j_2}}{\#S_n} = \frac{\#S_{n-2}}{\#S_n}.$$

Hence, if  $\inf_I \ker \mathbf{i} = \inf_I \ker \mathbf{j}$ , we have

$$\begin{aligned} & Q(\hat{\mathbf{1}})Q(P_{i(s_1)j(s_1)}) \cdots Q(P_{i(s_b)j(s_b)})Q(\hat{\mathbf{1}}) \\ &= \frac{(\#S_{n-1}/\#S_n)^2 (\#S_{n-2}/\#S_n)^{b-1}}{(\#S_{n-1}/\#S_n)^b} Q(\hat{\mathbf{1}}) \\ &= \frac{(\#S_{n-2})^{b-1}}{\#S_n (\#S_{n-1})^{b-2}} Q(\hat{\mathbf{1}}) = \frac{1}{n(n-1)^{b-1}} Q(\hat{\mathbf{1}}). \end{aligned}$$

It proves the lemma.  $\square$

LEMMA 3.12. *Let  $u_{ij}(i, j \in [n])$  and  $p$  be the generators of  $\mathcal{A}[I; n]$ . Then for any  $k \in \mathbb{N}$ ,  $\pi \in I(k)$  and  $\mathbf{i}, \mathbf{j} \in [n]^k$ ,*

$$(3.9) \quad \sum_{\substack{\mathbf{r} \in [n]^k, \\ \inf_I \ker \mathbf{r} = \pi}} u_{\mathbf{r}\mathbf{j}} p = \delta(\pi, \inf_I \ker_I \mathbf{j}) p.$$

$$(3.10) \quad \sum_{\substack{\mathbf{s} \in [n]^k, \\ \inf_I \ker \mathbf{s} = \pi}} u_{\mathbf{i}\mathbf{s}} p = \delta(\inf_I \ker \mathbf{i}, \pi) p,$$

PROOF. We give the proof only for the equation (3.9); the same proof runs for the other. The proof is by induction on  $|\pi|$ . In the case  $|\pi| = 1$ , we have  $\pi = \mathbf{1}_k$ . Then for any  $\mathbf{r} \in [n]^k$ , it holds that  $\inf_I \ker \mathbf{r} = \pi$  if and only if  $\ker \mathbf{r} = \mathbf{1}_k$ . This gives the equation (3.9).

Let  $b \in \mathbb{N}$ . Assume (3.9) holds in the case  $|\pi| = b$ . In the case  $|\pi| = b + 1$ , write  $\pi = \{V_1 < V_2 < \dots < V_{b+1}\}$ . Set  $v = \max V_b$ .

$$\pi = \begin{array}{ccccccc} \boxed{1} & \boxed{\dots} & \boxed{\dots} & \dots & \boxed{\dots} & \boxed{\dots} & \boxed{k} \\ & V_1 & & V_2 & & V_b & v & V_{b+1} \end{array}$$

Then the left hand side of (3.9) is equal to

$$(3.11) \quad \sum_{\substack{\mathbf{r} \in [n]^{[k] \setminus V_{b+1}}, \\ \inf_I \ker \mathbf{r} = \pi|_{[k] \setminus V_{b+1}}}} (u_{r_1 j_1} u_{r_2 j_2} \dots u_{r_v j_v} \sum_{\substack{r' \in [n], \\ r' \neq r_v}} u_{r' j_{v+1}} u_{r' j_{v+2}} \dots u_{r' j_k} p).$$

It follows that

$$\begin{aligned} & \sum_{\substack{r' \in [n], \\ r' \neq r_v}} u_{r' j_{v+1}} u_{r' j_{v+2}} \dots u_{r' j_k} p \\ & = \delta(\ker(\mathbf{j}|_{V_{b+1}}), \mathbf{1}_{V_{b+1}}) p - u_{r_v j_{v+1}} u_{r_v j_{v+2}} \dots u_{r_v j_k} p. \end{aligned}$$

Then by the assumption of induction, (3.11) is equal to

$$(3.12) \quad \delta(\pi|_{[k] \setminus V_{b+1}}, \inf_I \ker(\mathbf{j}|_{[k] \setminus V_{b+1}})) p \cdot \delta(\mathbf{1}_{V_{b+1}}, \ker(\mathbf{j}|_{V_{b+1}})) - R,$$

where

$$R = \sum_{\substack{\mathbf{r} \in [n]^{[k] \setminus V_{b+1}}, \\ \inf_I \ker \mathbf{r} = \pi|_{[k] \setminus V_{b+1}}}} u_{r_1 j_1} u_{r_2 j_2} \cdots u_{r_v j_v} \cdot u_{r_{v+1} j_{v+1}} u_{r_{v+2} j_{v+2}} \cdots u_{r_k j_k} p.$$

For any multi-index  $\mathbf{r} \in [n]^{[k] \setminus V_{b+1}}$ , set  $\tilde{\mathbf{r}} \in [n]^k$  by  $\tilde{r}_m := r_m$  if  $m \leq v$ , and  $\tilde{r}_m := r_v$ , otherwise. Then  $\inf_I \ker \mathbf{r} = \pi|_{[k] \setminus V_{b+1}}$  if and only if  $\inf_I \ker \tilde{\mathbf{r}} = \tilde{\pi}$ , where  $\tilde{\pi} := \pi \vee (\uparrow^{\otimes(v-1)} \otimes \square \otimes \uparrow^{\otimes(k-v-1)})$ . We see that the partition  $\tilde{\pi}$  is drawn as the following figure.

$$\tilde{\pi} = \begin{array}{ccccccc} \boxed{1} & \boxed{\cdots} & \boxed{\cdots} & \cdots & \boxed{\cdots} & \boxed{\cdots} & \boxed{k} \\ & V_1 & V_2 & \cdots & V_b \cup V_{b+1} & & \end{array}$$

Since  $|\tilde{\pi}| = b$ , applying the assumption of induction yields  $R = \delta(\tilde{\pi}, \inf_I \ker \mathbf{j})p$ . Hence (3.12) is equal to

$$\begin{aligned} & [\delta(\pi|_{[k] \setminus V_{b+1}}, \inf_I \ker(\mathbf{j}|_{[k] \setminus V_{b+1}})) \cdot \delta(\mathbf{1}_{V_{b+1}}, \ker(\mathbf{j}|_{V_{b+1}})) - \delta(\tilde{\pi}, \inf_I \ker \mathbf{j})]p \\ & = \delta(\pi, \inf_I \ker \mathbf{j})p. \end{aligned}$$

This is the desired conclusion.  $\square$

**PROPOSITION 3.13.** *The functional  $h_I$  is a Haar state and the triplet  $(\pi_s, L^2(S_n), \hat{1})$  is the GNS-representation of the pair  $(\mathcal{A}_p[I; n], h_I)$ .*

