J. Math. Sci. Univ. Tokyo
23 (2016), 867–919.

Large Time Asymptotics of Solutions for the Third-Order Nonlinear Schrödinger Equation

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Abstract. We study the large time asymptotics of small amplitude solutions to the Cauchy problem for the third-order nonlinear Schrödinger equation with zero mass condition which is considered as a complex version of the modified Korteweg-de Vries equation. We apply the factorization of the free evolution group to improve the time decay estimate of solutions obtained in the previous work.

1. Introduction

We consider the Cauchy problem for the third-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t v + \alpha\partial_x^2 v + i\beta\partial_x^3 v = \mu\left|v\right|^2 v + i\lambda\partial_x\left|v\right|^2 v, \ t>0, \ x \in \mathbf{R}, \\ v\left(0,x\right) = v_0\left(x\right), \ x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \lambda, \mu \in \mathbf{R}$. This equation arises in the context of high-speed soliton transmission in long-haul optical communication system [14]. Also it can be considered as a particular form of the higher order nonlinear Schrödinger equation introduced by [36] to describe the nonlinear propagation of pulses through optical fibers. This equation also represents the propagation of pulses by taking higher dispersion effects into account than those given by the Schrödinger equation (see [17], [32], [37], [45]). The higher order nonlinear Schrödinger equations have been widely studied recently. For the local and global well-posedness of the Cauchy problem we refer to [5], [6], [35] and references therein. The dispersive blow-up was obtained in [3].

We exclude the term with the second derivative changing $v = v_1 e^{\frac{\alpha i}{3\beta}x}$

$$i\partial_t v_1 + i\beta\partial_x^3 v_1 + \frac{i\alpha^2}{3\beta}\partial_x v_1 - \frac{2\alpha^3}{27\beta^2}v_1 = \left(\mu + \frac{\alpha\lambda}{3\beta}\right)\left|v_1\right|^2 v_1 + i\lambda\partial_x\left(\left|v_1\right|^2 v_1\right).$$

2010 *Mathematics Subject Classification.* 35Q35, 35Q55.
Key words: Third-order nonlinear Schrödinger equation, large time behavior of solutions, complex modified Korteweg-de Vries equation.

After that we exclude the third and fourth terms on the left-hand side of the above equation by changing $v_1(t,x) = v_2\left(t, x - \frac{\alpha^2}{3\beta}t\right) e^{-\frac{2it\alpha^3}{27\beta^2}}$, then we get

$$i\partial_t v_2 + i\beta\partial_x^3 v_2 = \left(\mu + \frac{\alpha\lambda}{3\beta}\right)|v_2|^2 v_2 + i\lambda\partial_x\left(|v_2|^2 v_2\right).$$

Without loss of generality we take $\beta = -\frac{1}{3}$

$$\partial_t v_2 - \frac{1}{3}\partial_x^3 v_2 = -i(\mu - \alpha\lambda)|v_2|^2 v_2 + \lambda\partial_x\left(|v_2|^2 v_2\right).$$

Note that the symbol $\Lambda(\xi) = \frac{1}{3}\xi^3$ degenerates at $\xi = 0$, hence the free evolution group performs more slow time decay like $O\left(t^{-\frac{1}{3}}\right)$, in general. Therefore the first term of the right hand side is a difficult term with respect to small data global existence. To avoid this difficulty we choose $\mu = \alpha\lambda$, so we consider

$$\begin{cases} \mathcal{L}v = \lambda\partial_x\left(|v|^2 v\right), \ t > 0, \ x \in \mathbf{R}, \\ v(0,x) = v_0(x), \ x \in \mathbf{R}, \end{cases} \tag{1.2}$$

where $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$. Then the solution of (1.1) is represented by

$$v\left(t, x + \alpha^2 t\right) e^{-\frac{2it\alpha^3}{3} - \alpha i x}$$

through the solution v of (1.2). Next we assume that the initial data are such that $\widehat{v_0}(0) = 0$. By changing $v = \partial_x u$, we get the equation in the potential form

$$\begin{cases} \mathcal{L}u = \lambda|u_x|^2 u_x, \ t > 0, \ x \in \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{1.3}$$

This is our target equation in the present paper. Note that equation (1.3) or (1.2) can be considered as a complex version of the modified Korteweg-de Vries (KdV) equation. KdV equation was introduced in [33] to describe unidirectional propagation of nonlinear dispersive long waves. In [25], it was also pointed out that the Alfvén waves are also described by the modified KdV equation $\mathcal{L}u = \partial_x u^3$. Concerning the solitary wave solutions of higher order dispersive equations, there are a lot of works (see, e.g., [24], [30]). However the large time asymptotic behavior of small solutions was not studied well. As far as we know the large time asymptotics of solutions to $\mathcal{L}u = \partial_x u^2$ is still an open problem. For the modified KdV equation $\mathcal{L}u = \partial_x u^3$ we studied the large time asymptotics of small solutions and showed that the small amplitude real-valued solutions are stable in the neighborhood of the self-similar solutions (see [22]). Cauchy problem (1.3) was intensively studied by many authors. The existence and uniqueness of solutions to (1.3) were proved in [16], [18], [26], [27], [29], [28], [34], [39], [41], [46] and the smoothing properties of solutions were studied in [4], [9], [12], [26], [27], [29], [28], [41]. The blow-up effect for a special class of slowly decaying solutions of the Cauchy problem (1.3) was found in [2].

The large time asymptotics of solutions to the generalized KdV equation $\mathcal{L}u = \partial_x |u|^{\rho-1} u$ was studied in [7], [19], [29], [31], [38], [40], [43], [44] for different values of ρ in the super critical region $\rho > 3$. For the special cases of the KdV equation itself $\mathcal{L}u = \partial_x \left(u^2\right)$ and of the modified KdV equation $\mathcal{L}u = \partial_x \left(u^3\right)$, the Cauchy problem was solved by the Inverse Scattering Transform (IST) method and thus the large time asymptotic behavior of solutions was studied (see [1], [13]). It was known that solutions of the modified KdV equation $\mathcal{L}u = \partial_x \left(u^3\right)$ decay with the same speed as in the corresponding linear case, i.e. $\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{3}}$ as $t \to \infty$ (see [13]). The IST method depends essentially on the nonlinearity in the equation. It is not applicable if we slightly change the nonlinear term like $a(t)\,\partial_x \left(u^3\right)$ with $|a(t)| \le C$. So it is very important to develop alternative methods for studying the large time asymptotics of solutions. In [20] we obtained the large time asymptotics of solutions to equation

$$\mathcal{L}v = \partial_x \left(v^3\right) \tag{1.4}$$

in the case of small real-valued initial data $v_0 \in \mathbf{H}^{1,1}$ with zero total mass assumption $\int_{\mathbf{R}} v_0(x)dx = 0$. More precisely we have the asymptotics

$$\begin{aligned} v(t,x) &= t^{-\frac{1}{3}}\theta(x)\,\mathrm{Re}Ai\left(xt^{-\frac{1}{3}}\right) W\left(\left(\frac{x}{t}\right)^{\frac{1}{2}}\right) \\ &\quad \times \exp\left(-\frac{3}{4}i\left|W\left(\left(\frac{x}{t}\right)^{\frac{1}{2}}\right)\right|^2 \log t\right) \\ &\quad + O\left(t^{\gamma-\frac{1}{2}}\right) \end{aligned} \tag{1.5}$$

for large time t, where $\gamma > 0$, $W \in \mathbf{L}^\infty$ satisfying $W(0) = 0$ is uniquely defined by the data v_0, and

$$Ai(x) = \frac{1}{\pi}\int_0^\infty e^{ix\xi - \frac{i}{3}\xi^3} d\xi$$

is the Airy function. Note that the estimate of the remainder term in formula (1.5) is not sufficiently exact, since the main term decays as $O\left(t^{-\frac{1}{2}}\right)$ in the domain $x \sim t$ due to the known estimate

$$\left|\theta(x)\,\mathrm{Re}Ai\left(xt^{-\frac{1}{3}}\right)\right| \le C\left\langle xt^{-\frac{1}{3}}\right\rangle^{-\frac{1}{4}}.$$

We could not obtain more exact estimate for the remainder since the method of paper [20] is based on the $\mathbf{L}^2$ - estimate of the operator $\mathcal{P} = \partial_x x + 3t\partial_t$. By the identity $\mathcal{P} - \partial_x\mathcal{J} = 3t\mathcal{L}$ with $\mathcal{J} = x + t\partial_x^2$, $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$, we see that the $\mathbf{L}^2$ - estimate of $\mathcal{P}$ is related with the $\mathbf{L}^2$ - estimate of solutions multiplying by the operator $\partial_x\mathcal{J}$. However to get more accurate estimate for the remainder in asymptotics (1.5) we need the $\mathbf{L}^2$ - estimate of $\mathcal{J}u(t)$. In the present paper we fill this gap by applying the factorization technique, which allows us to obtain a priori estimate of the norm $\|\mathcal{J}u(t)\|_{\mathbf{L}^2}$. We also use the operator $\mathcal{P}$ here in order to avoid the derivative loss in equation (1.3). Another advantage is that we are able here to find the large time asymptotics for the complex solution $u(t)$ of (1.3).

We are now in a position to state our result.

THEOREM 1.1. *Assume that the initial data $v_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,1}$ with a sufficiently small norm $\|v_0\|_{\mathbf{H}^2\cap\mathbf{H}^{1,1}} \le \varepsilon$ and $\int v_0 dx = 0$. Then there exists a unique global solution $e^{\frac{1}{3}it(-i\partial_x)^3}v \in \mathbf{C}\left([0,\infty);\mathbf{H}^2\cap\mathbf{H}^{1,1}\right)$ of the Cauchy problem (1.4) satisfying the time decay estimates*

$$\|v(t)\|_{\mathbf{L}^\infty} + \|\partial_x v(t)\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{2}}.$$

Moreover there exists a unique modified final state $y_F \in \mathbf{L}^\infty$ and a unique real valued function $\Phi_F \in \mathbf{L}^\infty$ such that the asymptotics

$$\begin{aligned}&\partial_x^{j-1}v(t) = t^{-\frac{1}{2}}\theta(x)\,A\left(t,\frac{x}{t}\right)\varkappa^j y_F(\varkappa)\,H(y_F,\Phi_F)\\&+t^{-\frac{1}{2}}\theta(x)\,\overline{A\left(t,\frac{x}{t}\right)}\varkappa^j y_F(-\varkappa)\,\overline{H(y_F,\Phi_F)}\\&+O\left(\varepsilon t^{-\frac{1}{2}-\frac{1}{12}+\gamma}\right)\end{aligned}$$

hold uniformly with respect to $x \in \mathbf{R}$*, where* $j = 1, 2$, $\varkappa = (x/t)^{\frac{1}{2}}, \gamma > 0$,

$$H(y_F, \Phi_F) = e^{\frac{2}{3}it\varkappa^3 - \frac{i\lambda}{2}\varkappa\Phi_F(\varkappa) + \left(-\frac{i\lambda}{2}\varkappa^2\left(\frac{1}{2}|y_F(\varkappa)|^2 + |y_F(-\varkappa)|^2\right)\log t\right)}$$

and $A(t)$ *has the asymptotics*

$$A\left(t, \frac{x}{t}\right) = \frac{1}{\sqrt{2i}}\theta_+(x)|t|^{\frac{1}{6}}\left\langle t^{-\frac{1}{2}}x^{\frac{3}{2}}\right\rangle^{-\frac{1}{6}} + O\left(|t|^{\frac{1}{6}}\left\langle t^{-\frac{1}{2}}x^{\frac{3}{2}}\right\rangle^{-\frac{7}{6}}\right).$$

Theorem 1.1 is a direct consequence of the following theorem by virtue of the relation $\partial_x u = v$.

THEOREM 1.2. *Assume that the initial data* $u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{2,1}$*, with a sufficiently small norm* $\|u_0\|_{\mathbf{H}^3 \cap \mathbf{H}^{2,1}} \le \varepsilon$*. Then there exists a unique global solution* $e^{\frac{1}{3}it(-i\partial_x)^3}u \in \mathbf{C}\left([0,\infty); \mathbf{H}^3 \cap \mathbf{H}^{2,1}\right)$ *of the Cauchy problem (1.3) satisfying the time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{3}},$$
$$\|\partial_x u(t)\|_{\mathbf{L}^\infty} + \left\|\partial_x^2 u(t)\right\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{2}}.$$

Moreover there exists a unique modified final state $y_F \in \mathbf{L}^\infty$ *and a unique real valued function* $\Phi_F \in \mathbf{L}^\infty$ *such that the asymptotics*

$$\begin{aligned}\partial_x^j u(t) &= t^{-\frac{1}{2}}\theta(x) A\left(t, \frac{x}{t}\right)\varkappa^j y_F(\varkappa) H(y_F, \Phi_F) \\ &+ t^{-\frac{1}{2}}\theta(x)\overline{A\left(t, \frac{x}{t}\right)}\varkappa^j y_F(-\varkappa)\overline{H(y_F, \Phi_F)} \\ &+ O\left(\varepsilon t^{-\frac{1}{2} - \frac{1}{12} + \gamma}\right)\end{aligned} \tag{1.6}$$

hold uniformly with respect to $x \in \mathbf{R}$*, where* $j = 1, 2$, $\varkappa = (x/t)^{\frac{1}{2}}, \gamma > 0, H(y_F, \Phi_F)$ *and* $A(t)$ *are the same ones given in Theorem 1.1.*

REMARK 1.1. By the asymptotics of $A(t)$ which will be shown below, we have

$$\begin{aligned} & A\left(t, \frac{x}{t}\right)\left(\frac{x}{t}\right)^{\frac{1}{2}} \\ = \; & \frac{1}{\sqrt{2i}}\theta_+(x)|t|^{\frac{1}{6}}\left\langle t^{-\frac{1}{2}}x^{\frac{3}{2}}\right\rangle^{-\frac{1}{6}}\left(\frac{x}{t}\right)^{\frac{1}{2}} + O\left(|t|^{\frac{1}{6}}\left\langle t^{-\frac{1}{2}}x^{\frac{3}{2}}\right\rangle^{-\frac{7}{6}}\left(\frac{x}{t}\right)^{\frac{1}{2}}\right)\end{aligned}$$

For $x \sim t$ we have

$$|A(t)\varkappa| \le C$$

which implies, that the second term of the right-hand side of (1.6) is a remainder in the region $x \sim t$. If the initial data u_0 are real-valued functions, then $y_F(-\xi) = \overline{y_F(\xi)}$, which implies the asymptotic of solutions is

$$\partial_x u(t) = 2t^{-\frac{1}{2}}\theta(x)\,\mathrm{Re}A(t)\,\varkappa y_F(\varkappa)\,H(y_F,\Phi_F) + O\left(\varepsilon t^{-\frac{1}{2}-\frac{1}{12}+\gamma}\right).$$

The main term of the large time asymptotics is the same as in our previous work [20] and the estimate of the remainder term in (1.6) is more exact (see formula (1.5) and note that the real-valued solutions for equations (1.3) and (1.4) are related by $v(t) = \partial_x u(t)$).

We now introduce the factorization formula for equation (1.3). We have for the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}E\mathcal{F}$, where the multiplication factor $E = e^{-it\Lambda(\xi)}$, $\Lambda(\xi) = \frac{1}{3}\xi^3$. Then we find

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{F}^{-1}E\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it\left(\frac{x}{t}\xi-\Lambda(\xi)\right)}\phi(\xi)\,d\xi \\ &= \mathcal{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi,\end{aligned}$$

where the dilation operator $\mathcal{D}_t\phi = |t|^{-\frac{1}{2}}\phi\left(xt^{-1}\right)$. Note that there are two stationary points in the integral $\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi$ for $x > 0$, which are defined by the two roots $\xi = -\sqrt{x}$ and $\xi = \sqrt{x}$ of the equation $\Lambda'(\xi) = x$. Denote $\theta_-(x) = 1$ for $x \le 0$ and $\theta_-(x) = 0$ for $x > 0$, and $\theta_+(x) = 1-\theta_-(x)$. We define the cut off function $\chi(x) \in \mathbf{C}^2(\mathbf{R})$ such that $\chi(x) = 0$ for $x \le -\frac{1}{3}$, $\chi(x) = 1$ for $x \ge \frac{1}{3}$, and such that $\chi(x) + \chi(-x) \equiv 1$. Then we decompose

$$\begin{aligned}&\mathcal{U}(t)\mathcal{F}^{-1}\phi = \theta_+(x)\,\mathcal{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,\chi\left(\xi x^{-\frac{1}{2}}\right)d\xi \\ &+\theta_+(x)\,\mathcal{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{-it(x\xi-\Lambda(\xi))}\phi(-\xi)\,\chi\left(\xi x^{-\frac{1}{2}}\right)d\xi \\ &+\theta_-(x)\,\mathcal{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi.\end{aligned}$$

Define the operators

$$\begin{aligned}\mathcal{V}(t)\phi &= \theta_+(tx)\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{-itS(x,\xi)}\phi(\xi)\chi\left(\xi x^{-\frac{1}{2}}\right)d\xi,\\ \mathcal{W}(t)\phi &= \theta_-(tx)\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi,\end{aligned}$$

where $S(x,\xi)=\Lambda(\xi)-\Lambda(\sqrt{x})-x(\xi-\sqrt{x})$. Denote $M=e^{it\left(x\sqrt{x}-\Lambda\left(\sqrt{x}\right)\right)}$, then we get

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi=\mathcal{D}_t\left(M\mathcal{V}(t)\phi+\overline{M}\mathcal{V}(-t)\mathcal{D}_{-1}\phi+\mathcal{W}(t)\phi\right).$$

Denote $\mathcal{D}_t^{-1}\phi=|t|^{\frac{1}{2}}\phi(xt)$, also we decompose the inverse operator

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)\phi &= \overline{E}\mathcal{F}\phi=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-it\left(\frac{x}{t}\xi-\Lambda(\xi)\right)}\phi(x)\,dx\\ &= \sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} e^{-it(\xi x-\Lambda(\xi))}\mathcal{D}_t^{-1}\phi(x)\,dx.\end{aligned}$$

Define the new dependent variable $\widehat{\varphi}=\mathcal{F}\mathcal{U}(-t)u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L}=\partial_t\mathcal{F}\mathcal{U}(-t)$, applying the operator $\mathcal{F}\mathcal{U}(-t)$ to equation (1.3) we get

$$\partial_t\widehat{\varphi}=\partial_t\mathcal{F}\mathcal{U}(-t)u=\mathcal{F}\mathcal{U}(-t)\mathcal{L}u=\lambda\mathcal{F}\mathcal{U}(-t)\left(|u_x|^2u_x\right).$$

Then since

$$\begin{aligned}u_x &= \mathcal{U}(t)\partial_x\mathcal{F}^{-1}\widehat{\varphi}=\mathcal{U}(t)\mathcal{F}^{-1}i\xi\widehat{\varphi}\\ &= \mathcal{D}_t\left(M\psi_++\overline{M}\psi_-+r\right),\end{aligned}$$

where $\psi_+=\mathcal{V}(t)i\xi\widehat{\varphi}$, $\psi_-=\mathcal{V}(-t)\mathcal{D}_{-1}i\xi\widehat{\varphi}$, $r=\mathcal{W}(t)i\xi\widehat{\varphi}$, we find the following decomposition

$$\begin{aligned}&\partial_t\widehat{\varphi}=\lambda\mathcal{F}\mathcal{U}(-t)\left(|u_x|^2u_x\right)\\ =&\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}} dxe^{-it(\xi x-\Lambda(\xi))}\mathcal{D}_t^{-1}\left|\mathcal{D}_t\left(M\psi_++\overline{M}\psi_-+r\right)\right|^2\\ &\times\mathcal{D}_t\left(M\psi_++\overline{M}\psi_-+r\right)\\ =&\sqrt{\frac{1}{2\pi|t|}}\int_{\mathbf{R}} dxe^{-it(\xi x-\Lambda(\xi))}\left|M\psi_++\overline{M}\psi_-+r\right|^2\\ &\times\left(M\psi_++\overline{M}\psi_-+r\right).\end{aligned}$$

Since

$$\begin{aligned} & \left|M\psi_+ + \overline{M}\psi_- + r\right|^2 \left(M\psi_+ + \overline{M}\psi_- + r\right) \\ = \;& \theta_+ (x) \left|M\psi_+ + \overline{M}\psi_-\right|^2 \left(M\psi_+ + \overline{M}\psi_-\right) + \theta_- (x) \left|r\right|^2 r \end{aligned}$$

we get

$$\begin{aligned} \partial_t \widehat{\varphi} &= \frac{\lambda}{\sqrt{2\pi \left|t\right|}} \int_0^\infty dx e^{-it(\xi x - \Lambda(\xi))} \sum_{j=1}^4 I_j \\ & + \frac{\lambda}{\sqrt{2\pi \left|t\right|}} \int_{-\infty}^0 dx e^{-it(\xi x - \Lambda(\xi))} \left|r\right|^2 r. \end{aligned}$$

where

$$\begin{aligned} I_1 &= M^3 \overline{\psi_-} \psi_+^2, I_2 = M \left(\left|\psi_+\right|^2 + 2\left|\psi_-\right|^2\right) \psi_+, \\ I_3 &= \overline{M} \left(\left|\psi_-\right|^2 + 2\left|\psi_+\right|^2\right) \psi_-, I_4 = \overline{M}^3 \overline{\psi_+} \psi_-^2. \end{aligned}$$

Define the operators

$$\begin{aligned} \mathcal{V}^* (t) \phi &= \sqrt{\frac{\left|t\right|}{2\pi}} \int_0^\infty e^{itS(x,\xi)} \phi(x)\, dx, \\ \mathcal{W}^* (t) \phi &= \sqrt{\frac{\left|t\right|}{2\pi}} \int_{-\infty}^0 e^{-it(\xi x - \Lambda(\xi))} \phi(x)\, dx. \end{aligned}$$

Thus we get the equation

$$\partial_t \widehat{\varphi} = \lambda \left|t\right|^{-1} (N_1 + N_2) + \lambda \left|t\right|^{-1} e^{\frac{8}{9} it\Lambda(\xi)} (N_3 + N_4) + \lambda \left|t\right|^{-1} N_5, \tag{1.7}$$

where

$$\begin{aligned} & N_1 = \mathcal{D}_{-1} \mathcal{V}^* (-t) \left(\left|\psi_-\right|^2 + 2\left|\psi_+\right|^2\right) \psi_-, \\ & N_2 = \mathcal{V}^* (t) \left(\left|\psi_+\right|^2 + 2\left|\psi_-\right|^2\right) \psi_+, \\ & N_3 = \mathcal{D}_{-3} \mathcal{V}^* (-3t) \left(\overline{\psi_+} \psi_-^2\right),\ N_4 = \mathcal{D}_3 \mathcal{V}^* (3t) \left(\overline{\psi_-} \psi_+^2\right), \\ & N_5 = \mathcal{W}^* \left|r\right|^2 r. \end{aligned} \tag{1.8}$$

In order to get an optimal time decay estimate of solutions in the uniform norm we need the estimate of $\widehat{\varphi}$ in $\mathbf{L}^\infty$ - norm and the estimate for the derivative $\partial_\xi \widehat{\varphi}$ in $\mathbf{L}^2$ - norm. Note that $\|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} = \|\mathcal{J}u\|_{\mathbf{L}^2}$ and for the operator $\mathcal{J} = \mathcal{U}(t)\, x \mathcal{U}(-t)$ we have

$$\begin{aligned} \mathcal{J} &= \mathcal{U}(t)\, x\mathcal{U}(-t) = \mathcal{F}^{-1} e^{-it\Lambda(\xi)} i\partial_\xi e^{it\Lambda(\xi)} \mathcal{F} \\ &= \mathcal{F}^{-1}\left(i\partial_\xi - t\Lambda'(\xi)\right)\mathcal{F} = x - t\Lambda'(-i\partial_x) = x + t\partial_x^2. \end{aligned}$$

The operator $\mathcal{J}$ is a generalization of $\mathcal{J}_s = e^{\frac{it}{2}\partial_x^2} x e^{-\frac{it}{2}\partial_x^2} = x + it\partial_x$, and $\mathcal{J}_s$ was widely used in the study of the large time behavior of solutions to the nonlinear Schrödinger equations (see [23]).

Our main task is to estimate each term of the right-hand side of equation (1.7) in $\mathbf{L}^\infty$ and $\mathbf{H}^1$ norms. We organize the rest of our paper as follows. In Section 2, we state the uniform estimates for the decomposition operators $\mathcal{V}$ and $\mathcal{W}$ (Lemma 2.1). The uniform estimates of $\mathcal{V}^*(t)$ and $\mathcal{W}^*(t)$ are obtained in Lemma 2.2. We prove $\mathbf{L}^2$ - estimates of the derivatives of $\mathcal{V}$ (Lemma 3.4) and $\mathcal{W}$ (Lemma 3.6) in Section 3. Lemmas 3.1 - 3.3 are prepared for proving Lemma 3.4, Lemma 3.5 is used in the proof of Lemma 3.6. The $\mathbf{L}^2$ - estimates of the derivatives of $\mathcal{V}^*$ and $\mathcal{W}^*$ (Lemma 4.1) are obtained in Section 4. In Section 5 we find the main term of the large time asymptotic behavior of the nonlinearity in equation (1.7) (see Lemma 5.1). In Section 6 we obtain a priori estimates of solutions to (1.3). Finally Section 7 is devoted to the proof of Theorem 1.2.

2. Estimates in the Uniform Norm

Define the kernel

$$A(t,x) = \theta_+(x)\sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{-itS(x,\xi)} \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right) d\xi,$$

where the cut off function $\widetilde{\chi}(z) \in \mathbf{C}^2(\mathbf{R})$ is such that $\widetilde{\chi}(z) = 1$ for $\frac{2}{3} \le z \le \frac{3}{2}$, and $\widetilde{\chi}(z) = 0$ for $z \le \frac{1}{3}$ or $z \ge 3$. Changing the variable of integration $\xi = y\sqrt{x}$ we get $S(x,\xi) = x^{\frac{3}{2}} G(y)$, where $G(y) = \Lambda(y) - y + \frac{2}{3}$ and

$$A(t,x) = \theta_+(x)\, |x|^{\frac{1}{2}} \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{-itx^{\frac{3}{2}} G(y)} \widetilde{\chi}(y)\, dy.$$

To compute the asymptotics of the function $A(t,x)$ for large $\left|tx^{\frac{3}{2}}\right|$ we apply the stationary phase method (see [15], p. 163). We have the asymptotics

$$\begin{aligned} & \int_{\mathbf{R}} e^{-izG(y)} f(y)\,dy \\ (2.1) \qquad &= e^{-izG(y_0)} f(y_0) \sqrt{\frac{2\pi}{|zG''(y_0)|}} e^{-i\frac{\pi}{4}\operatorname{sgn} G''(y_0) z} + O\left(|z|^{-\frac{3}{2}}\right) \end{aligned}$$

for $|z| \to \infty$, where the stationary point y_0 is defined by $G'(y_0) = 0$. Since $G'(y) = y^2 - 1$, we find $y_0 = 1$ and by virtue of formula (2.1) we get

$$A(t,x) = \frac{1}{\sqrt{2}} \theta_+(x) |t|^{\frac{1}{6}} \left\langle tx^{\frac{3}{2}} \right\rangle^{-\frac{1}{6}} e^{-i\frac{\pi}{4}\frac{t}{|t|}} + O\left(|t|^{\frac{1}{6}} \left\langle tx^{\frac{3}{2}} \right\rangle^{-\frac{7}{6}}\right)$$

for $\left|tx^{\frac{3}{2}}\right| \to \infty$, $x > 0$. In particular we have

$$|A(t,x)| \le C|t|^{\frac{1}{6}} \left\langle tx^{\frac{3}{2}} \right\rangle^{-\frac{1}{6}}$$

for all $t \in \mathbf{R}$, $x > 0$.

In the next lemma we obtain the estimates of the operators

$$\begin{aligned} \mathcal{V}(t)\phi &= \theta_+(x) \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{-itS(x,\xi)} \phi(\xi) \chi\left(\xi x^{-\frac{1}{2}}\right) d\xi, \\ \mathcal{W}(t)\phi &= \theta_-(x) \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{it(x\xi - \Lambda(\xi))} \phi(\xi)\, d\xi, \end{aligned}$$

where $S(x,\xi) = \Lambda(\xi) - \Lambda(\sqrt{x}) - x(\xi - \sqrt{x})$.