**PROOF.** Our proof starts with the observation that the functional  $\omega \circ \pi_s$  satisfies the Haar invariance on  $V_n^s$ . For any  $k \in \mathbb{N}$ ,  $\mathbf{i}, \mathbf{j} \in [n]^k$ ,

$$\begin{aligned} (id \otimes \omega \circ \pi_s) \Delta(pu_{\mathbf{i}\mathbf{j}}p) &= \sum_{\mathbf{s} \in [n]^k} pu_{\mathbf{i}\mathbf{s}}p \cdot \frac{\delta(\inf_I \ker \mathbf{s}, \inf_I \ker \mathbf{j})}{n(n-1)^{|\inf_I \ker \mathbf{s}|-1}} \\ &= \frac{1}{n(n-1)^{|\inf_I \ker \mathbf{j}|-1}} \sum_{\substack{\mathbf{s} \in [n]^k, \\ \inf_I \ker \mathbf{s} = \inf_I \ker \mathbf{j}}} pu_{\mathbf{i}\mathbf{s}}p. \end{aligned}$$

By the equation (3.9), we have for any interval partition  $\pi \in I(k)$ ,

$$\sum_{\substack{\mathbf{s} \in [n]^k, \\ \inf_I \ker \mathbf{s} = \pi}} pu_{\mathbf{i}\mathbf{s}}p = \delta(\inf_I \ker \mathbf{i}, \pi)p.$$

From this, we obtain the half of the Haar invariance of  $\omega \circ \pi_s$ . Similar arguments can be applied to the other invariance. By the uniqueness of the Haar functional (Lemma 3.2), we have proven the proposition.  $\square$

**THEOREM 3.14.** *For any  $n \in \mathbb{N}$ ,  $Beq_s(n)$  and  $\mathcal{B}_s(n)$  admit the unique Haar states. We write them  $h_s$  and  $h_{\mathcal{B}_s}$ , respectively. Furthermore, we have  $h_{\mathcal{B}_s} \circ \alpha = h_s$ .*

**PROOF.** The existence of a Haar state follows immediately from Proposition 3.13. The uniqueness follows from Proposition 3.2.  $\square$

**LEMMA 3.15.** *Assume the index  $x$  be  $o$  or  $h$ . Let  $u_{ij}(i, j \in [n])$  and  $p$  be the generators of  $\mathcal{A}[I_x; n]$ . Then for any  $k \in \mathbb{N}$ ,  $\pi \in I_x(2k)$  and multi-indices  $\mathbf{i}, \mathbf{j} \in [n]^{2k}$  with  $\prod^{\otimes k} \leq \ker \mathbf{i}, \ker \mathbf{j}$ , it holds that*

$$(3.13) \quad \sum_{\substack{\mathbf{r} \in [n]^{2k}, \\ \inf_{I_x} \ker \mathbf{r} = \pi}} u_{\mathbf{r}\mathbf{j}} p = \delta(\pi, \inf_{I_x} \ker \mathbf{j}) p,$$

$$(3.14) \quad \sum_{\substack{\mathbf{s} \in [n]^{2k}, \\ \inf_{I_x} \ker \mathbf{s} = \pi}} u_{\mathbf{i}\mathbf{s}} p = \delta(\inf_{I_x} \ker \mathbf{i}, \pi) p.$$

**PROOF.** We only prove the first equation. In the case of  $x = o$ , we have  $\pi = \prod^{\otimes k}$  and  $\inf_{I_o} \ker \rho = \prod^{\otimes k}$  for any  $\rho \in P(2k)$  with  $\rho \geq \prod^{\otimes k}$ . Hence the first equation follows from the definition.

In the case of  $x = h$ , the proof is by induction on  $|\pi|$ . In the case  $|\pi| = 1$ , we have  $\pi = \mathbf{1}_{2k}$ . Then for any  $\mathbf{r} \in [n]^{2k}$ , it holds that  $\inf_{I_h} \ker \mathbf{r} = \pi$  if and only if  $\ker \mathbf{r} = \mathbf{1}_{2k}$ . This gives (3.13).

Let  $b \in \mathbb{N}$ . Assume the first equation holds in the case  $|\pi| = b$ . In the case  $|\pi| = b + 1$ , write  $\pi = \{V_1 < V_2 < \dots < V_{b+1}\}$ . Set  $v = \max V_b$ .

$$\pi = \begin{array}{ccccccc} \boxed{1} \boxed{2} \cdots \boxed{\quad} & \boxed{\quad} \boxed{\quad} \cdots \boxed{\quad} & \cdots & \boxed{\quad} \boxed{\quad} \cdots \boxed{\quad} & \boxed{\quad} \boxed{\quad} \cdots \boxed{\quad} & \boxed{\quad} \boxed{\quad} \cdots \boxed{\quad} & \boxed{2k} \\ & V_1 & & V_2 & & \cdots & & V_b & & v & & & & & V_{b+1} \end{array}$$

Then the left hand side of (3.13) is equal to

$$\sum_{\substack{\mathbf{r} \in [n]^{[2k] \setminus V_{b+1}}, \\ \inf_{I_h} \ker \mathbf{r} = \pi|_{[k] \setminus V_{b+1}}} (u_{r_1 j_1} u_{r_2 j_2} \cdots u_{r_v j_v} \sum_{\substack{r' \in [n], \\ r' \neq r_v}} u_{r' j_{v+1}} u_{r' j_{v+2}} \cdots u_{r' j_{2k}} p).$$

Since  $|V_{b+1}|$  is even, it follows that

$$\sum_{\substack{r' \in [n], \\ r' \neq r_v}} u_{r'j_{v+1}} u_{r'j_{v+2}} \cdots u_{r'j_{2k}} p = \delta(\ker(\mathbf{j}|_{V_{b+1}}), \mathbf{1}_{V_{b+1}}) p - u_{r_v j_{v+1}} u_{r_v j_{v+2}} \cdots u_{r_v j_{2k}} p.$$

By the assumption of induction, (3.13) is equal to

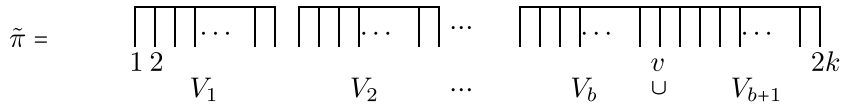
$$(3.15) \quad \delta(\pi|_{[k] \setminus V_{b+1}}, \inf_{I_h} \ker(\mathbf{j}|_{[k] \setminus V_{b+1}})) p \cdot \delta(\mathbf{1}_{V_{b+1}}, \ker(\mathbf{j}|_{V_{b+1}})) - R,$$

where

$$R = \sum_{\substack{\mathbf{r} \in [n]^{[2k] \setminus V_{b+1}}, \\ \inf_{I_h} \ker \mathbf{r} = \pi|_{[2k] \setminus V_{b+1}}}} u_{r_1 j_1} u_{r_2 j_2} \cdots u_{r_v j_v} \cdot u_{r_v j_{v+1}} u_{r_v j_{v+2}} \cdots u_{r_v j_{2k}} p.$$

For any multi-index  $\mathbf{r} \in [n]^{[2k] \setminus V_{b+1}}$ , set  $\tilde{\mathbf{r}} \in [n]^{2k}$  by  $\tilde{\mathbf{r}}_m := \mathbf{r}_m$  if  $m \leq v$ ,  $\tilde{\mathbf{r}}_m := r_v$ , otherwise.

Set  $\tilde{\pi} := \pi \vee (\uparrow^{\otimes(v-1)} \otimes \square \otimes \uparrow^{\otimes(2k-v-1)})$ . The partition  $\tilde{\pi}$  can be drawn as the following figure.



Then  $\inf_{I_h} \ker \mathbf{r} = \pi|_{[2k] \setminus V_{b+1}}$  if and only if  $\inf_{I_h} \ker \tilde{\mathbf{r}} = \tilde{\pi}$ . Since  $|\tilde{\pi}| = b$ , applying the assumption of induction yields  $R = \delta(\tilde{\pi}, \inf_{I_h} \ker \mathbf{j}) p$ . Hence (3.15) is equal to

$$\begin{aligned} & [\delta(\pi|_{[k] \setminus V_{b+1}}, \inf_{I_h} \ker(\mathbf{j}|_{[2k] \setminus V_{b+1}})) \cdot \delta(\mathbf{1}_{V_{b+1}}, \ker(\mathbf{j}|_{V_{b+1}})) - \delta(\tilde{\pi}, \inf_{I_h} \ker \mathbf{j})] p \\ & = \delta(\pi, \inf_{I_h} \ker \mathbf{j}) p. \end{aligned}$$

This is the desired conclusion.  $\square$

Let us construct  $*$ -representations of  $\mathcal{A}[I_o; n]$ ,  $\mathcal{A}[I_h; n]$ , which give us Haar states. We set a one dimensional projection  $R$  and self-adjoint opera-

tors  $F_i \in M_{n+1}(\mathbb{C})$  ( $i \leq n$ ) by the following: for  $k, l \leq n + 1$ ,

$$R(k, l) = \begin{cases} 1, & \text{if } k = l = n + 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$F_i(k, l) = \begin{cases} 1, & \text{if } (k, l) = (i, n + 1), (n + 1, i), \\ 0, & \text{otherwise.} \end{cases}$$

For any  $i, r \in [n]$  with  $i \neq r$  we have

$$(3.16) \quad RF_i^2 = R, \quad RF_iF_r = 0.$$

Set  $F_{ij} = F_i \otimes F_j$ . We set operators

$$P^o := R \otimes R, \quad U_{ij}^o := \frac{1}{\sqrt{n}}F_{ij},$$

$$P^h := Q(\hat{1}) \otimes P^o, \quad U_{ij}^h := Q(\hat{P}_{ij}) \otimes F_{ij}.$$

LEMMA 3.16. *The following relations define a  $*$ -homomorphism  $\pi_o: \mathcal{A}[I_o; n] \rightarrow M_{n+1}(\mathbb{C}) \otimes M_{n+1}(\mathbb{C})$  and a  $*$ -homomorphism  $\pi_h: \mathcal{A}[I_o; n] \rightarrow B(L^2(S_n)) \otimes M_{n+1}(\mathbb{C}) \otimes M_{n+1}(\mathbb{C})$ .*

$$\pi_x(p^x) = P^x, \quad \pi_x(u_{ij}^x) = U_{ij}^x.$$

PROOF. The proof is straightforward.  $\square$

LEMMA 3.17. *Let  $l, n \in \mathbb{N}$ . If  $l$  is odd then for any  $\mathbf{i}, \mathbf{j} \in [n]^l$ , we have*

$$(3.17) \quad \pi_o(pu_{\mathbf{i}\mathbf{j}}p) = \pi_h(pu_{\mathbf{i}\mathbf{j}}p) = 0.$$

*If  $l$  is even and  $l = 2k$ , then for any  $\mathbf{i}, \mathbf{j} \in [n]^{2k}$ , we have*

$$(3.18) \quad \pi_o(pu_{\mathbf{i}\mathbf{j}}p) = \zeta(\Pi^{\otimes k}, \ker \mathbf{i})\zeta(\Pi^{\otimes k}, \ker \mathbf{j})\frac{1}{n^k} \cdot P_o.$$

$$(3.19) \quad \pi_h(pu_{\mathbf{i}\mathbf{j}}p) = \zeta(\Pi^{\otimes k}, \ker \mathbf{i})\zeta(\Pi^{\otimes k}, \ker \mathbf{j})\delta(\inf_{I_h} \ker \mathbf{i}, \inf_{I_h} \ker \mathbf{j})$$

$$\times \frac{1}{n(n-1)^{|\inf_{I_h} \ker \mathbf{i}|-1}} \cdot P_h.$$

PROOF. The first and the second equations follow directly from (3.16). We prove the last equation. If  $i \neq r$ , or  $j \neq s$ , we have  $P_h U_{ij}^h U_{rs}^h = 0$ . Hence if  $\zeta(\Gamma^{\otimes k}, \ker \mathbf{i}) = 0$  or  $\zeta(\Gamma^{\otimes k}, \ker \mathbf{j}) = 0$  then  $\pi_h(pu_{\mathbf{ij}}) = 0$ .

Assume  $\zeta(\Gamma^{\otimes k}, \ker \mathbf{i}) = 1$  and  $\zeta(\Gamma^{\otimes k}, \ker \mathbf{j}) = 1$ . Then  $\inf_I \ker \mathbf{i} = \inf_{I_h} \ker \mathbf{i}$  and  $\inf_I \ker \mathbf{j} = \inf_{I_h} \ker \mathbf{j}$ . We check that  $P_h U_{ij}^h{}^2 = Q(\hat{1})Q(\hat{P}_{ij})^2 \otimes R \otimes R = P_h(Q(\hat{P}_{ij})^2 \otimes 1 \otimes 1)$ . By (3.8),

$$\begin{aligned} \pi_h(pu_{\mathbf{ij}}p) &= Q(\hat{1})Q(\hat{P}_{i_1 j_1})^2 \cdots Q(\hat{P}_{i_{2k-1} j_{2k-1}})^2 Q(\hat{1}) \otimes R \otimes R \\ &= \frac{\delta(\inf_I \ker \mathbf{i}, \inf_I \ker \mathbf{j})}{n(n-1)^{|\inf_I \ker \mathbf{i}|-1}} Q(\hat{1}) \otimes R \otimes R. \\ &= \frac{\delta(\inf_{I_h} \ker \mathbf{i}, \inf_{I_h} \ker \mathbf{j})}{n(n-1)^{|\inf_{I_h} \ker \mathbf{i}|-1}} P_h. \end{aligned}$$