LEMMA 2.1. *Let $j = 0, 1, 2$, $0 \le \alpha \le \frac{11}{4} - j$. Then the following estimates are valid for all $|t| \ge 1$*

$$\begin{aligned} & \left\| x^{\frac{\alpha}{2}} \left(\mathcal{V}(t) \xi^j \phi - A(t) x^{\frac{j}{2}} \phi\left(x^{\frac{1}{2}}\right) \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ \le \;& C|t|^{\frac{1}{6} - \frac{1}{3}\min(j+\alpha,1)} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + C|t|^{-\frac{1}{3}\min\left(j+\alpha,\frac{3}{4}\right)} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \end{aligned}$$

and

$$\begin{aligned} \left\| |x|^{\frac{\alpha}{2}} \mathcal{W}(t) \xi^j \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \le\;& C|t|^{\frac{1}{6} - \frac{1}{3}\min\left(j+\alpha,\frac{3}{2}\right)} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \\ & + C|t|^{-\frac{1}{3}\min\left(j+\alpha,\frac{3}{2}\right)} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}. \end{aligned}$$

Remark 2.1. In particular, we have the following estimates

$$\begin{aligned}
&\left\| x^{\frac{1}{4}} \langle x\rangle^{\frac{3}{4}} \mathcal{V}(t)\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\| \langle x\rangle^{\frac{1}{2}} \mathcal{V}(t)\xi\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
&+ \left\| \langle x\rangle^{\frac{1}{4}} \mathcal{V}(t)\xi^2\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
&+ |t|^{\frac{1}{6}} \left\| \left\langle xt^{\frac{2}{3}} \right\rangle^{\frac{1}{4}} \mathcal{W}(t)\xi\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} + \left\| \left\langle xt^{\frac{2}{3}} \right\rangle^{\frac{1}{2}} \mathcal{W}(t)\xi\widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \\
\leq\ & C \left\| \langle\xi\rangle^2 \phi \right\|_{\mathbf{L}^\infty} + C|t|^{-\frac{1}{6}} \left\| \langle\xi\rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Since we have

$$\begin{aligned}
&\left\| x^{\frac{1}{4}} \mathcal{V}(t)\xi^2\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
\leq\ & \left\| x^{\frac{1}{4}} A(t)x\phi\left(x^{\frac{1}{2}}\right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + C|t|^{-\frac{1}{6}} \left\| \langle\xi\rangle^2\phi \right\|_{\mathbf{L}^\infty} + C|t|^{-\frac{1}{4}} \left\| \langle\xi\rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}
\end{aligned}$$

and by the estimate of $A(t)$

$$\begin{aligned}
&\left\| x^{\frac{1}{4}} A(t)x\phi\left(x^{\frac{1}{2}}\right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
\leq\ & C \left\| x^{\frac{1}{4}} |t|^{\frac{1}{6}} \left\langle tx^{\frac{3}{2}} \right\rangle^{-\frac{1}{6}} x\phi\left(x^{\frac{1}{2}}\right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq C \left\| \langle\xi\rangle^2\phi \right\|_{\mathbf{L}^\infty}
\end{aligned}$$

we have the desired estimate.

Proof. We have

$$\begin{aligned}
&\mathcal{V}(t)\xi^j\phi - A(t)x^{\frac{j}{2}}\phi\left(x^{\frac{1}{2}}\right) \\
=\ & \sqrt{\frac{|t|}{2\pi}} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} e^{-itS(x,\xi)} \left(\xi^j\phi(\xi) - x^{\frac{j}{2}}\phi\left(x^{\frac{1}{2}}\right)\right) \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right) d\xi \\
&+ \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{-itS(x,\xi)}\phi(\xi)\chi_1\left(\xi x^{-\frac{1}{2}}\right)\xi^j d\xi = I_1 + I_2
\end{aligned}$$

for $x > 0$, $|t| \geq 1$, where $\chi_1(z) = \chi(z) - \widetilde{\chi}(z)$. Integrating by parts via the identity

$$e^{-itS(x,\xi)} = H_1\partial_\xi\left(\left(\xi - x^{\frac{1}{2}}\right)e^{-itS(x,\xi)}\right) \tag{2.2}$$

with $H_1 = \left(1 - it\left(\xi - x^{\frac{1}{2}}\right)\partial_\xi S(x,\xi)\right)^{-1}$ we get

$$\begin{aligned} I_1 &= C|t|^{\frac{1}{2}} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} e^{-itS(x,\xi)} \left(\xi^j \phi(\xi) - x^{\frac{j}{2}} \phi\left(x^{\frac{1}{2}}\right)\right) \\ &\quad \times \left(\xi - x^{\frac{1}{2}}\right) \partial_\xi \left(H_1 \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\right) d\xi \\ &\quad + C|t|^{\frac{1}{2}} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} e^{-itS(x,\xi)} \left(\xi - x^{\frac{1}{2}}\right) H_1 \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right) \partial_\xi \left(\xi^j \phi(\xi)\right) d\xi. \end{aligned}$$

Since $S(x,\xi) = \Lambda(\xi) - \Lambda\left(x^{\frac{1}{2}}\right) - x\left(\xi - x^{\frac{1}{2}}\right)$, we find $\partial_\xi S(x,\xi) = \Lambda'(\xi) - x$ and $\partial_\xi^2 S(x,\xi) = \Lambda''(\xi)$. Using the estimates

$$\left|\xi^j \phi(\xi) - x^{\frac{j}{2}} \phi\left(x^{\frac{1}{2}}\right)\right| \le C x^{\frac{j}{2}} \left|\xi - x^{\frac{1}{2}}\right|^{\frac{1}{2}} \|\partial_\xi \phi\|_{\mathbf{L}^2}$$

in the domain $\frac{1}{3}x^{\frac{1}{2}} \le \xi \le 3x^{\frac{1}{2}}$,

$$\left|\left(\xi - x^{\frac{1}{2}}\right) H_1 \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\right| \le \frac{C\left|\xi - x^{\frac{1}{2}}\right|}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2}$$

and

$$\left|\left(\xi - x^{\frac{1}{2}}\right) \partial_\xi \left(H_1 \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\right)\right| \le \frac{C}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2},$$

by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left|x^{\frac{\alpha}{2}} I_1\right| &\le C|t|^{\frac{1}{2}} x^{\frac{\alpha+j-1}{2}} \langle x\rangle^{-1} \left\|\langle\xi\rangle^2 \phi\right\|_{\mathbf{L}^\infty} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} \frac{\left|\xi - x^{\frac{1}{2}}\right| d\xi}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2} \\ &+ C|t|^{\frac{1}{2}} x^{\frac{\alpha+j}{2}} \langle x\rangle^{-1} \left\|\langle\xi\rangle^2 \phi_\xi\right\|_{\mathbf{L}^2} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} \frac{\left|\xi - x^{\frac{1}{2}}\right|^{\frac{1}{2}} d\xi}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2} \\ &+ C|t|^{\frac{1}{2}} x^{\frac{\alpha+j}{2}} \langle x\rangle^{-1} \left\|\langle\xi\rangle^2 \phi_\xi\right\|_{\mathbf{L}^2} \left(\int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} \frac{\left(\xi - x^{\frac{1}{2}}\right)^2 d\xi}{\left(1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2\right)^2}\right)^{\frac{1}{2}}. \end{aligned}$$

Then changing $\xi = x^{\frac{1}{2}} y$ we obtain

$$\left| x^{\frac{\alpha}{2}} I_1 \right| \leq C |t|^{\frac{1}{2}} x^{\frac{\alpha+j+1}{2}} \langle x \rangle^{-1} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \int_{\frac{1}{3}}^{3} \frac{|y-1| \, dy}{1 + |t| x^{\frac{3}{2}} (y-1)^2}$$

$$+ C |t|^{\frac{1}{2}} x^{\frac{\alpha+j+1}{2}+\frac{1}{4}} \langle x \rangle^{-1} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \int_{\frac{1}{3}}^{3} \frac{|y-1|^{\frac{1}{2}} \, dy}{1 + |t| x^{\frac{3}{2}} (y-1)^2}$$

$$+ C |t|^{\frac{1}{2}} x^{\frac{\alpha+j+1}{2}+\frac{1}{4}} \langle x \rangle^{-1} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \left(\int_{\frac{1}{3}}^{3} \frac{(y-1)^2 \, dy}{\left(1 + |t| x^{\frac{3}{2}} (y-1)^2\right)^2} \right)^{\frac{1}{2}}.$$

Since

$$\int_0^2 \frac{y dy}{1 + zy^2} \leq C \langle z \rangle^{-1} \log \langle z \rangle$$

and

$$\int_0^2 \frac{y^{\frac{1}{2}} dy}{1 + zy^2} + \left(\int_0^2 \frac{y^2 dy}{(1 + zy^2)^2} \right)^{\frac{1}{2}}$$

$$\leq \quad C z^{-\frac{3}{4}} \int_0^{2z^{\frac{1}{2}}} y^{\frac{1}{2}} \langle y \rangle^{-2} dy + C z^{-\frac{3}{4}} \left(\int_0^{2z^{\frac{1}{2}}} y^2 \langle y \rangle^{-4} dy \right)^{\frac{1}{2}} \leq C \langle z \rangle^{-\frac{3}{4}}$$

for all $z > 0$, we get

$$\left| x^{\frac{\alpha}{2}} I_1 \right| \leq C |t|^{\frac{1}{2}} x^{\frac{\alpha+j+1}{2}} \langle x \rangle^{-1} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \left\langle t x^{\frac{3}{2}} \right\rangle^{-1} \log \left\langle t x^{\frac{3}{2}} \right\rangle$$

$$+ C |t|^{\frac{1}{2}} x^{\frac{\alpha+j+1}{2}+\frac{1}{4}} \langle x \rangle^{-1} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \left\langle t x^{\frac{3}{2}} \right\rangle^{-\frac{3}{4}}$$

$$\leq \quad C |t|^{\frac{1}{6} - \frac{1}{3} \min(j+\alpha, 1)} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + C |t|^{-\frac{1}{3} \min\left(j+\alpha, \frac{3}{4}\right)} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}$$

for all $x > 0$, $|t| \geq 1$.

To estimate I_2 we integrate by parts via the identity

$$e^{-itS(x,\xi)} = H_2 \partial_\xi \left(\xi e^{-itS(x,\xi)} \right) \tag{2.3}$$

with $H_2 = \left(1 + it\xi \left(x - \xi^2\right)\right)^{-1}$, we find

$$I_2 = C |t|^{\frac{1}{2}} \int_{\mathbf{R}} e^{-itS(x,\xi)} \phi(\xi) \xi \partial_\xi \left(H_2 \chi_1 \left(\xi x^{-\frac{1}{2}} \right) \xi^j \right) d\xi$$

$$+ C |t|^{\frac{1}{2}} \int_{\mathbf{R}} e^{-itS(x,\xi)} H_2 \chi_1 \left(\xi x^{-\frac{1}{2}} \right) \xi^{j+1} \phi_\xi(\xi) \, d\xi.$$

We find the estimates

$$\left|H_2\chi_1\left(\xi x^{-\frac{1}{2}}\right)\xi^{j+1}\right| \le C\left|\xi\right|^{j+1}\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^{-1}$$

and

$$\left|\xi\partial_\xi\left(H_2\chi_1\left(\xi x^{-\frac{1}{2}}\right)\xi^{j}\right)\right| \le C\left|\xi\right|^{j}\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^{-1}$$

in the domain $|\xi| < \frac{2}{3}x^{\frac{1}{2}}$, or $\xi > \frac{3}{2}x^{\frac{1}{2}}$. Hence we obtain

$$\begin{aligned}\left|x^{\frac{\alpha}{2}}I_2\right| &\le C\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}\int_{\mathbf{R}}\frac{|t|^{\frac{1}{2}}|x|^{\frac{\alpha}{2}}|\xi|^j\langle\xi\rangle^{-2}d\xi}{1+|t|\,|\xi|\left(\xi^2+|x|\right)}\\ &+C\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\left(\int_{\mathbf{R}}\frac{|t|\,|x|^{\alpha}|\xi|^{2+2j}\langle\xi\rangle^{-4}d\xi}{\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Next by choosing $\nu = \min(j,1)$ we have

$$\begin{aligned}&\int_{\mathbf{R}}\frac{|t|^{\frac{1}{2}}|x|^{\frac{\alpha}{2}}|\xi|^j\langle\xi\rangle^{-2}d\xi}{1+|t|\,|\xi|\left(\xi^2+|x|\right)} \le C\int_{\mathbf{R}}\frac{|t|^{\frac{1}{2}}|x|^{\frac{\alpha}{2}}|\xi|^{\nu}d\xi}{1+|t|\,|\xi|\left(\xi^2+|x|\right)}\\ &= C|t|^{\frac{1}{2}}|x|^{\frac{\alpha+\nu+1}{2}}\left(\int_0^1\frac{|y|^{\nu}dy}{1+|t|\,|x|^{\frac{3}{2}}y}+\int_1^\infty\frac{|y|^{\nu}dy}{1+|t|\,|x|^{\frac{3}{2}}y^3}\right)\\ &\le C|t|^{\frac{1}{6}-\frac{1}{3}\min(j+\alpha,1)}\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbf{R}}\frac{|t|\,|x|^{\alpha}|\xi|^{2+2j}\langle\xi\rangle^{-4}d\xi}{\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^2} \le C\int_{\mathbf{R}}\frac{|t|\,|x|^{\alpha}|\xi|^{2+2\nu}d\xi}{\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^2}\\ &= C|t|\,|x|^{\alpha+\nu+\frac{3}{2}}\left(\int_0^1\frac{|y|^{2+2\nu}dy}{\left(1+|t|\,|x|^{\frac{3}{2}}y\right)^2}+\int_1^\infty\frac{|y|^{2+2\nu}dy}{\left(1+|t|\,|x|^{\frac{3}{2}}y^3\right)^2}\right)\\ &\le C|t|^{-\frac{2}{3}\min\left(j+\alpha,\frac{3}{4}\right)}.\end{aligned}$$

Therefore we find the estimate

$$\begin{aligned}\left|x^{\frac{\alpha}{2}}I_2\right| &\le C|t|^{\frac{1}{6}-\frac{1}{3}\min(j+\alpha,1)}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}\\ &+C|t|^{-\frac{1}{3}\min\left(j+\alpha,\frac{3}{4}\right)}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}.\end{aligned}$$

Next we estimate the operator $\mathcal{W}(t)\,\xi^j\phi$. Integrating by parts via (2.3), we find

$$\mathcal{W}(t)\,\xi^j\phi = C\,|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,\xi\partial_\xi\left(H_2\xi^j\right)d\xi$$
$$+C\,|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}H_2\xi^{j+1}\phi_\xi(\xi)\,d\xi$$

for $x\le 0$.The estimates

$$\left|H_2\xi^{j+1}\right|\le C\,|\xi|^{1+j}\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^{-1}$$

and

$$\left|\xi\partial_\xi\left(H_2\xi^j\right)\right|\le C\,|\xi|^j\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^{-1}$$

hold for $x\le 0$. Hence we obtain

$$\left||x|^{\frac{\alpha}{2}}\,\mathcal{W}(t)\,\xi^j\phi\right|\le C\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}\int_{\mathbf{R}}\frac{|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}}\,|\xi|^j\,\langle\xi\rangle^{-2}\,d\xi}{1+|t|\,|\xi|\,(\xi^2+|x|)}$$
$$+C\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\left(\int_{\mathbf{R}}\frac{|t|\,|x|^{\alpha}\,|\xi|^{2+2j}\,\langle\xi\rangle^{-4}\,d\xi}{(1+|t|\,|\xi|\,(\xi^2+|x|))^2}\right)^{\frac{1}{2}}.$$

Therefore as above we have

$$\begin{aligned}\left||x|^{\frac{\alpha}{2}}\,\mathcal{W}(t)\,\xi^j\phi\right| &\le C\,|t|^{\frac{1}{6}-\min\left(\frac{j+\alpha}{3},\frac{1}{4}\right)}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}\\ &\quad+C\,|t|^{-\min\left(\frac{j+\alpha}{3},\frac{1}{4}\right)}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}.\end{aligned}$$

Lemma 2.1 is proved. □

Next we define the kernel

$$\widetilde{A}(t,\xi)=\theta_+(\xi)\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{itS(x,\xi)}\chi\left(x\xi^{-2}\right)dx.$$

Changing the variable of integration $x=\xi^2y$, we get $S(x,\xi)=\xi^3\widetilde{G}(y)$, where $\widetilde{G}(y)=\frac{2}{3}y^{\frac{3}{2}}-y+\frac{1}{3}$ and

$$\widetilde{A}(t,\xi)=\xi^2\theta_+(\xi)\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{it\xi^3\widetilde{G}(y)}\chi(y)\,dy.$$

To compute the asymptotics of the function $\widetilde{A}(t,\xi)$ for large $\left|t\xi^3\right|$ we apply the formula (2.1)

$$\widetilde{A}(t,\xi) = \sqrt{2\xi} e^{i\frac{\pi}{4}\frac{t}{|t|}} + O\left(\xi^{\frac{1}{2}} \left\langle t\xi^3\right\rangle^{-1}\right)$$

as $\left|t\xi^3\right| \to \infty$, $\xi > 0$. In particular we have the estimate $\left|\widetilde{A}(t,\xi)\right| \le C\langle\xi\rangle^{\frac{1}{2}}$ for all $t \in \mathbf{R}$, $\xi > 0$.

For the operator

$$\mathcal{W}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}} \int_{-\infty}^{0} e^{-it(\xi x - \Lambda(\xi))} \phi(x)\, dx$$

we will use the estimate

$$\|\mathcal{W}^*\phi\|_{\mathbf{L}^\infty} \le C|t|^{\frac{1}{2}} \left\| \int_{-\infty}^{0} e^{-it\xi x}\phi(x)\, dx \right\|_{\mathbf{L}^\infty} \le C|t|^{\frac{1}{2}} \|\phi\|_{\mathbf{L}^1(\mathbf{R}_-)}.$$

In the next lemma we estimate the operator

$$\mathcal{V}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}} \int_{0}^{\infty} e^{itS(x,\xi)} \phi(x)\, dx.$$

LEMMA 2.2. *Let* $\alpha \in \left[\frac{3}{8}, \frac{3}{4}\right]$. *Then the estimate*

$$\left\|\mathcal{V}^*(t)\phi - \widetilde{A}(t)\phi\left(\xi^2\right)\right\|_{\mathbf{L}^\infty} \le C|t|^{-\frac{1}{6}} \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + C|t|^{\frac{2}{3}\alpha - \frac{1}{2}} \|x^\alpha \phi_x\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

is true for all $|t| \ge 1$.

PROOF. We have

$$\begin{aligned}
& \mathcal{V}^*(t)\phi - \widetilde{A}(t)\phi\left(\xi^2\right) \\
= \; & \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{itS(x,\xi)} \left(\phi(x) - \phi\left(\xi^2\right)\right) \theta_+(\xi)\, \widetilde{\chi}\left(x\xi^{-2}\right) dx \\
& + \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{itS(x,\xi)} \phi(x) \left(1 - \theta_+(\xi)\, \widetilde{\chi}\left(x\xi^{-2}\right)\right) dx \\
= \; & I_1 + I_2.
\end{aligned}$$

In the first integral I_1 using the identity

$$e^{itS(x,\xi)} = H_3 \partial_x \left(\left(x - \xi^2 \right) e^{itS(x,\xi)} \right)$$

with $H_3 = \left(1 + it \left(x - \xi^2 \right) \partial_x S \left(x, \xi \right) \right)^{-1}$, we integrate by parts

$$I_1 = C \left| t \right|^{\frac{1}{2}} \theta_+ \left(\xi \right) \int_0^\infty e^{itS(x,\xi)} \left(\phi \left(x \right) - \phi \left(\xi^2 \right) \right) \left(x - \xi^2 \right) \partial_x \left(H_3 \widetilde{\chi} \left(x \xi^{-2} \right) \right) dx$$
$$+ C \left| t \right|^{\frac{1}{2}} \theta_+ \left(\xi \right) \int_0^\infty e^{itS(x,\xi)} \left(x - \xi^2 \right) H_3 \widetilde{\chi} \left(x \xi^{-2} \right) \phi_x \left(x \right) dx.$$

Then using the estimates

$$\left| \phi \left(x \right) - \phi \left(\xi^2 \right) \right| \leq \left| \int_x^{\xi^2} x^a x^{-\alpha} \phi_x dx \right| \leq C \left| \xi \right|^{\frac{1}{2} - 2\alpha} \left| x^{\frac{1}{2}} - \xi \right|^{\frac{1}{2}} \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)},$$

$$\left| \left(x - \xi^2 \right) H_3 \widetilde{\chi} \left(x \xi^{-2} \right) \right| \leq \frac{C \left| \xi \right| \left| x^{\frac{1}{2}} - \xi \right|}{1 + \left| t \right| \left| \xi \right| \left(x^{\frac{1}{2}} - \xi \right)^2}$$

and

$$\left| \left(x - \xi^2 \right) \partial_x \left(H_3 \widetilde{\chi} \left(x \xi^{-2} \right) \right) \right| \leq \frac{C}{1 + \left| t \right| \left| \xi \right| \left(x^{\frac{1}{2}} - \xi \right)^2}$$

in the domain $\frac{1}{3} \xi^2 \leq x \leq 3 \xi^2$, we find

$$\begin{aligned}
& \left| I_1 \right| \leq C \left| t \right|^{\frac{1}{2}} \theta_+ \left(\xi \right) \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \int_{\frac{1}{3}\xi^2}^{3\xi^2} \frac{\left| \xi \right|^{\frac{1}{2} - 2\alpha} \left| x^{\frac{1}{2}} - \xi \right|^{\frac{1}{2}} dx}{1 + \left| t \right| \xi \left(x^{\frac{1}{2}} - \xi \right)^2} \\
& + C \left| t \right|^{\frac{1}{2}} \theta_+ \left(\xi \right) \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_{\frac{1}{3}\xi^2}^{3\xi^2} \frac{\left| \xi \right|^{2 - 4\alpha} \left(x^{\frac{1}{2}} - \xi \right)^2 dx}{\left(1 + \left| t \right| \xi \left(x^{\frac{1}{2}} - \xi \right)^2 \right)^2} \right)^{\frac{1}{2}} \\
\leq \;& C \left| t \right|^{\frac{1}{2}} \xi^{3 - 2\alpha} \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \times \left(\int_0^2 \frac{y^{\frac{1}{2}} dy}{1 + \left| t \right| \xi^3 y^2} + \left(\int_0^2 \frac{y^2 dy}{\left(1 + \left| t \right| \xi^3 y^2 \right)^2} \right)^{\frac{1}{2}} \right) \\
\leq \;& C \left| t \right|^{\frac{1}{2}} \xi^{3 - 2\alpha} \left\langle t \xi^3 \right\rangle^{-\frac{3}{4}} \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left| t \right|^{\frac{2}{3}\alpha - \frac{1}{2}} \left\| x^\alpha \phi_x \right\|_{\mathbf{L}^2(\mathbf{R}_+)}
\end{aligned}$$

for $\alpha \in \left[\frac{3}{8}, \frac{3}{2}\right]$.

In the second integral I_2, using the identity

$$e^{itS(x,\xi)} = H_4 \partial_x \left(x e^{itS(x,\xi)} \right)$$

with $H_4 = \left(1 + itx\partial_x S\left(x, \xi\right)\right)^{-1}$ we integrate by parts

$$\begin{aligned} I_2 &= C \left|t\right|^{\frac{1}{2}} \int_0^\infty e^{it\widetilde{S}(x,\xi)} \phi\left(x\right) x \partial_x \left(H_4 \left(1 - \theta_+\left(\xi\right) \widetilde{\chi}\left(x\xi^{-2}\right)\right)\right) dx \\ &+ C \left|t\right|^{\frac{1}{2}} \int_0^\infty e^{it\widetilde{S}(x,\xi)} x H_4 \left(1 - \theta_+\left(\xi\right) \widetilde{\chi}\left(x\xi^{-2}\right)\right) \phi_x\left(x\right) dx. \end{aligned}$$

Then using the estimates

$$\left| H_4 \left(1 - \theta_+\left(\xi\right) \widetilde{\chi}\left(x\xi^{-2}\right)\right)\right| \leq \frac{C}{1 + \left|t\right| x \left(x^{\frac{1}{2}} + \left|\xi\right| \right)}$$

and

$$\left| x\partial_x \left(H_4 \left(1 - \theta_+\left(\xi\right) \widetilde{\chi}\left(x\xi^{-2}\right)\right)\right)\right| \leq \frac{C}{1 + \left|t\right| x \left(x^{\frac{1}{2}} + \left|\xi\right| \right)}$$

for $0 < x < \frac{2}{3}\xi^2$, $\xi > 0$, or $x > \frac{3}{2}\xi^2$, $\xi > 0$, or $\xi < 0$, we get

$$\begin{aligned} \left|I_2\right| &\leq C \left|t\right|^{\frac{1}{2}} \left\|\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_0^\infty \frac{dx}{1 + \left|t\right| x \left(x^{\frac{1}{2}} + \left|\xi\right| \right)} \\ &+ C \left|t\right|^{\frac{1}{2}} \left\|x^\alpha \phi_x\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_0^\infty \frac{x^{2-2\alpha} dx}{\left(1 + \left|t\right| x \left(x^{\frac{1}{2}} + \xi \right)\right)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Changing the variable $x = \xi^2 y$, we obtain

$$\begin{aligned} & \int_0^\infty \frac{dx}{1 + \left|t\right| x \left(x^{\frac{1}{2}} + \left|\xi\right| \right)} \\ \leq \; & C\xi^2 \int_0^1 \frac{dy}{1 + \left|t\right| \xi^3 y} + C\xi^2 \int_1^\infty \frac{dy}{1 + \left|t\right| \xi^3 y^{\frac{3}{2}}} \\ \leq \; & C\xi^2 \left(\left|t\right| \xi^3\right)^{-1} \int_0^{\left|t\right|\xi^3} \frac{dy}{1+y} + C\xi^2 \left(\left|t\right| \xi^3\right)^{-\frac{2}{3}} \int_{\left(\left|t\right|\xi^3\right)^{\frac{2}{3}}}^\infty \frac{dy}{1 + y^{\frac{3}{2}}} \\ \leq \; & C\xi^2 \left\langle t\xi^3 \right\rangle^{-1} \log \left\langle t\xi^3 \right\rangle + C \left|t\right|^{-\frac{2}{3}} \left\langle t\xi^3 \right\rangle^{-\frac{1}{3}} \\ \leq \; & C \left|t\right|^{-\frac{2}{3}} \left\langle t\xi^3 \right\rangle^{-\frac{1}{3}} \log \left\langle t\xi^3 \right\rangle \end{aligned}$$

and

$$
\begin{aligned}
&\int_0^\infty \frac{x^{2-2\alpha}dx}{\left(1+tx\left(x^{\frac{1}{2}}+\xi\right)\right)^2} \\
\leq\ & C\xi^{6-4\alpha}\left(\int_0^1 \frac{y^{2-2\alpha}dy}{\left(1+|t|\,\xi^3 y\right)^2}+\int_1^\infty \frac{y^{2-2\alpha}dy}{\left(1+|t|\,\xi^3 y^{\frac{3}{2}}\right)^2}\right) \\
\leq\ & C\xi^{6-4\alpha}\left\langle |t|\,\xi^3\right\rangle^{2\alpha-3}\int_0^{\left\langle |t|\xi^3\right\rangle}\frac{y^{2-2\alpha}dy}{(1+y)^2} \\
& +C\xi^{6-4\alpha}\left(|t|\,\xi^3\right)^{\frac{2}{3}(2\alpha-3)}\int_{(|t|\xi^3)^{\frac{2}{3}}}^\infty \frac{y^{2-2\alpha}dy}{\left(1+y^{\frac{3}{2}}\right)^2} \\
\leq\ & C\xi^{6-4\alpha}\left\langle |t|\,\xi^3\right\rangle^{2\alpha-3+\max(0,1-2\alpha)}+C\,|t|^{\frac{4}{3}\alpha-2}\left\langle t\xi^3\right\rangle^{-\frac{4}{3}\alpha}\leq C\,|t|^{\frac{4}{3}\alpha-2}
\end{aligned}
$$

if $\alpha\in\left(0,\frac{3}{2}\right)$. Therefore

$$
|I_2|\leq C\,|t|^{-\frac{1}{6}}\,\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}+C\,|t|^{\frac{2}{3}\alpha-\frac{1}{2}}\,\|x^\alpha\phi_x\|_{\mathbf{L}^2(\mathbf{R}_+)}\,.
$$

Lemma 2.2 is proved. □

3. Estimates for Derivatives of $\mathcal{V}$ and $\mathcal{W}$

In the next lemma we estimate an auxiliary integral operator

$$
\mathcal{I}_1\phi=|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{-itS(x,\xi)}\psi\left(x,\xi\right)\phi\left(\xi\right)d\xi.
$$

Denote

$$
t_a=\begin{cases} 1, & a>1, \\ \sqrt{\log\langle t\rangle}, & a=1, \\ |t|^{\frac{1-a}{3}}, & a<1 \end{cases}.
$$

Lemma 3.1. *Let* $j=0,1,2,\ 1-\nu\leq\alpha<\min\left(1+2\delta,2-\nu\right),\ \nu=\min\left(j,1\right).$ *Suppose that*

$$
\left|\left(x\partial_x\right)^k\psi\left(x,\xi\right)\right|\leq\frac{C\,|\xi|^j}{|\xi|+|x|^{\frac{1}{2}}}
$$

for all $\xi, x \in \mathbf{R}$, $k = 0, 1, 2$. *Then the estimate*

$$\left\| |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_1 \phi \right\|_{\mathbf{L}^2} \leq C t_{\alpha+\nu} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^2}$$

is true for all $t \geq 1$.