This finishes proof.  $\square$

*Notation 3.18.* We define states  $\omega_o$  on  $M_{n+1}(\mathbb{C})$  and  $\omega_h$  on  $B(L^2(S_n)) \otimes M_{n+1}(\mathbb{C})$  by

$$\omega_o := \frac{\text{tr}_{n+1}(P_o \cdot)}{\text{tr}_{n+1} P_o}, \quad \omega_h := \frac{\omega \otimes \text{tr}_{n+1}(P_h \cdot)}{\omega \otimes \text{tr}_{n+1}(P_h)}.$$

PROPOSITION 3.19. For  $x = o, h$ , each state  $\omega_x \circ \pi_x$  is a Haar state. Furthermore,  $h_{I_x} = \omega_x \circ \pi_x$ .

PROOF. If  $l \in \mathbb{N}$  is odd, by (3.17),  $(id \otimes (\omega \circ \pi_x)) \Delta(pu_{\mathbf{ij}}p) = 0 = \omega_x(pu_{\mathbf{ij}}p)$ , where  $\mathbf{i}, \mathbf{j} \in [n]^l$  and  $x = o, h$ .

Assume  $l \in \mathbb{N}$  is even and set  $l = 2k$ . By (3.18) and (3.19), we have

$$\begin{aligned} (id \otimes (\omega_o \circ \pi_o)) \Delta(pu_{\mathbf{ij}}p) &= \sum_{\mathbf{s} \in [n]^{2k}} pu_{\mathbf{is}}p \cdot \zeta(\Gamma^{\otimes k}, \ker \mathbf{s}) \zeta(\Gamma^{\otimes k}, \ker \mathbf{j}) \frac{1}{n^k} \\ &= \frac{\zeta(\Gamma^{\otimes k}, \ker \mathbf{j})}{n^k} \sum_{\substack{\mathbf{s} \in [n]^k, \\ \Gamma^{\otimes k} \leq \ker \mathbf{s}}} pu_{\mathbf{is}}p. \\ (id \otimes (\omega_h \circ \pi_h)) \Delta(pu_{\mathbf{ij}}p) &= \sum_{\mathbf{s} \in [n]^{2k}} pu_{\mathbf{is}}p \cdot \delta(\inf_{I_h} \ker \mathbf{s}, \inf_{I_h} \ker \mathbf{j}) \end{aligned}$$



$$\begin{aligned} & \times \frac{\zeta(\Pi^{\otimes k}, \ker \mathbf{s})\zeta(\Pi^{\otimes k}, \ker \mathbf{j})}{n(n-1)^{|\inf_{I_h} \ker \mathbf{s}|-1}} \\ &= \frac{\zeta(\Pi^{\otimes k}, \ker \mathbf{j})}{n(n-1)^{|\inf_{I_h} \ker \mathbf{j}|-1}} \sum_{\substack{\mathbf{s} \in [n]_k, \\ \inf_{I_h} \ker \mathbf{s} = \inf_{I_h} \ker \mathbf{j}}} pu_{\mathbf{is}p}. \end{aligned}$$

By (3.13), we obtain the half of the Haar invariance of  $\omega_x \circ \pi_x$  ( $x = o, h$ ). Similar arguments can be applied to the other invariance. By the uniqueness of the Haar functional (Lemma 3.2), we have proven the proposition.  $\square$

**THEOREM 3.20.** *For any  $n \in \mathbb{N}$ ,  $Beq_o(n)$  and  $Beq_h(n)$  admit the unique Haar states. We write them  $h_o$  and  $h_h$ , respectively. In particular, we have  $h_o \circ \iota_n = h_{I_o}$  and  $h_h \circ \iota_n = h_{I_h}$ .*

**PROOF.** As  $\|U_{ij}^x\|_n \leq 1$ , we can extend  $\pi_x$  to  $Beq_x$  ( $x = 0, h$ ), which proves the theorem.  $\square$

### 4. Boolean De Finetti Theorems

Let  $(M, \varphi)$  be a pair of a von Neumann algebra and a normal state with faithful GNS-representation and consider an infinite sequence  $(x_j)_{j \in \mathbb{N}}$  of self-adjoint elements  $x_j \in M$ . We may assume  $M \subseteq B(H)$ , and  $\varphi$  is implemented by  $\Omega \in H$ , which is a cyclic vector for  $M$ . Throughout this section we suppose  $ev_x(\mathcal{P}_\infty^o)$  is  $\sigma$ -weakly dense in  $M$ , where  $ev_x$  is the evaluation map (see Notation 1.15. for the definition).

#### 4.1. Combinatorial part

At first we show the purely combinatorial part of Boolean de Finetti theorems.

**PROPOSITION 4.1.** *Assume  $D$  be a blockwise category of interval partitions. Let  $E: M \rightarrow N$  be a  $\varphi$ -preserving conditional expectation. Suppose  $(x_j)_{j \in J}$  are Boolean independent and identically distributed over  $(E, N)$ , and  $K_k^E[x_1, x_1, \dots, x_1] = 0$ , for all  $k \in \mathbb{N} \setminus L_D$ . Then  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[D]$ -invariant.*

PROOF. By the moments-cumulants formula, we have for any  $\mathbf{j} \in [n]^k$  and  $k \in \mathbb{N}$ ,

$$\begin{aligned}
 & (\varphi \circ \text{ev}_x \otimes \text{id}) \circ \Psi_n(X_{j_1} \cdots X_{j_k}) \\
 &= \sum_{\mathbf{i} \in [n]^k} \varphi(x_{i_1} \cdots x_{i_k}) \otimes pu_{i_1 j_1} \cdots u_{i_k j_k} p \\
 &= \sum_{\mathbf{i} \in [n]^k} \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{i}}} K_E^{(\pi)}[x_1, \dots, x_1] \otimes pu_{i_1 j_1} \cdots u_{i_k j_k} p \\
 &= \sum_{\pi \in D(k)} K_E^{(\pi)}[x_1, \dots, x_1] \otimes \sum_{\substack{\mathbf{i} \in [n]^k \\ \pi \leq \ker \mathbf{i}}} pu_{i_1 j_1} \cdots u_{i_k j_k} p \\
 &= \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{j}}} K_E^{(\pi)}[x_1, \dots, x_1] \otimes p \\
 &= \varphi \circ \text{ev}_x(X_{j_1} \cdots X_{j_k}) \otimes p. \quad \square
 \end{aligned}$$

**4.2. Observations on the conditional expectations**

To prove the opposite direction, we observe properties of the conditional expectations. Throughout this section, we assume  $D$  is a blockwise category of interval partitions.