PROOF. We have

$$\begin{aligned} & \left\| |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_1 \phi \right\|_{\mathbf{L}^2}^2 = C |t| \int_{\mathbf{R}} |x|^{\alpha} \langle x \rangle^{-2\delta} dx \int_{\mathbf{R}} e^{itS(x,\xi)} \psi(x,\xi) \overline{\phi(\xi)} d\xi \\ & \times \int_{\mathbf{R}} e^{-itS(x,\zeta)} \psi(x,\zeta) \phi(\zeta) d\zeta \\ = \ & C \int_{\mathbf{R}} d\xi e^{it\Lambda(\xi)} \overline{\phi(\xi)} \int_{\mathbf{R}} d\zeta e^{-it\Lambda(\zeta)} \phi(\zeta) K(t,\xi,\zeta), \end{aligned}$$

where

$$K(t,\xi,\zeta) = |t| \int_{\mathbf{R}} e^{-itx(\xi-\zeta)} \psi(x,\xi) \psi(x,\zeta) |x|^{\alpha} \langle x \rangle^{-2\delta} dx.$$

We integrate by parts two times via the identity

$$e^{-itx(\xi-\zeta)} = H_5 \partial_x \left(x e^{-itx(\xi-\zeta)} \right)$$

with $H_5 = (1 - itx(\xi - \zeta))^{-1}$. Then we have

$$\begin{aligned} & K(t,\xi,\zeta) \\ = \ & |t| \int_{\mathbf{R}} e^{-itx(\xi-\zeta)} x \partial_x \left(H_5 x \partial_x \left(H_5 \psi(x,\xi) \psi(x,\zeta) |x|^{\alpha} \langle x \rangle^{-2\delta} \right) \right) dx. \end{aligned}$$

Using the estimate

$$\begin{aligned} & \left| x \partial_x \left(H_5 x \partial_x \left(H_5 \psi(x,\xi) \psi(x,\zeta) |x|^{\alpha} \langle x \rangle^{-2\delta} \right) \right) \right| \\ \leq \ & \frac{C |\xi|^j |\zeta|^j |x|^{\alpha} \langle x \rangle^{-2\delta}}{(1 + |tx| |\xi - \zeta|)^2 \left(|\xi| + |x|^{\frac{1}{2}} \right) \left(|\zeta| + |x|^{\frac{1}{2}} \right)}, \end{aligned}$$

we get with $\nu = \min(1, j)$

$$\begin{aligned}
& |K(t,\xi,\zeta)| \\
\leq\ & C|t| \int_{\mathbf{R}} \left| x\partial_x \left(H_5 x \partial_x \left(H_5 \psi(x,\xi)\psi(x,\zeta)|x|^{\alpha}\langle x\rangle^{-2\delta}\right)\right)\right| dx \\
\leq\ & C|t| \int_{\mathbf{R}} \frac{|\xi|^j |\zeta|^j |x|^{\alpha}\langle x\rangle^{-2\delta} dx}{(1+|tx||\xi-\zeta|)^2 \left(|\xi|+|x|^{\frac{1}{2}}\right)\left(|\zeta|+|x|^{\frac{1}{2}}\right)} \\
\leq\ & C|t|\langle\xi\rangle\langle\zeta\rangle \int_0^1 \frac{x^{\alpha+\nu-1}dx}{(1+|t||x||\xi-\zeta|)^2} \\
& +C|t|\langle\xi\rangle^2\langle\zeta\rangle^2 \int_1^{\infty} \frac{x^{\alpha-2\delta-1}dx}{(1+|tx||\xi-\zeta|)^2} \\
\leq\ & C|t|\langle\xi\rangle^2\langle\zeta\rangle^2\langle\xi-\zeta\rangle^{-1}\langle(\xi-\zeta)t\rangle^{-\alpha-\nu} \\
& +C|t|\langle\xi\rangle^2\langle\zeta\rangle^2(|\xi-\zeta|t)^{2\delta-\alpha}\langle(\xi-\zeta)t\rangle^{\alpha-2\delta-2} \\
\leq\ & C|t|\langle\xi\rangle^2\langle\zeta\rangle^2\langle\xi-\zeta\rangle^{-1}(|\xi-\zeta|t)^{2\delta-\alpha}\langle(\xi-\zeta)t\rangle^{-\nu-2\delta}.
\end{aligned}$$

Since

$$|t|\left\|\langle\xi\rangle^{-1}(|\xi||t|)^{2\delta-\alpha}\langle\xi t\rangle^{-\nu-2\delta}\right\|_{\mathbf{L}^1} = C\int_{\mathbf{R}}\left\langle\xi t^{-1}\right\rangle^{-1}|\xi|^{2\delta-\alpha}\langle\xi\rangle^{-\nu-2\delta}d\xi$$

$$\leq\ C\int_0^1 \xi^{2\delta-\alpha}d\xi + C\int_1^{|t|}\xi^{-\alpha-\nu}d\xi + C|t|\int_{|t|}^{\infty}\xi^{-\alpha-\nu-1}d\xi \leq Ct^2_{\alpha+\nu}$$

for $|t| \geq 1$, if $1-\nu \leq \alpha < \min(1+2\delta, 2-\nu)$, then by the Young inequality we obtain

$$\begin{aligned}
& \left\||x|^{\frac{\alpha}{2}}\mathcal{I}_1\phi\right\|^2_{\mathbf{L}^2(\mathbf{R})} \leq C|t|\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^2} \\
& \times\left\|\int_{\mathbf{R}}\langle\xi-\zeta\rangle^{-1}(|\xi-\zeta|t)^{2\delta-\alpha}\langle(\xi-\zeta)t\rangle^{-\nu-2\delta}\langle\zeta\rangle^2|\phi(\zeta)|d\zeta\right\|_{\mathbf{L}^2} \\
\leq\ & C|t|\left\|\langle\xi\rangle^2\phi\right\|^2_{\mathbf{L}^2}\left\|\langle\xi\rangle^{-1}(|\xi||t|)^{2\delta-\alpha}\langle\xi t\rangle^{-\nu-2\delta}\right\|_{\mathbf{L}^1} \\
\leq\ & Ct^2_{\alpha+\nu}\left\|\langle\xi\rangle^2\phi\right\|^2_{\mathbf{L}^2}.
\end{aligned}$$

Lemma 3.1 is proved. □

In the next lemma we estimate the operator

$$\mathcal{I}_2\phi = \theta_+(x)\sqrt{|t|}\int_{\mathbf{R}} e^{-itS(x,\xi)}\psi(x,\xi)\widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\phi(\xi)d\xi.$$

LEMMA 3.2. *Let* $j = 0, 1, 2,\ 1 - \nu \leq \alpha < \min(1 + 2\delta, 2 - \nu),\ \nu = \min(j, 1),\ \delta \geq 0$. *Suppose that*

$$|\psi(x,\xi)| + \left| \left(\xi - x^{\frac{1}{2}}\right) \partial_\xi \psi(x,\xi) \right| \leq C\xi^{j-2}$$

for $0 < \frac{1}{3}x^{\frac{1}{2}} < \xi < 3x^{\frac{1}{2}}$. *Then the estimate*

$$\begin{aligned} \left\| x^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_2 \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C t_{\alpha+\nu-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \\ &\quad + C |t|^{-\frac{1}{3}(\alpha+\nu-1)} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \end{aligned}$$

is true for all $t \geq 1$.

PROOF. We use the identity (2.2) and integrate by parts to find

$$\begin{aligned} \mathcal{I}_2 \phi &= C |t|^{\frac{1}{2}} \int_{\mathbf{R}} e^{-itS(x,\xi)} \phi(\xi) \left(\xi - x^{\frac{1}{2}}\right) \partial_\xi \left(H_1 \psi(x,\xi) \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\right) d\xi \\ &\quad + C |t|^{\frac{1}{2}} \int_{\mathbf{R}} e^{-itS(x,\xi)} \left(\xi - x^{\frac{1}{2}}\right) H_1 \psi(x,\xi) \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right) \phi_\xi(\xi) \, d\xi \end{aligned}$$

for $x > 0$. Using the estimates

$$\left| \left(\xi - x^{\frac{1}{2}}\right) H_1 \psi(x,\xi) \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right) \right| \leq \frac{C x^{\frac{j}{2}-1} \left|\xi - x^{\frac{1}{2}}\right|}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2}$$

and

$$\left| \left(\xi - x^{\frac{1}{2}}\right) \partial_\xi \left(H_1 \psi(x,\xi) \widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)\right) \right| \leq \frac{C x^{\frac{j}{2}-1}}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2}$$

in the domain $\frac{1}{3}x^{\frac{1}{2}} < \xi < 3x^{\frac{1}{2}}$, by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left| x^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_2 \phi \right| &\leq C |t|^{\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} \frac{x^{\frac{\alpha+\nu}{2}-1} \langle x \rangle^{-\delta-\frac{1}{2}} d\xi}{1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2} \\ &\quad + C |t|^{\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \left(\int_{\frac{1}{3}x^{\frac{1}{2}}}^{3x^{\frac{1}{2}}} \frac{x^{\alpha+\nu-2} \langle x \rangle^{-2\delta-1} \left(\xi - x^{\frac{1}{2}}\right)^2 d\xi}{\left(1 + |t| x^{\frac{1}{2}} \left(\xi - x^{\frac{1}{2}}\right)^2\right)^2} \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq C|t|^{\frac{1}{2}} x^{\frac{\alpha+\nu}{2}-\frac{1}{2}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} \int_0^2 \frac{dy}{1+|t| x^{\frac{3}{2}} y^2}$$

$$+ C|t|^{\frac{1}{2}} x^{\frac{\alpha+\nu}{2}-\frac{1}{4}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \left(\int_0^2 \frac{y^2 dy}{\left(1+|t| x^{\frac{3}{2}} y^2\right)^2} \right)^{\frac{1}{2}}$$

$$\leq C|t|^{\frac{1}{2}} x^{\frac{\alpha+\nu}{2}-\frac{1}{2}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\langle t x^{\frac{3}{2}} \right\rangle^{-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty}$$

$$+ C|t|^{\frac{1}{2}} x^{\frac{\alpha+\nu}{2}-\frac{1}{4}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\langle t x^{\frac{3}{2}} \right\rangle^{-\frac{3}{4}} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}$$

for all $x > 0$, $|t| \geq 1$. Since

$$\left\| |t|^{\frac{1}{2}} x^{\frac{\alpha+\nu-1}{2}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\langle t x^{\frac{3}{2}} \right\rangle^{-\frac{1}{2}} \right\|^2_{\mathbf{L}^2(\mathbf{R}_+)}$$

$$= |t|^{1-\frac{2}{3}(\alpha+\nu)} \int_0^\infty y^{\alpha+\nu-1} \left\langle y |t|^{-\frac{2}{3}} \right\rangle^{-2\delta-1} \langle y \rangle^{-\frac{3}{2}} dy$$

$$\leq C|t|^{1-\frac{2}{3}(\alpha+\nu)} \int_0^1 y^{\alpha+\nu-1} dy + C|t|^{-\frac{2}{3}\left(\alpha+\nu-\frac{3}{2}\right)} \int_1^{|t|^{\frac{2}{3}}} y^{\alpha+\nu-\frac{5}{2}} dy$$

$$+ C|t|^{1-\frac{2}{3}(\alpha+\nu-2\delta-1)} \int_{|t|^{\frac{2}{3}}}^\infty y^{\alpha+\nu-2\delta-\frac{7}{2}} dy \leq C t^2_{\alpha+\nu-\frac{1}{2}},$$

and

$$\left\| |t|^{\frac{1}{2}} x^{\frac{\alpha+\nu}{2}-\frac{1}{4}} \langle x \rangle^{-\delta-\frac{1}{2}} \left\langle t x^{\frac{3}{2}} \right\rangle^{-\frac{3}{4}} \right\|^2_{\mathbf{L}^2(\mathbf{R}_+)}$$

$$= |t|^{-\frac{2}{3}(\alpha+\nu-1)} \int_0^\infty y^{\alpha+\nu-\frac{1}{2}} \langle y \rangle^{-\frac{9}{4}} dy$$

$$\leq C|t|^{-\frac{2}{3}(\alpha+\nu-1)} \int_0^1 y^{\alpha+\nu-\frac{1}{2}} dy + C|t|^{-\frac{2}{3}(\alpha+\nu-1)} \int_1^\infty y^{\alpha+\nu-\frac{11}{4}} dy$$

$$\leq C|t|^{-\frac{2}{3}(\alpha+\nu-1)},$$

therefore

$$\left\| x^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_2 \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t_{\alpha+\nu-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + C|t|^{-\frac{1}{3}(\alpha+\nu-1)} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}$$

Lemma 3.2 is proved. □

Next we estimate

$$\mathcal{I}_3\phi = \sqrt{|t|}\int_{\mathbf{R}} e^{-itS(x,\xi)}\phi(\xi)\,\psi(x,\xi)\,\chi_1\left(|\xi|\,x^{-\frac{1}{2}}\right)d\xi$$

for $x > 0$.

LEMMA 3.3. *Let* $j = 0,1,2,\ 1-\nu \le \alpha < \min(1+2\delta, 2-\nu)\,,\ \nu = \min(j,1)\,,\ \delta \ge 0$. *Suppose that*

$$|\psi(x,\xi)| + |\xi\partial_\xi\psi(x,\xi)| \le \left(|\xi| + |x|^{\frac{1}{2}}\right)^{j-2}$$

for $|\xi| < \frac{1}{3}x^{\frac{1}{2}}$ *or* $|\xi| > 3x^{\frac{1}{2}}$. *Then the estimate*

$$\begin{aligned}\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\delta}\mathcal{I}_3\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} &\le Ct_{\alpha+\nu-\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}\\ &\quad + C|t|^{-\frac{1}{3}(\alpha+\nu-1)}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\end{aligned}$$

is true for all $t \ge 1$.

PROOF. We use the identity (2.3) and integrate by parts

$$\begin{aligned}\mathcal{I}_3\phi &= C|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{-itS(x,\xi)}\phi(\xi)\,\xi\partial_\xi\left(H_2\psi(x,\xi)\,\chi_1\left(|\xi|\,x^{-\frac{1}{2}}\right)\right)d\xi\\ &\quad + C|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{-itS(x,\xi)}\xi H_2\psi(x,\xi)\,\chi_1\left(|\xi|\,x^{-\frac{1}{2}}\right)\phi_\xi(\xi)\,d\xi.\end{aligned}$$

Using the estimates

$$\left|\xi H_2\psi(x,\xi)\,\chi_1\left(|\xi|\,x^{-\frac{1}{2}}\right)\right| \le \frac{C|\xi|\left(|\xi| + |x|^{\frac{1}{2}}\right)^{j-2}}{1+|t|\,|\xi|\,(\xi^2+|x|)}$$

and

$$\left|\xi\partial_\xi\left(H_2\psi(x,\xi)\,\chi_1\left(|\xi|\,x^{-\frac{1}{2}}\right)\right)\right| \le \frac{C\left(|\xi| + |x|^{\frac{1}{2}}\right)^{j-2}}{1+|t|\,|\xi|\,(\xi^2+|x|)}$$

for $|\xi| < \frac{1}{3}x^{\frac{1}{2}}$ or $|\xi| > 3x^{\frac{1}{2}}$, we obtain

$$\left| |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_3 \phi \right| \leq C |t|^{\frac{1}{2}} |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \|\phi\|_{\mathbf{L}^\infty} \int_{\mathbf{R}} \frac{\left(|\xi| + |x|^{\frac{1}{2}} \right)^{j-2} d\xi}{1 + |t| |\xi| (\xi^2 + |x|)}$$
$$+ C |t|^{\frac{1}{2}} |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \|\phi_\xi\|_{\mathbf{L}^2} \left(\int_{\mathbf{R}} \frac{\xi^2 \left(|\xi| + |x|^{\frac{1}{2}} \right)^{2j-4} d\xi}{(1 + |t| |\xi| (\xi^2 + |x|))^2} \right)^{\frac{1}{2}}$$
$$\leq C |t|^{\frac{1}{2}} |x|^{\frac{\alpha+j-1}{2}} \langle x \rangle^{-\delta} \|\phi\|_{\mathbf{L}^\infty} \int_0^\infty \frac{\langle y \rangle^{j-2} dy}{1 + |t| |x|^{\frac{3}{2}} y \langle y \rangle^2}$$
$$+ C |t|^{\frac{1}{2}} |x|^{\frac{\alpha+j}{2} - \frac{1}{4}} \langle x \rangle^{-\delta} \|\phi_\xi\|_{\mathbf{L}^2} \left(\int_0^\infty \frac{y^2 \langle y \rangle^{2j-4} dy}{\left(1 + |t| |x|^{\frac{3}{2}} y \langle y \rangle^2 \right)^2} \right)^{\frac{1}{2}}$$
$$\leq C |t|^{\frac{1}{2}} |x|^{\frac{\alpha+j-1}{2}} \langle x \rangle^{-\delta} \left\langle |t| |x|^{\frac{3}{2}} \right\rangle^{-\frac{1}{2}} \|\phi\|_{\mathbf{L}^\infty}$$
$$+ C |t|^{\frac{1}{2}} |x|^{\frac{\alpha+j}{2} - \frac{1}{4}} \langle x \rangle^{-\delta} \left\langle |t| |x|^{\frac{3}{2}} \right\rangle^{-\frac{3}{4}} \|\phi_\xi\|_{\mathbf{L}^2}.$$

Therefore as above

$$\left\| |x|^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \mathcal{I}_3 \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C t_{\alpha+\nu-\frac{1}{2}} \|\phi\|_{\mathbf{L}^\infty} + C |t|^{-\frac{1}{3}(\alpha+\nu-1)} \|\phi_\xi\|_{\mathbf{L}^2}.$$

Lemma 3.3 is proved. □

In the next lemma we estimate the derivatives of $\mathcal{V}$.

LEMMA 3.4. *Let* $j = 0, 1, 2$, $1 - \nu \leq \alpha < \min(1 + 2\delta, 2 - \nu)$, $\nu = \min(j, 1)$, $\delta \geq 0$. *Then the estimates*

$$\left\| x^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \partial_x \mathcal{V}(t) \xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$
$$\leq C t_{\alpha+\nu-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + C t_{\alpha+\nu} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}$$

$$\left\| x^{\frac{\alpha}{2}} \langle x \rangle^{-\delta} \left[x^{\frac{1}{2}} \mathcal{V}(t) - \mathcal{V}(t) \xi \right] \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$
$$\leq C |t|^{-1} \left(t_{\alpha-\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + t_\alpha \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2} \right)$$

and

$$\begin{aligned}&\left\| x^{\frac{\alpha}{2}} \langle x\rangle^{-\frac{1}{2}-\delta} \partial_t \mathcal{V}(t)\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\leq C|t|^{-1}\left(t_{\alpha+\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}+t_{\alpha+1}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\right)\end{aligned}$$

are valid for all $|t|\geq 1$.

PROOF. We have $\partial_x S(x,\xi)=-\left(\xi-x^{\frac{1}{2}}\right)$ and $\partial_\xi S(x,\xi)=\xi^2-x$, hence $\partial_\xi e^{-itS(x,\xi)}=-it\left(\xi^2-x\right)e^{-itS(x,\xi)}$

$$\partial_x e^{-itS(x,\xi)}=-\frac{1}{\xi+x^{\frac{1}{2}}}\partial_\xi e^{-itS(x,\xi)},$$

then integrating by parts we get

$$\begin{aligned}&\partial_x\mathcal{V}(t)\xi^j\phi=\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}}e^{-itS(x,\xi)}\phi_\xi(\xi)\psi_1(x,\xi)\,d\xi \\ &+\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}}e^{-itS(x,\xi)}\phi(\xi)\psi_2(x,\xi)\left(\widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)+\chi_1\left(\xi x^{-\frac{1}{2}}\right)\right)d\xi \\ =\;& I_1+J_1\end{aligned}$$

for $x>0$, where

$$\psi_1(x,\xi)=\chi\left(\xi x^{-\frac{1}{2}}\right)\frac{\xi^j}{\xi+x^{\frac{1}{2}}}$$

and

$$\psi_2(x,\xi)=\partial_\xi\left(\chi\left(\xi x^{-\frac{1}{2}}\right)\frac{\xi^j}{\xi+x^{\frac{1}{2}}}\right)+\partial_x\chi\left(\xi x^{-\frac{1}{2}}\right).$$

We apply Lemma 3.1, then we get the estimate

$$\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\delta}I_1\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\leq Ct_{\alpha+\nu}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}.$$

Next by Lemma 3.2 and Lemma 3.3 we find

$$\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\delta}J_1\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\leq Ct_{\alpha+\nu-\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}+C|t|^{-\frac{1}{3}(\alpha+\nu-1)}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}.$$

Thus the first estimate of the lemma is true. Consider the second estimate. Since $it\left(x^{\frac{1}{2}}-\xi\right)=\partial_x S(x,\xi)$ we obtain

$$\begin{aligned}
&it\left[x^{\frac{1}{2}}\mathcal{V}(t)-\mathcal{V}(t)\xi\right]\phi=\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}}e^{-itS(x,\xi)}\phi_\xi(\xi)\psi_1(x,\xi)\,d\xi\\
&+\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}}e^{-itS(x,\xi)}\phi(\xi)\widetilde{\psi_2}(x,\xi)\left(\widetilde{\chi}\left(\xi x^{-\frac{1}{2}}\right)+\chi_1\left(\xi x^{-\frac{1}{2}}\right)\right)d\xi\\
=\;&I_2+J_2
\end{aligned}$$

for $x>0$, where $\widetilde{\psi_2}(x,\xi)=\partial_\xi\left(\chi\left(\xi x^{-\frac{1}{2}}\right)\frac{1}{\xi+x^{\frac{1}{2}}}\right)$. We apply Lemmas 3.1 - 3.3 to get the estimate

$$\begin{aligned}
&\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\delta}\left[x^{\frac{1}{2}}\mathcal{V}(t)-\mathcal{V}(t)\xi\right]\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\leq\;& C|t|^{-1}\left(t_{\alpha-\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}+t_\alpha\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\right).
\end{aligned}$$

Thus the second estimate of the lemma is true. Consider the third estimate. Since $S(x,\xi)=\frac{1}{3}\left(2x^{\frac{1}{2}}+\xi\right)\left(\xi-x^{\frac{1}{2}}\right)^2$, we obtain

$$\begin{aligned}
&\langle x\rangle^{-\frac{1}{2}}\partial_t\mathcal{V}(t)\phi=\frac{1}{2|t|}\langle x\rangle^{-\frac{1}{2}}\mathcal{V}(t)\phi\\
&-i\sqrt{\frac{|t|}{2\pi}}\int_{\mathbf{R}}e^{-itS(x,\xi)}S(x,\xi)\phi(\xi)\chi\left(\xi x^{-\frac{1}{2}}\right)d\xi\\
=\;&\frac{1}{2|t|}\langle x\rangle^{-\frac{1}{2}}\mathcal{V}(t)\phi-\frac{2}{3}x^{\frac{1}{2}}\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(x,\xi)}\partial_\xi\left(\langle\xi\rangle\phi(\xi)\right)\psi_4(x,\xi)\,d\xi\\
&-\frac{1}{3}\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(x,\xi)}\partial_\xi\left(\xi\langle\xi\rangle\phi(\xi)\right)\psi_4(x,\xi)\,d\xi\\
&-\frac{2}{3}x^{\frac{1}{2}}\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(x,\xi)}\langle\xi\rangle\phi(\xi)\psi_5(x,\xi)\,d\xi\\
&-\frac{1}{3}\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(x,\xi)}\xi\langle\xi\rangle\phi(\xi)\psi_5(x,\xi)\,d\xi\\
=\;&I_3+I_4+J_3+J_4
\end{aligned}$$

for $x>0$, where

$$\psi_4(x,\xi)=\frac{\xi-x^{\frac{1}{2}}}{3t\langle x\rangle^{\frac{1}{2}}\left(\xi+x^{\frac{1}{2}}\right)\langle\xi\rangle}\chi\left(\xi x^{-\frac{1}{2}}\right),$$

and

$$\psi_5(x,\xi) = \partial_\xi \frac{\xi - x^{\frac{1}{2}}}{3t\langle x\rangle^{\frac{1}{2}}\left(\xi + x^{\frac{1}{2}}\right)\langle\xi\rangle}.$$

We apply Lemma 3.1, then we get the estimate

$$\begin{aligned} &\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\frac{1}{2}-\delta} I_3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\frac{1}{2}-\delta} I_4\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C|t|^{-1} t_{\alpha+1}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}. \end{aligned}$$

Next by Lemma 3.2 and Lemma 3.3 we find

$$\begin{aligned} &\left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\frac{1}{2}-\delta} J_3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\|x^{\frac{\alpha}{2}}\langle x\rangle^{-\frac{1}{2}-\delta} J_4\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C|t|^{-1}\left(t_{\alpha+\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty} + t_{\alpha+1}\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2}\right). \end{aligned}$$

Lemma 3.4 is proved. □

Next we estimate

$$\mathcal{I}_4\phi = \sqrt{|t|}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\psi(x,\xi)\,d\xi.$$

LEMMA 3.5. *Let $0 \leq \alpha < \frac{3}{2}$. Suppose that*

$$|\psi(x,\xi)| + |\xi\partial_\xi\psi(x,\xi)| \leq |\xi|\left(|\xi| + |x|^{\frac{1}{2}}\right)^{-2}$$

for $x < 0$, $\xi \in \mathbf{R}$. Then the estimate

$$\left\||x|^{\frac{\alpha}{2}}\mathcal{I}_4\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C|t|^{\frac{1}{6}-\frac{\alpha}{3}}\|\phi\|_{\mathbf{L}^\infty} + C|t|^{-\frac{\alpha}{3}}\|\phi_\xi\|_{\mathbf{L}^2}$$

is true for all $t \geq 1$.

PROOF. We use the identity (2.3) and integrate by parts

$$\begin{aligned} \mathcal{I}_4\phi &= C|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,\xi\partial_\xi\left(H_2\psi(x,\xi)\right)d\xi \\ &\quad + C|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\xi H_2\psi(x,\xi)\,\phi_\xi(\xi)\,d\xi. \end{aligned}$$

Using the estimates

$$\left|\xi H_2\psi\left(x,\xi\right)\right| \leq \frac{C\xi^2\left(|\xi|+|x|^{\frac{1}{2}}\right)^{-2}}{1+|t|\,|\xi|\left(\xi^2+|x|\right)}$$

and

$$\left|\xi\partial_\xi\left(H_2\psi\left(x,\xi\right)\right)\right| \leq \frac{C\,|\xi|\left(|\xi|+|x|^{\frac{1}{2}}\right)^{-2}}{1+|t|\,|\xi|\left(\xi^2+|x|\right)}$$

for $x<0$, $\xi\in\mathbf{R}$, we obtain

$$\begin{aligned}
&\left||x|^{\frac{\alpha}{2}}\,\mathcal{I}_4\phi\right| \leq C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}}\,\|\phi\|_{\mathbf{L}^\infty}\int_{\mathbf{R}}\frac{|\xi|\left(|\xi|+|x|^{\frac{1}{2}}\right)^{-2}d\xi}{1+|t|\,|\xi|\left(\xi^2+|x|\right)}\\
&+C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}}\,\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_{\mathbf{R}}\frac{\xi^4\left(|\xi|+|x|^{\frac{1}{2}}\right)^{-4}d\xi}{\left(1+|t|\,|\xi|\left(\xi^2+|x|\right)\right)^2}\right)^{\frac{1}{2}}\\
\leq\;\; & C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}}\,\|\phi\|_{\mathbf{L}^\infty}\int_0^\infty\frac{y\,\langle y\rangle^{-2}\,dy}{1+|t|\,|x|^{\frac{3}{2}}\,y\,\langle y\rangle^2}\\
&+C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}+\frac{1}{4}}\,\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_0^\infty\frac{y^4\,\langle y\rangle^{-4}\,dy}{\left(1+|t|\,|x|^{\frac{3}{2}}\,y\,\langle y\rangle^2\right)^2}\right)^{\frac{1}{2}}\\
\leq\;\; & C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}}\left\langle|t|\,|x|^{\frac{3}{2}}\right\rangle^{-1}\|\phi\|_{\mathbf{L}^\infty}+C\,|t|^{\frac{1}{2}}\,|x|^{\frac{\alpha}{2}+\frac{1}{4}}\left\langle|t|\,|x|^{\frac{3}{2}}\right\rangle^{-1}\|\phi_\xi\|_{\mathbf{L}^2}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\||x|^{\frac{\alpha}{2}}\,\mathcal{I}_4\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C\,|t|^{\frac{1}{2}}\,\|\phi\|_{\mathbf{L}^\infty}\left\||x|^{\frac{\alpha}{2}}\left\langle|t|\,|x|^{\frac{3}{2}}\right\rangle^{-1}\right\|_{\mathbf{L}^2(\mathbf{R}_-)}\\
&+C\,|t|^{\frac{1}{2}}\,\|\phi_\xi\|_{\mathbf{L}^2}\left\||x|^{\frac{\alpha}{2}+\frac{1}{4}}\left\langle|t|\,|x|^{\frac{3}{2}}\right\rangle^{-1}\right\|_{\mathbf{L}^2(\mathbf{R}_-)}\\
\leq\;\; & C\,|t|^{\frac{1}{6}-\frac{\alpha}{3}}\,\|\phi\|_{\mathbf{L}^\infty}+C\,|t|^{-\frac{\alpha}{3}}\,\|\phi_\xi\|_{\mathbf{L}^2}
\end{aligned}$$

if $0\leq\alpha<\frac{3}{2}$. Lemma 3.5 is proved. $\square$

Next we estimate the derivative of $\mathcal{W}\left(t\right)$.

LEMMA 3.6. *Let $0 \le \alpha < 1$. Then the estimate*

$$\left\| |x|^{\frac{\alpha}{2}} \partial_x \mathcal{W}(t) \xi\phi \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C|t|^{\frac{1}{6}-\frac{\alpha}{3}} \|\phi\|_{\mathbf{L}^\infty} + Ct_{\alpha+1} \left\| \langle\xi\rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}$$

is valid for all $|t| \ge 1$.