*Notation 4.2.*

- (1) Denote by  $\mathcal{P}_\infty^{o, \Psi_n}$  the fixed point algebra of the coaction  $\Psi_n$ , that is,

$$\mathcal{P}_\infty^{o, \Psi_n} := \{f \in \mathcal{P}_\infty^o \mid \Psi_n(f) = f \otimes p\}.$$

- (2) Define a linear map  $\mathcal{E}_n: \mathcal{P}_\infty^o \rightarrow \mathcal{P}_\infty^o$  by  $\mathcal{E}_n := (\text{id} \otimes h) \circ \Psi_n$ .
- (3) For  $\pi \in P(k)$ , we set

$$X_\pi := \sum_{\mathbf{j} \in [n]^k, \pi \leq \ker \mathbf{j}} X_{j_1} \cdots X_{j_k}.$$

PROPOSITION 4.3. *The following hold:*

(1)  $\Psi_n$  is  $\mathcal{P}_\infty^{o,\Psi_n}$ - $\mathcal{P}_\infty^{o,\Psi_n}$  bilinear map : for each  $f \in \mathcal{P}_\infty^{o,\Psi_n}$  and  $g \in \mathcal{P}_\infty^o$ ,

$$\Psi_n(fg) = (f \otimes \text{id})\Psi_n(g), \Psi_n(gf) = \Psi_n(g)(f \otimes \text{id}).$$

(2)  $\mathcal{E}_n$  is a conditional expectation with respect to the embedding  $\mathcal{P}_\infty^{o,\Psi_n} \hookrightarrow \mathcal{P}_\infty^o$ .

PROOF. By (3.5), it follows that  $\mathcal{P}_\infty^{o,\Psi_n} = \text{Span}\{X_\pi \in \mathcal{P}_\infty^o \mid \pi \in D(k), k \in \mathbb{N}\}$ . For any  $\mathbf{j} \in [n]^k, \pi \in D(l)$  and  $k, l \in \mathbb{N}$ ,

$$\Psi_n(X_{i_1} \cdots X_{i_k} X_\pi) = \Psi_n(X_{i_1} \cdots X_{i_k})(X_\pi \otimes \text{id})$$

by the direct computation. The symmetric proof shows  $\Psi_n$  is a  $\mathcal{P}_\infty^{o,\Psi_n}$ - $\mathcal{P}_\infty^{o,\Psi_n}$  bilinear map.

Next, we prove that  $\mathcal{E}_n$  is a conditional expectation.  $\mathcal{E}_n$  is also  $\mathcal{P}_\infty^{o,\Psi_n}$ - $\mathcal{P}_\infty^{o,\Psi_n}$  bilinear map since so is  $\Psi_n$ . Clearly we have  $\mathcal{E}_n[f] = (\text{id} \otimes h)(f \otimes p) = f$  for any  $f \in \mathcal{P}_\infty^{o,\Psi_n}$ . The proof is completed by showing that  $\Psi_n \circ \mathcal{E}_n = \mathcal{E}_n[\cdot] \otimes p$ . Let  $\nu$  be the natural isomorphism  $V_n^D \otimes \mathbb{C} \rightarrow V_n^D$ . Then

$$\begin{aligned} \Psi_n \circ \mathcal{E}_n[f] &= (\text{id} \otimes \nu) \circ (\Psi_n \circ \text{id}) \circ (\text{id} \otimes h) \circ \Psi_n \\ &= (\text{id} \otimes \nu) \circ (\text{id} \otimes \text{id} \otimes h) \circ (\Psi_n \otimes \text{id}) \circ \Psi_n. \end{aligned}$$

As  $\Psi_n$  is a linear coaction, the right-hand side is equal to  $(\text{id} \otimes \nu) \circ (\text{id} \otimes \text{id} \otimes h) \circ (\text{id} \otimes \Delta) \circ \Psi_n$ . By the invariance property of the Haar functional  $h$ , this is equal to  $(\text{id} \otimes \nu) \circ \iota \circ (\text{id} \otimes h) \circ \Psi_n$ , where  $\iota$  is the embedding  $\mathcal{P}_\infty^o \otimes \mathbb{C} \hookrightarrow \mathcal{P}_\infty^o \otimes V_n^D \otimes \mathbb{C}$ ;  $\iota(f \otimes \lambda) = f \otimes p \otimes \lambda$ . By the easy computation, this is equal to  $\mathcal{E}_n[\cdot] \otimes p$ .  $\square$

Using the invariance of the joint distribution, we see that the conditional expectation is connected with the  $L^2$ -conditional expectation.

LEMMA 4.4. Suppose  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[D]$ -invariant for a blockwise category  $D$  of interval partitions, or  $\text{Beq}_x$ -invariant for  $x = s, o, h$ . Then  $\mathcal{E}_n$  preserves  $\varphi \circ \text{ev}_x$  for any  $n \in \mathbb{N}$ . Moreover for any  $f \in \mathcal{P}_\infty^o$ , we have

$$e_n \text{ev}_x(f) e_n = \text{ev}_x(\mathcal{E}_n(f)) e_n,$$

where  $e_n$  is the orthogonal projection onto  $\overline{\text{ev}_x(\mathcal{P}^{\Psi_n})\Omega}$ .

PROOF. By definition  $\mathcal{A}_p[D]$ -invariance implies that  $\mathcal{E}_n$  preserves  $\varphi \circ \text{ev}_x$ . Assume *Beq<sub>x</sub>*-invariance. Since  $h_{I_x} = h_x \circ \iota_n$ , we have  $\mathcal{E}_n = (\text{id} \otimes (h_x \circ \iota_n)) \Psi_n = (\text{id} \otimes h_x) \Phi_n$ . The *Beq<sub>x</sub>*-invariance implies that  $\mathcal{E}_n$  preserves  $\varphi \circ \text{ev}_x$ . For any  $\pi, \sigma \in D(k)$  and  $f \in \mathcal{P}_\infty^o$ , we have

$$\begin{aligned} \langle X_\pi \Omega, \text{ev}_x \circ \mathcal{E}_n(f) f_\sigma \Omega \rangle &= \varphi(\text{ev}_x \circ \mathcal{E}_n(X_\pi^* f X_\sigma)) = \varphi(\text{ev}_x(X_\pi^* f X_\sigma)) \\ &= \langle X_\pi \Omega, \text{ev}_x(f) f_\sigma \Omega \rangle, \end{aligned}$$

which completes the proof.  $\square$

In [1], a noncommutative martingale convergence theorem of cumulants plays an important role in the proof of de Finetti theorems. Since  $\varphi$  is not faithful, we modify this convergence theorem.

PROPOSITION 4.5. *Let  $(M \subseteq B(H), \Omega \in H)$  be a pair of a von Neumann algebra and a cyclic vector. Assume  $M$  is  $\sigma$ -weakly generated by a sequence  $(x_n)_{n \in \mathbb{N}}$  of self-adjoint elements. Let  $q \in M$  be a non-zero projection and  $L := qMq$ , set a conditional expectation  $E_L := q \cdot q : M \rightarrow L$ . Let  $(\mathcal{B}_n)_{n \in \mathbb{N}}$  be a decreasing sequence of  $*$ -subalgebras of  $\mathcal{P}_\infty^o$ , and denote by  $e_n$  the orthogonal projections onto the closed subspaces  $\overline{\text{ev}_x(\mathcal{B}_n)\Omega}$ . Set*

$$B_\infty := \bigcap_{n \in \mathbb{N}} \overline{\text{ev}_x(\mathcal{B}_n)^{\sigma w}}.$$

We assume the following conditions:

- (1) *There is a  $\varphi \circ \text{ev}_x$  preserving conditional expectation  $\mathcal{E}_n : \mathcal{P}_\infty^o \rightarrow \mathcal{B}_n$  for each  $n \in \mathbb{N}$ .*
- (2)  $\overline{B_\infty \Omega} = \overline{L \Omega}$ .

Then for any  $\pi \in I(k)$ ,  $k \in \mathbb{N}$ , and  $f_1, \dots, f_k \in \mathcal{P}_\infty^o$ , we have

$$\begin{aligned} s\text{-}\lim_{n \rightarrow \infty} \text{ev}_x(\mathcal{E}_n^\pi[f_1, \dots, f_k])e_n &= E_L^\pi[f_1(x), \dots, f_k(x)], \\ s\text{-}\lim_{n \rightarrow \infty} \text{ev}_x(K_\pi^{\mathcal{E}_n}[f_1, \dots, f_k])e_n &= K_\pi^{E_L}[f_1(x), \dots, f_k(x)], \end{aligned}$$

where we write  $f(x) = \text{ev}_x(f)$  for  $f \in \mathcal{P}_\infty^o$ .

PROOF. By condition (1),  $e_n \text{ev}_x(f) e_n = \text{ev}_x(\mathcal{E}_n(f)) e_n$ . By condition (2),  $s\text{-}\lim_{n \rightarrow \infty} e_n = q$ , and  $s\text{-}\lim_{n \rightarrow \infty} \text{ev}_x(\mathcal{E}_n(f)) e_n = q \text{ev}_x(f) q =$

$E_L[\text{ev}_x(f)]$ . It holds that  $\text{ev}_x \circ \mathcal{E}_n^\pi[f_1, \dots, f_k]e_n = \prod_{V \in \pi}^{\rightarrow} e_n \text{ev}_x(\prod_{j \in V}^{\rightarrow} f_j)e_n$ , for any  $\pi \in I(k)$ . Hence

$$\begin{aligned} s\text{-}\lim_{n \rightarrow \infty} \text{ev}_x \circ \mathcal{E}_n^\pi[f_1, \dots, f_k]e_n &= \prod_{V \in \pi}^{\rightarrow} E_L[\prod_{j \in V}^{\rightarrow} f_j(x)] \\ &= E_L^\pi[f_1(x), f_2(x), \dots, f_k(x)]. \end{aligned}$$

Partitioned cumulants are linear combinations of partitioned conditional expectations, which proves the statement.  $\square$

PROPOSITION 4.6. *For any  $k \in \mathbb{N}$ ,  $\pi \in D(k)$  and sufficiently large  $n$  such that the Gram matrix is invertible, we have*

$$\mathcal{E}_n^\pi[X_1, \dots, X_1] = \frac{1}{n^{|\pi|}} \sum_{\substack{i \in [n]^k \\ \pi \leq \ker i}} X_{i_1} X_{i_2} \cdots X_{i_k}.$$

PROOF. This follows by a similar proof to that in [1, Prop.4.7], which is induction on  $|\pi|$ .  $\square$

LEMMA 4.7. *Let  $M$  be a von Neumann algebra. Fix a nonzero projection  $e \in M$ . Set a conditional expectation  $E: M \rightarrow N = eMe$  by  $E(y) = eye$ . Let  $k \in \mathbb{N}$  with  $k \geq 2$  and  $\pi \in I(k)$ . Assume that  $l \in \mathbb{N}$  satisfies  $l < k$  and  $l \sim^\pi l + 1$ . Then for any  $b \in N, y_1, \dots, y_k \in M$ ,*

$$(4.1) \quad K_\pi^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k] = 0.$$

PROOF. In the case  $k = 2$ , it holds  $I(2) = \{\square\}$  and  $K_2^E[y_1 b, y_2] = E[y_1 b y_2] - E[y_1 b]E[y_2] = e y_1 b y_2 e - e y_1 b e y_2 e = 0$  as  $b = be$ .

Let  $k \geq 3$ . Assume (4.1) holds for any  $\pi \in I(k - 1)$ . Since  $b = be$ ,  $E[y_1 \dots y_l b y_{l+1} \dots y_k] = E[y_1 \dots y_l b]E[y_{l+1} \dots y_k]$ . The moments-cumulants formula and the assumption of induction imply that

$$\begin{aligned} &K_k^E[y_1 \dots, y_l b, y_{l+1}, \dots, y_k] \\ &= E[y_1 \dots y_l b y_{l+1} \dots y_k] - \sum_{\pi \in I(k), \pi \neq \mathbf{1}_k} K_\pi^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k] \\ &= E[y_1 \dots y_l b]E[y_{l+1} \dots y_k] - \sum_{\pi \in I(k), l \not\sim^\pi l+1} K_\pi^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k]. \end{aligned}$$

We have  $\{\pi \in I(k) \mid l \not\sim^\pi l+1\} = \{\sigma \otimes \rho \mid \sigma \in I(l), \rho \in I(k-l)\}$ . Then

$$\begin{aligned} & K_{\sigma \otimes \rho}^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k] \\ &= \prod_{V \in \sigma \otimes \rho}^{\rightarrow} K_{(V)}^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k] \\ &= \prod_{V_1 \in \sigma}^{\rightarrow} K_{(V_1)}^E[y_1, \dots, y_l b] \prod_{V_2 \in \rho}^{\rightarrow} K_{(V_2)}^E[y_{l+1}, \dots, y_k] \\ &= K_{\sigma}^E[y_1, \dots, y_l b] K_{\rho}^E[y_{l+1}, \dots, y_k]. \end{aligned}$$

Hence  $E[y_1 \dots y_l b] E[y_{l+1} \dots y_k] - \sum_{\pi \in I(k), l \not\sim^\pi l+1} K_{\pi}^E[y_1, \dots, y_l b, y_{l+1}, \dots, y_k] = 0$ . Induction on  $k$  proves the lemma.  $\square$