PROOF. Applying the identity

$$\partial_x e^{it(x\xi-\Lambda(\xi))} = -\frac{\xi}{\xi^2 - x}\partial_\xi e^{it(x\xi-\Lambda(\xi))},$$

we find

$$\partial_x \mathcal{W}(t)\xi\phi = \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi_\xi(\xi) \frac{\xi^2}{\xi^2-x} d\xi$$
$$+ \sqrt{\frac{|t|}{2\pi}} \int_{\mathbf{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\partial_\xi \left(\frac{\xi^2}{\xi^2-x}\right) d\xi = I_5 + J_5$$

for $x < 0$. We apply Lemma 3.1 with $\psi(x,\xi) = \frac{\xi}{\xi^2-x}$, then we get the estimate

$$\left\| |x|^{\frac{\alpha}{2}} I_5 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le Ct_{\alpha+1} \left\| \langle\xi\rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}.$$

if $0 \le \alpha < 1$. Also by Lemma 3.5 with $\psi(x,\xi) = \partial_\xi \left(\frac{\xi^2}{\xi^2-x}\right)$ we find

$$\left\| |x|^{\frac{\alpha}{2}} J_5 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C|t|^{\frac{1}{6}-\frac{\alpha}{3}} \|\phi\|_{\mathbf{L}^\infty} + C|t|^{-\frac{\alpha}{3}} \|\phi_\xi\|_{\mathbf{L}^2}.$$

Lemma 3.6 is proved. □

4. Estimates for Derivatives of $\mathcal{V}^*$ and $\mathcal{W}^*$

Denote $\mathcal{A}_0 = \frac{1}{t}\partial_x$, $\mathcal{A} = \overline{M}\frac{1}{t}\partial_x M = \mathcal{A}_0 + ix^{\frac{1}{2}}$, $M = e^{it\left(x\sqrt{x}-\Lambda\left(\sqrt{x}\right)\right)}$, then we obtain

$$\begin{aligned} i\xi\mathcal{V}^*\phi &= \sqrt{\frac{|t|}{2\pi}} e^{it\Lambda(\xi)} \int_0^\infty i\xi e^{-itx\xi} M\phi(x)\,dx \\ &= -\sqrt{\frac{|t|}{2\pi}} e^{it\Lambda(\xi)} \int_0^\infty M\phi(x)\,\mathcal{A}_0 e^{-itx\xi} dx \\ &= \sqrt{\frac{|t|}{2\pi}} e^{it\Lambda(\xi)} \frac{1}{t}\phi(0) + \mathcal{V}^*\mathcal{A}\phi. \end{aligned}$$

Since $\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ and $\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^\infty} \le C|t|^{\frac{1}{2}}\|\phi\|_{\mathbf{L}^1(\mathbf{R}_+)}$, by the Riesz interpolation theorem (see [42], p. 52) we have

$$\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^p} \le C|t|^{\frac{1}{2}-\frac{1}{p}}\|\phi\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbf{R}_+)}$$

for $2 \le p \le \infty$.

Lemma 4.1. *Let $\delta > \frac{5}{2}$. Then the estimates*

$$\left\|\langle\xi\rangle^{-\delta}\partial_\xi\mathcal{V}^*\phi\right\|_{\mathbf{L}^2} = |t|\left\|\langle\xi\rangle^{-\delta}\left(\xi^2\mathcal{V}^* - \mathcal{V}^*x\right)\phi\right\|_{\mathbf{L}^2}$$
$$\le C|t|^{\frac{1}{2}}\left(|\phi(0)| + |\mathcal{A}\phi(0)|\right) + C\|t\mathcal{A}_0\mathcal{A}\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}t\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

$$\left\|\langle\xi\rangle^{-\delta}\left(\xi^3\mathcal{V}^* - \mathcal{V}^*x^{\frac{3}{2}}\right)\phi\right\|_{\mathbf{L}^2}$$
$$\le C|t|^{-\frac{1}{2}}\left(|\phi(0)| + |\mathcal{A}\phi(0)| + \left|\mathcal{A}^2\phi(0)\right|\right)$$
$$+C\left\|\mathcal{A}_0\mathcal{A}^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\|x\mathcal{A}_0\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

$$\left\|\langle\xi\rangle^{-\delta}\partial_t\mathcal{V}^*\phi\right\|_{\mathbf{L}^2} \le C|t|^{-\frac{1}{2}}\left(|\phi(0)| + |\mathcal{A}\phi(0)| + \left|\mathcal{A}^2\phi(0)\right|\right)$$
$$+C\left\|\mathcal{A}_0\mathcal{A}^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$
$$+C\|x\mathcal{A}_0\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C|t|^{-\frac{1}{2}}\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

and

$$\left\|\langle\xi\rangle^{-\delta}\partial_\xi\mathcal{W}^*\phi\right\|_{\mathbf{L}^2} \le C\|\partial_x\phi\|_{\mathbf{L}^2} + C|t|\|x\phi\|_{\mathbf{L}^2}$$

are true for all $|t| \ge 1$.

Proof. We have $\partial_\xi S(x,\xi) = \Lambda'(\xi) - x$. Hence

$$\partial_\xi\mathcal{V}^*(t)\phi = it\xi^2\mathcal{V}^*(t)\phi - it\mathcal{V}^*(t)x\phi = it\left(\xi^2\mathcal{V}^* - \mathcal{V}^*x\right)\phi.$$

Then applying $i\xi\mathcal{V}^*\phi = \sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0) + \mathcal{V}^*\mathcal{A}\phi$ we get

$$\partial_\xi\mathcal{V}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\xi\phi(0) - iti\xi\mathcal{V}^*\mathcal{A}\phi - it\mathcal{V}^*(t)x\phi$$
$$= \sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\xi\phi(0) - \sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}i\mathcal{A}\phi(0) - it\mathcal{V}^*\mathcal{A}^2\phi - it\mathcal{V}^*(t)x\phi.$$

Next $\mathcal{A}=\mathcal{A}_0+ix^{\frac{1}{2}}$, hence $\mathcal{A}^2=\mathcal{A}_0\mathcal{A}+ix^{\frac{1}{2}}\mathcal{A}_0-x$

$$\partial_\xi \mathcal{V}^*(t)\phi=\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\xi\phi(0)-\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}i\mathcal{A}\phi(0)$$
$$-it\mathcal{V}^*\mathcal{A}_0\mathcal{A}\phi-it\mathcal{V}^*ix^{\frac{1}{2}}\mathcal{A}_0\phi.$$

Then by estimate $\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^2}\le C\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ we obtain

$$\begin{aligned}
&\left\|\langle\xi\rangle^{-\delta}\partial_\xi\mathcal{V}^*\phi\right\|_{\mathbf{L}^2}\\
\le\ & C|t|^{\frac{1}{2}}(|\phi(0)|+|\mathcal{A}\phi(0)|)\left\|\langle\xi\rangle^{1-\delta}\right\|_{\mathbf{L}^2}\\
&+C|t|\left\|\langle\xi\rangle^{-\delta}\mathcal{V}^*\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}+C|t|\left\|\langle\xi\rangle^{-\delta}\mathcal{V}^*x^{\frac{1}{2}}\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le\ & C|t|^{\frac{1}{2}}(|\phi(0)|+|\mathcal{A}\phi(0)|)+C\|t\mathcal{A}_0\mathcal{A}\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}+C\left\|x^{\frac{1}{2}}t\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

Similarly applying $i\xi\mathcal{V}^*\phi=\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0)+\mathcal{V}^*\mathcal{A}\phi$ we consider

$$\left(\xi^3\mathcal{V}^*-\mathcal{V}^*x^{\frac{3}{2}}\right)\phi=-i\xi^2\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0)$$
$$-\xi\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\mathcal{A}\phi(0)+i\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\mathcal{A}^2\phi(0)$$
$$+\mathcal{V}^*\left(i\mathcal{A}^3-x^{\frac{3}{2}}\right)\phi.$$

Next $\mathcal{A}=\mathcal{A}_0+ix^{\frac{1}{2}}$, hence $i\mathcal{A}^3-x^{\frac{3}{2}}=i\mathcal{A}_0\mathcal{A}^2-x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}-ix\mathcal{A}_0$, therefore

$$\left(\xi^3\mathcal{V}^*-\mathcal{V}^*x^{\frac{3}{2}}\right)\phi=-i\xi^2\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0)$$
$$+\xi\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\mathcal{A}\phi(0)+i\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\mathcal{A}^2\phi(0)$$
$$+\mathcal{V}^*\left(i\mathcal{A}_0\mathcal{A}^2-x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}-ix\mathcal{A}_0\right)\phi.$$

Then by estimate $\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^2}\le C\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ we obtain

$$\left\|\langle\xi\rangle^{-\delta}\left(\xi^3\mathcal{V}^*-\mathcal{V}^*x^{\frac{3}{2}}\right)\phi\right\|_{\mathbf{L}^2}\le C|t|^{-\frac{1}{2}}|\phi(0)|\left\|\xi^2\langle\xi\rangle^{-\delta}\right\|_{\mathbf{L}^2}$$
$$+C|t|^{-\frac{1}{2}}|\mathcal{A}\phi(0)|\left\|\xi\langle\xi\rangle^{-\delta}\right\|_{\mathbf{L}^2}+C|t|^{-\frac{1}{2}}\left|\mathcal{A}^2\phi(0)\right|\left\|\langle\xi\rangle^{-\delta}\right\|_{\mathbf{L}^2}$$

$$
\begin{aligned}
&+ C\left\|\mathcal{A}_0\mathcal{A}^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq\ & C|t|^{-\frac{1}{2}}|\phi(0)| + C|t|^{-\frac{1}{2}}|\mathcal{A}\phi(0)| + C|t|^{-\frac{1}{2}}\left|\mathcal{A}^2\phi(0)\right| \\
&+ C\left\|\mathcal{A}_0\mathcal{A}^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}
\end{aligned}
$$

if $\delta > \frac{5}{2}$. Next we consider

$$
\begin{aligned}
\partial_t\mathcal{V}^*\phi &= \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{itS(x,\xi)}iS(x,\xi)\,\phi(x)\,dx \\
&= \frac{i}{3}\left(\xi^3\mathcal{V}^* - \mathcal{V}^*x^{\frac{3}{2}}\right)\phi - i\left(\xi\mathcal{V}^* - \mathcal{V}^*x^{\frac{1}{2}}\right)x\phi.
\end{aligned}
$$

So that

$$
\begin{aligned}
\left\|\langle\xi\rangle^{-\delta}\partial_t\mathcal{V}^*\phi\right\|_{\mathbf{L}^2} \leq\ & \left\|\langle\xi\rangle^{-\delta}\left(\xi^3\mathcal{V}^* - \mathcal{V}^*x^{\frac{3}{2}}\right)\phi\right\|_{\mathbf{L}^2} \\
&+ \left\|\langle\xi\rangle^{-\delta}\left(\xi\mathcal{V}^* - \mathcal{V}^*x^{\frac{1}{2}}\right)x\phi\right\|_{\mathbf{L}^2}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
i\xi\mathcal{V}^*\phi &= -\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0) - \mathcal{V}^*\mathcal{A}\phi \\
&= -\sqrt{\frac{|t|}{2\pi}}e^{it\Lambda(\xi)}\frac{1}{t}\phi(0) - \mathcal{V}^*\mathcal{A}_0\phi - i\mathcal{V}^*x^{\frac{1}{2}}\phi.
\end{aligned}
$$

Hence

$$
i\left(\xi\mathcal{V}^* - \mathcal{V}^*x^{\frac{1}{2}}\right)x\phi = -\mathcal{V}^*\mathcal{A}_0x\phi
$$

and

$$
\left\|\langle\xi\rangle^{-\delta}\left(\xi\mathcal{V}^* - \mathcal{V}^*x^{\frac{1}{2}}\right)x\phi\right\|_{\mathbf{L}^2} \leq \left\|\langle\xi\rangle^{-\delta}\mathcal{V}^*\mathcal{A}_0x\phi\right\|_{\mathbf{L}^2} \leq \left\|\mathcal{A}_0x\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
$$

Therefore

$$
\begin{aligned}
&\left\|\langle\xi\rangle^{-\delta}\partial_t\mathcal{V}^*\phi\right\|_{\mathbf{L}^2} \leq C|t|^{-\frac{1}{2}}|\phi(0)| + C|t|^{-\frac{1}{2}}|\mathcal{A}\phi(0)| + C|t|^{-\frac{1}{2}}\left|\mathcal{A}^2\phi(0)\right| \\
&+C\left\|\mathcal{A}_0\mathcal{A}^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\mathcal{A}_0\mathcal{A}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\mathcal{A}_0\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ \left\|\mathcal{A}_0x\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}
\end{aligned}
$$

if $\delta > \frac{5}{2}$.

Finally we integrate by parts

$$\partial_\xi \mathcal{W}^* (t) \phi = \sqrt{\frac{|t|}{2\pi}} e^{it\Lambda(\xi)} \int_{-\infty}^{0} it\xi^2 e^{-it\xi x} \phi (x) \, dx$$
$$-it\sqrt{\frac{|t|}{2\pi}} \int_{-\infty}^{0} e^{-it(\xi x - \Lambda(\xi))} x\phi (x) \, dx = \xi \mathcal{W}^* (t) \phi_x - it\mathcal{W}^* (t) x\phi.$$

Then as above

$$\left\| \langle \xi \rangle^{-\delta} \partial_\xi \mathcal{W}^* \phi \right\|_{\mathbf{L}^2} \le C \left\| \partial_x \phi \right\|_{\mathbf{L}^2} + C |t| \left\| x\phi \right\|_{\mathbf{L}^2}.$$

Lemma 4.1 is proved. □

5. Estimate for the Nonlinearity

In the next lemma we find the large time asymptotic behavior of the nonlinearities in equation (1.7). Define the norm

$$\|\phi\|_{\mathbf{Z}} = \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}} \left\| \langle \xi \rangle^2 \phi_\xi \right\|_{\mathbf{L}^2}.$$

LEMMA 5.1. *The following asymptotics are true*

$$N_1 = -i\xi^2 \theta_- (\xi) \left(\frac{1}{2} |\widehat{\varphi} (-|\xi|)|^2 + |\widehat{\varphi} (|\xi|)|^2 \right) \widehat{\varphi} (-|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right),$$
$$N_2 = i\xi^2 \theta_+ (\xi) \left(\frac{1}{2} |\widehat{\varphi} (|\xi|)|^2 + |\widehat{\varphi} (-|\xi|)|^2 \right) \widehat{\varphi} (|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right),$$
$$N_3 = -\frac{i}{18} \xi^2 \theta_- (\xi) \mathcal{D}_3 \overline{\widehat{\varphi} (|\xi|)} \widehat{\varphi}^2 (-|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right),$$
$$N_4 = \frac{i}{18} \xi^2 \theta_+ (\xi) \mathcal{D}_3 \overline{\widehat{\varphi} (-|\xi|)} \widehat{\varphi}^2 (|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right)$$

and

$$N_5 = O\left(|t|^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right)$$

for all $t \ge 1$ uniformly with respect to $\xi \in \mathbf{R}$, where $\widehat{\varphi} (t) = \mathcal{F}\mathcal{U} (-t) u (t)$ and $N_j, j = 1, 2, \cdots, 5$ are defined in (1.8).