### 4.3. Boolean de Finetti theorems

LEMMA 4.8. *Assume that  $\|x_j\| \leq \|x_1\|$  for any  $j \in \mathbb{N}$ . Let  $D$  be one of  $I, I_o, I_h, I_b$ . For any  $k \in \mathbb{N}$ ,  $\sigma \in D(k)$  and  $n_0, n \in \mathbb{N}$  with  $n_0 \leq n$ , set an element in  $\mathcal{P}_{\geq n_0}^o$  by*

$$f_{\sigma}^{n_0, n} := \sum_{\pi \in D(k)} \frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i} \in [n_0, n]^k \\ \pi \leq \ker \mathbf{i}}} X_{i_1} X_{i_2} \cdots X_{i_k} \mu_{I(k)}(\pi, \sigma).$$

Then we have

$$\begin{aligned} (4.2) \quad & \|\text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \text{ev}_x \circ K_{\sigma}^{\mathcal{E}_n}[X_1, \dots, X_1]\| \\ & \rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}. \end{aligned}$$

$$(4.3) \quad \|\text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \text{ev}_x(f_{\sigma}^{n_0, n})\| \rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}.$$

PROOF. By Proposition 1.10 and Lemma 4.6, we have for sufficiently large  $n$ ,

$$\begin{aligned} \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] &= \sum_{\mathbf{i} \in [n]^k} X_{i_1} X_{i_2} \cdots X_{i_k} Q_{\mathbf{j}}^{(k)} \\ &= \sum_{\mathbf{i} \in [n]^k} X_{i_1} X_{i_2} \cdots X_{i_k} \sum_{\substack{\pi, \sigma \in D(k) \\ \pi \leq \ker \mathbf{i}, \sigma \leq \ker \mathbf{j}}} W_{k, n}(\pi, \sigma) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \left( \frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i} \in [n]^k \\ \pi \leq \ker \mathbf{i}}} X_{i_1} X_{i_2} \cdots X_{i_k} \right) n^{|\pi|} W_{k,n}(\pi, \sigma) \\
 &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \mathcal{E}_n^\pi[X_1, \dots, X_1] n^{|\pi|} W_{k,n}(\pi, \sigma).
 \end{aligned}$$

By the moments-cumulants formula Proposition 1.22, we have

$$\begin{aligned}
 &\mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_\sigma^{\mathcal{E}_n}[X_1, \dots, X_1] \\
 &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \mathcal{E}_n^\pi[X_1, \dots, X_1] n^{|\pi|} W_{k,n}(\pi, \sigma) \\
 &\quad - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \mathcal{E}_n^\pi[X_1, \dots, X_1] \mu_{I(k)}(\pi, \sigma) \\
 &= \sum_{\pi \in D(k)} \left[ \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} n^{|\pi|} W_{k,n}(\pi, \sigma) - \mu_{I(k)}(\pi, \sigma) \right] \mathcal{E}_n^\pi[X_1, \dots, X_1].
 \end{aligned}$$

$$\begin{aligned}
 &\| \text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \text{ev}_x \circ K_\sigma^{\mathcal{E}_n}[X_1, \dots, X_1] \| \\
 &\leq \max_{\pi \in D(k)} \left[ \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} |n^{|\pi|} W_{k,n}(\pi, \sigma) - \mu_{I(k)}(\pi, \sigma)| \right] \\
 &\quad \times \sum_{\pi \in D(k)} \| \text{ev}_x \circ \mathcal{E}_n^\pi[X_1, \dots, X_1] \| \\
 &\leq \max_{\pi \in D(k)} \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} |n^{|\pi|} W_{k,n}(\pi, \sigma) - \mu_{I(k)}(\pi, \sigma)| \cdot |D(k)| \cdot \|x_1\|^k.
 \end{aligned}$$

By the Weingarten estimate in Proposition 1.12,

$$\max_{\pi \in D(k)} \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} |n^{|\pi|} W_{k,n}(\pi, \sigma) - \mu_{I(k)}(\pi, \sigma)| = O\left(\frac{1}{n}\right) \text{ (as } n \rightarrow \infty\text{)}.$$

Therefore, we have (4.2).

For any  $n_0 \in \mathbb{N}$ , we have

$$K_\sigma^{\mathcal{E}_n}[X_1, \dots, X_1] - f_\sigma^{n_0, n} = \sum_{\pi \in D(k)} \frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i} \in [n]^k \setminus [n_0, n]^k \\ \pi \leq \ker \mathbf{i}}} X_{i_1} X_{i_2} \cdots X_{i_k} \mu_{I(k)}(\pi, \sigma).$$

Now

$$\frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i} \in [n]^k \setminus [n_0, n]^k \\ \pi \leq \ker \mathbf{i}}} \|x_{i_1} x_{i_2} \cdots x_{i_k}\| \leq \frac{n^{|\pi|} - (n - n_0)^{|\pi|}}{n^{|\pi|}} \|x_1\|^k \rightarrow 0 \text{ (as } n \rightarrow \infty).$$

Hence

$$\begin{aligned} & \| \text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \text{ev}_x(f_\sigma^{n_0, n}) \| \\ & \leq \| \text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] - \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \text{ev}_x \circ K_\sigma^{\mathcal{E}_n}[X_1, \dots, X_1] \| \\ & \quad + \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \| \text{ev}_x \circ K_\sigma^{\mathcal{E}_n}[X_1, \dots, X_1] - \text{ev}_x(f_\sigma^{n_0, n}) \| \\ & \rightarrow 0 \text{ (as } n \rightarrow \infty). \quad \square \end{aligned}$$

Now we are prepared to prove our main theorem, de Finetti theorems for  $\mathcal{A}_p[I_x]$  and  $\text{Beq}_x$ .

**THEOREM 4.9.** *Let  $(M, \varphi)$  be a pair of a von Neumann algebra and a nondegenerate normal state. Assume  $M$  is generated by self-adjoint elements  $(x_j)_{j \in \mathbb{N}}$ . Consider the following three assertions.*

- (1) *The joint distribution of  $(x_j)_{j \in \mathbb{N}}$  is  $\mathcal{A}_p[I_x]$ -invariant.*
- (2) *The joint distribution of  $(x_j)_{j \in \mathbb{N}}$  is  $\text{Beq}_x$ -invariant.*
- (3) *The elements  $(x_j)_{j \in \mathbb{N}}$  are Boolean independent and identically distributed over  $(E_{\text{nut}}, M_{\text{nut}})$ , and for all  $k \in \mathbb{N} \setminus L_{I_x}$ , and  $b_1, \dots, b_k \in M_{\text{nut}} \cup \{1\}$ , it holds that*

$$K_k^{E_{\text{nut}}}[x_1 b_1, x_1 b_2, \dots, x_1] = 0.$$



Then for  $x = s, o, h$ , all assertions are equivalent. For  $x = b$ , (1) and (3) are equivalent.

PROOF. By Proposition 4.1, we have (3) implies (1). We prove each condition (1), (2) implies (3) in the case  $x = s, o, h$ , and prove (1) implies (3) in the case  $x = b$ . Let  $(H, \Omega)$  be the GNS-representation of  $(M, \varphi)$ . As  $\varphi$  is nondegenerate, we may assume  $M \subseteq B(H)$ . Set  $B_\infty := \overline{\bigcap_{n \in \mathbb{N}} \text{ev}_x(\mathcal{P}^{\Psi_n})^{\sigma w}}$ . At first, we prove  $\overline{B_\infty \Omega} = \overline{M_{\text{nut}} \Omega}$ . Since  $\mathcal{P}_{\geq n}^o \subseteq \mathcal{P}^{\Psi_n}$ , it is clear that  $\overline{B_\infty \Omega} \supseteq \overline{M_{\text{nut}} \Omega}$ . Let  $e_n$  be the orthogonal projection onto the subspace  $H_n := \text{ev}_x(\mathcal{P}^{\Psi_n})\Omega \subseteq H$ . Set  $e_\infty$  be the orthogonal projection onto  $\bigcap_{n \in \mathbb{N}} H_n = \overline{B_\infty \Omega}$ . The projections  $(e_n)_{n \in \mathbb{N}}$  strongly converges to  $e_\infty$ . To see  $\overline{B_\infty \Omega} \subseteq \overline{M_{\text{nut}} \Omega}$ , we only need to show that  $e_\infty x_{\mathbf{j}} \Omega \in \overline{M_{\text{nut}} \Omega}$  for any  $k \in \mathbb{N}, \mathbf{j} \in [n]^k$ . By Lemma 4.4, each condition (1), (2) implies  $\text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] \Omega = e_n x_{j_1} x_{j_2} \cdots x_{j_k} \Omega$ . As each condition (1), (2) implies that  $(x_j)_{j \in \mathbb{N}}$  are identically distributed, we have  $\|x_j\| = \|x_1\|$  for any  $j \in \mathbb{N}$ . Then by Lemma 4.8, it holds that  $\text{ev}_x(f_\sigma^{n_0, n}) \Omega$  converges to an element in  $\overline{\text{ev}_x(\mathcal{P}_{\geq n_0}^o) \Omega}$  as  $n \rightarrow \infty$ . We have

$$e_\infty x_{\mathbf{j}} \Omega = \lim_{n \rightarrow \infty} \text{ev}_x \circ \mathcal{E}_n[X_{j_1} X_{j_2} \cdots X_{j_k}] \Omega \in \bigcap_{n_0 \in \mathbb{N}} \overline{\text{ev}_x(\mathcal{P}_{\geq n_0}^o) \Omega} = \overline{M_{\text{nut}} \Omega}.$$

By Lemma 4.4,  $\mathcal{E}_n$  preserves  $\varphi \circ \text{ev}_x$  and by the modified martingale convergence theorem (see Proposition 4.5) and (4.2), we obtain for any  $j_1, \dots, j_k \in J, k \in \mathbb{N}$ ,

$$(4.4) \quad E_{\text{nut}}[x_{j_1} \cdots x_{j_k}] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_\sigma^{E_{\text{nut}}} [x_1, \dots, x_1].$$

The proof is completed by showing that for any  $b_0, \dots, b_k \in M_{\text{nut}} \cup \{1\}$ ,  $j_1, \dots, j_k \in J$ , and  $k \in \mathbb{N}$ ,

$$(4.5) \quad E_{\text{nut}}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_\sigma^{E_{\text{nut}}} [x_1 b_1, x_1 b_2, \dots, x_1].$$

We prove this by induction on  $\#\{l \in [k-1]; b_l \neq 1\}$ . In the case  $\#\{l \in [k-1]; b_l \neq 1\} = 1$ , the claim holds by (4.4). Pick any  $m \in \mathbb{N} \cup \{0\}$  with  $m \leq k-1$ . Assume that (4.5) is proved in the case that  $\#\{l \in$

$[k - 1]; b_l \neq 1\} < m$ . Consider the case  $\#\{l \in [k - 1]; b_l \neq 1\} = m$ . Let  $r = \max\{l \in [k - 1]; b_l \neq 1\}$ . Then by Lemma 4.7,

$$\begin{aligned} & \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_{\sigma}^{E_{\text{nut}}} [x_1 b_1, \dots, x_1 b_r, \dots, x_1] \\ &= \sum_{\substack{\sigma \in D(k), \sigma \leq \ker \mathbf{j} \\ r \not\leq \sigma}} K_{\sigma|_{[1,r]}}^{E_{\text{nut}}} [x_1 b_1, \dots, x_1] b_r K_{\sigma|_{[r+1,k]}}^{E_{\text{nut}}} [x_1 b_{r+1}, \dots, x_1]. \end{aligned}$$

By the property (D1), this equals to

$$\begin{aligned} & \sum_{\substack{\pi \in D(r) \\ \pi \leq \ker \mathbf{j}|_{[1,r]}}} K_{\pi}^{E_{\text{nut}}} [x_1 b_1, \dots, x_1] b_r \sum_{\substack{\rho \in D(k-r) \\ \rho \leq \ker \mathbf{j}|_{[r+1,k]}}} K_{\rho}^{E_{\text{nut}}} [x_1 b_{r+1}, \dots, x_1] \\ &= E_{\text{nut}} [x_{j_1} b_1 \cdots x_{j_r}] b_r E_{\text{nut}} [x_{j_{r+1}} b_{r+1} \cdots x_{j_k}] \\ &= E_{\text{nut}} [x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}]. \end{aligned}$$

By induction on  $m$ , (4.5) holds for any  $b_0, \dots, b_k \in M_{\text{nut}} \cup \{1\}$ , which proves (1).  $\square$

**COROLLARY 4.10.** *If the equivalent conditions in Theorem 4.9 are satisfied for one of  $x = o, h$  and  $b$ , then the following hold:*

- (o) *If  $x = o$ ,  $(x_j)_{j \in \mathbb{N}}$  form a  $M_{\text{nut}}$ -valued Boolean centered Bernoulli family.*
- (h) *If  $x = h$ ,  $(x_j)_{j \in \mathbb{N}}$  are Boolean independent, and have even and identically distributions, over  $M_{\text{nut}}$ .*
- (b) *If  $x = b$ ,  $(x_j)_{j \in \mathbb{N}}$  form a  $M_{\text{nut}}$ -valued Boolean shifted Bernoulli family.*

**PROOF.** The proof directly follows from Theorem 4.9.  $\square$

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