PROOF. By Lemma 2.2 with $\alpha = \frac{3}{8}$ we have

$$\mathcal{V}^*(t)\,\phi = \widetilde{A}(t)\,\phi\left(\xi^2\right) + C\,|t|^{-\frac{1}{6}}\,\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + C\,|t|^{-\frac{1}{4}}\left\|x^{\frac{3}{8}}\partial_x\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Then

$$\begin{aligned} N_2 &= \mathcal{V}^*(t)\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+ \\ &= \widetilde{A}(t)\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\left(\xi^2\right) + R_1 \\ &= \sqrt{2\xi}e^{i\frac{\pi}{4}\operatorname{sgn}t}\theta_+(\xi)\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\left(\xi^2\right) + R_1 + R_2, \end{aligned}$$

where

$$\begin{aligned} R_1 &= C\,|t|^{-\frac{1}{6}}\left\|\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\quad + C\,|t|^{-\frac{1}{4}}\left\|x^{\frac{3}{8}}\partial_x\left(\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \end{aligned}$$

and

$$\begin{aligned} R_2 &= O\left(\xi^{\frac{1}{2}}\left\langle t\xi^3\right\rangle^{-1}\right)\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\left(\xi^2\right) \\ &\le C\,|t|^{-\frac{1}{6}}\left\|\left(|\psi_+|^2 + 2\,|\psi_-|^2\right)\psi_+\left(\xi^2\right)\right\|_{\mathbf{L}^\infty}. \end{aligned}$$

Next by Lemma 2.1 with $\alpha = 0$, $j = 1$ we have

$$\mathcal{V}(t)\,\xi\phi = A(t)\,x^{\frac{1}{2}}\phi\left(x^{\frac{1}{2}}\right) + O\left(|t|^{-\frac{1}{12}}\,\|\widehat{\varphi}\|_{\mathbf{Z}}\right).$$

Since

$$A\left(t,\xi^2\right) = \frac{1}{\sqrt{2}}\,|t|^{\frac{1}{6}}\left\langle t\xi^3\right\rangle^{-\frac{1}{6}}e^{-i\frac{\pi}{4}\operatorname{sgn}t} + O\left(|t|^{\frac{1}{6}}\left\langle t\xi^3\right\rangle^{-\frac{7}{6}}\right)$$

we have

$$\begin{aligned} \psi_+\left(\xi^2\right) &= \mathcal{V}(t)\,i\xi\widehat{\varphi} = iA(t)\,|\xi|\,\widehat{\varphi}(|\xi|) + O\left(|t|^{-\frac{1}{12}}\,\|\widehat{\varphi}\|_{\mathbf{Z}}\right) \\ &= \frac{i\,|t|^{\frac{1}{6}}\,|\xi|}{\sqrt{2}\,\langle t\xi^3\rangle^{\frac{1}{6}}}e^{-i\frac{\pi}{4}\operatorname{sgn}t}\widehat{\varphi}(|\xi|) + O\left(|t|^{-\frac{1}{12}}\,\|\widehat{\varphi}\|_{\mathbf{Z}}\right) \end{aligned}$$

and

$$\begin{aligned} \psi_-\left(\xi^2\right) &= \mathcal{V}(-t)\,\mathcal{D}_{-1}i\xi\widehat{\varphi} = -iA(-t)\,|\xi|\,\widehat{\varphi}(-|\xi|) + O\left(|t|^{-\frac{1}{12}}\,\|\widehat{\varphi}\|_{\mathbf{Z}}\right) \\ &= \frac{-i\,|t|^{\frac{1}{6}}\,|\xi|}{\sqrt{2}\,\langle t\xi^3\rangle^{\frac{1}{6}}}e^{i\frac{\pi}{4}\operatorname{sgn}t}\widehat{\varphi}(-|\xi|) + O\left(|t|^{-\frac{1}{12}}\,\|\widehat{\varphi}\|_{\mathbf{Z}}\right). \end{aligned}$$

Therefore

$$|R_2| \le C|t|^{-\frac{1}{6}} \left\| \left(|\psi_+|^2 + 2|\psi_-|^2 \right) \psi_+ \left(\xi^2 \right) \right\|_{\mathbf{L}^\infty} \le C|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 .$$

Also by Lemma 3.4 with $j=1$, $\alpha = \frac{3}{4}$, $\delta = 0$ we have

$$\left\| x^{\frac{3}{8}} \partial_x \mathcal{V}(t) \xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C|t|^{\frac{1}{24}} \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + C \left\| \langle \xi \rangle^2 \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} .$$

Hence

$$\begin{aligned} R_1 &= C|t|^{-\frac{1}{6}} \left\| \left(|\psi_+|^2 + 2|\psi_-|^2 \right) \psi_+ \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\quad + C|t|^{-\frac{1}{4}} \left\| x^{\frac{3}{8}} \partial_x \left(\left(|\psi_+|^2 + 2|\psi_-|^2 \right) \psi_+ \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\le C|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 . \end{aligned}$$

Thus

$$\begin{aligned} & N_2 = \mathcal{V}^*(t) \left(|\psi_+|^2 + 2|\psi_-|^2 \right) \psi_+ \\ &= i\xi^2 \theta_+(\xi) \frac{|t|^{\frac{1}{2}} |\xi|^{\frac{3}{2}}}{2 \langle t\xi^3 \rangle^{\frac{1}{2}}} \left(|\widehat{\varphi}(|\xi|)|^2 + 2|\widehat{\varphi}(-|\xi|)|^2 \right) \widehat{\varphi}(|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right) \\ &= i\xi^2 \theta_+(\xi) \left(\frac{1}{2} |\widehat{\varphi}(|\xi|)|^2 + |\widehat{\varphi}(-|\xi|)|^2 \right) \widehat{\varphi}(|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right). \end{aligned}$$

In the same manner

$$\begin{aligned} & N_1 = \mathcal{D}_{-1} \mathcal{V}^*(-t) \left(|\psi_-|^2 + 2|\psi_+|^2 \right) \psi_- \\ &= \sqrt{2|\xi|} e^{-i\frac{\pi}{4}\mathrm{sgn}t} \theta_-(\xi) \mathcal{D}_{-1} \left(|\psi_-|^2 + 2|\psi_+|^2 \right) \psi_- + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right) \\ &= -i\xi^2 \theta_-(\xi) \left(\frac{1}{2} |\widehat{\varphi}(-|\xi|)|^2 + |\widehat{\varphi}(|\xi|)|^2 \right) \widehat{\varphi}(-|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right). \end{aligned}$$

Also

$$\begin{aligned} & N_4 = \mathcal{D}_3 \mathcal{V}^*(3t) \left(\overline{\psi_-} \psi_+^2 \right) \\ &= \mathcal{D}_3 \sqrt{2\xi} e^{i\frac{\pi}{4}\mathrm{sgn}t} \theta_+(\xi) \left(\overline{\psi_-} \psi_+^2 \right) \left(\xi^2 \right) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right) \\ &= \frac{i}{2} \mathcal{D}_3 \xi^2 \theta_+(\xi) \frac{|t|^{\frac{1}{2}} |\xi|^{\frac{3}{2}}}{\langle t\xi^3 \rangle^{\frac{1}{2}}} \overline{\widehat{\varphi}(-|\xi|)} \widehat{\varphi}^2(|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right) \\ &= \frac{i}{18} \xi^2 \theta_+(\xi) \mathcal{D}_3 \overline{\widehat{\varphi}(-|\xi|)} \widehat{\varphi}^2(|\xi|) + O\left(|t|^{-\frac{1}{12}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \right) \end{aligned}$$

and

$$\begin{aligned} N_3 &= \mathcal{D}_{-3}\mathcal{V}^*(-3t)\left(\overline{\psi_+}\psi_-^2\right) \\ &= \mathcal{D}_{-3}\sqrt{2|\xi|}e^{-i\frac{\pi}{4}\mathrm{sgnt}}\theta_+(\xi)\left(\overline{\psi_+}\psi_-^2\right)\left(\xi^2\right)+O\left(|t|^{-\frac{1}{12}}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right) \\ &= -\frac{i}{18}\xi^2\theta_-(\xi)\mathcal{D}_3\frac{|t|^{\frac{1}{2}}|\xi|^{\frac{3}{2}}}{\langle t\xi^3\rangle^{\frac{1}{2}}}\overline{\widehat{\varphi}(|\xi|)}\widehat{\varphi}^2(-|\xi|)+O\left(|t|^{-\frac{1}{12}}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right) \\ &= -\frac{i}{18}\xi^2\theta_-(\xi)\mathcal{D}_3\overline{\widehat{\varphi}(|\xi|)}\widehat{\varphi}^2(-|\xi|)+O\left(|t|^{-\frac{1}{12}}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right). \end{aligned}$$

Finally by estimate

$$\|\mathcal{W}^*\phi\|_{\mathbf{L}^\infty} \le C|t|^{\frac{1}{2}}\|\phi\|_{\mathbf{L}^1(\mathbf{R}_-)}$$

and using Lemma 2.1 with $\alpha=0$, $j=1$ and $\alpha=1$, $j=1$ we find

$$\begin{aligned} \left\|\left\langle xt^{\frac{2}{3}}\right\rangle^{\frac{1}{2}}\mathcal{W}(t)\xi\widehat{\varphi}\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} &\le C\|\mathcal{W}(t)\xi\phi\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \\ +C|t|^{\frac{1}{3}}\left\||x|^{\frac{1}{2}}\mathcal{W}(t)\xi\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} &\le C\|\widehat{\varphi}\|_{\mathbf{Z}}. \end{aligned}$$

Hence

$$\begin{aligned} N_5 &= \mathcal{W}^*|r|^2r = O\left(|t|^{\frac{1}{2}}\left\|r^3\right\|_{\mathbf{L}^1(\mathbf{R}_-)}\right) = O\left(|t|^{\frac{1}{2}}\|\mathcal{W}(t)\xi\widehat{\varphi}\|_{\mathbf{L}^3(\mathbf{R}_-)}^3\right) \\ &= O\left(|t|^{\frac{1}{2}}\left\|\left\langle xt^{\frac{2}{3}}\right\rangle^{-\frac{1}{2}}\right\|_{\mathbf{L}^3(\mathbf{R}_-)}^3\left\|\left\langle xt^{\frac{2}{3}}\right\rangle^{\frac{1}{2}}\mathcal{W}(t)\xi\widehat{\varphi}\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)}^3\right) \\ &= O\left(|t|^{-\frac{1}{6}}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right). \end{aligned}$$

Lemma 5.1 is proved. □

6. A Priori Estimates

Local existence and uniqueness of solutions to the Cauchy problem (1.3) can be established by the standard method.

Theorem 6.1. *Assume that the initial data* $u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{2,1}$. *Then for some* $T>0$ *there exists a unique local solution* u *of the Cauchy problem (1.3) such that* $\mathcal{U}(-t)u \in \mathbf{C}\left([0,T];\mathbf{H}^3 \cap \mathbf{H}^{2,1}\right)$.

We can take $T > 1$ if the data are small in $\mathbf{H}^3 \cap \mathbf{H}^{2,1}$ and we may assume that

$$\begin{aligned} & \|\mathcal{FU}(-t)\,\partial_x u(1)\|_{\mathbf{L}^\infty} + \left\|\mathcal{FU}(-t)\,\partial_x^2 u(1)\right\|_{\mathbf{L}^\infty} \\ & + \|\mathcal{J}u(1)\|_{\mathbf{H}^2} + \|u(1)\|_{\mathbf{H}^3} \le \varepsilon. \end{aligned} \tag{6.1}$$

To get the desired result, we prove a priori estimates of solutions uniformly in time. Define the following norm

$$\begin{aligned} \|u\|_{\mathbf{X}_T} &= \sup_{t\in[1,T]} \sum_{j=0}^{2} \left\|\mathcal{FU}(-t)\,\partial_x^j u(t)\right\|_{\mathbf{L}^\infty} \\ &\quad + \sup_{t\in[1,T]} t^{-\gamma} \left(\|\mathcal{J}u(t)\|_{\mathbf{H}^2} + \|u(t)\|_{\mathbf{H}^3}\right), \end{aligned}$$

where $\mathcal{J} = x + t\partial_x^2 = \mathcal{U}(t)\,x\mathcal{U}(-t)$, $\gamma > 0$ is sufficiently small.

LEMMA 6.2. *Assume that (6.1) holds. Then there exists an ε such that the estimate*

$$\|u\|_{\mathbf{X}_T} < C\varepsilon$$

is true for all $T > 1$.

PROOF. By the continuity of the norm $\|u\|_{\mathbf{X}_T}$ with respect to T, arguing by the contradiction we can find the first time $T > 0$ such that $\|u\|_{\mathbf{X}_T} = C\varepsilon$. Define the norm

$$\|\widehat{\varphi}\|_{\mathbf{W}} = \left\|\langle\xi\rangle^2\,\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-\gamma}\left\|\langle\xi\rangle^2\,\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + t^{-\gamma}\left\|\langle\xi\rangle^3\,\widehat{\varphi}\right\|_{\mathbf{L}^2}.$$

Note that by our assumption $\|u\|_{\mathbf{X}_T} = C\varepsilon$, it follows that

$$\|\widehat{\varphi}\|_{\mathbf{W}} \le C\varepsilon$$

for $t \in [1, T]$. By equation (1.3) we find

$$\frac{d}{dt}\|u\|_{\mathbf{H}^3} \le C\,\|u_x\|_{\mathbf{L}^\infty}\,\|u_{xx}\|_{\mathbf{L}^\infty}\,\|u\|_{\mathbf{H}^3} \le C\varepsilon^2 t^{-1}\,\|u\|_{\mathbf{H}^3}.$$

Then we get a priori estimate $\|u(t)\|_{\mathbf{H}^3} \le C\varepsilon t^{\gamma}$ if $\|\partial_x u(t)\|_{\mathbf{L}^\infty} + \left\|\partial_x^2 u(t)\right\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{2}}$. On the other hand, by Lemma 2.1 we have

$$\sum_{j=0}^{2}\left\|\partial_x^j u(t)\right\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{2}}.$$

Next we need the estimate for $\left\|\xi^{j}\widehat{\varphi}\right\|_{\mathbf{L}^{\infty}} = \left\|\mathcal{FU}\left(-t\right)\partial_x^j u\left(t\right)\right\|_{\mathbf{L}^{\infty}}$, $j = 0,1,2$. If $|\xi| \geq t^{\nu}$, then by the Sobolev embedding theorem we get

$$\begin{aligned}
\left\|\xi^{j}\widehat{\varphi}\right\|_{\mathbf{L}^{\infty}} &\leq C\left\|\langle\xi\rangle^{2}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}^{\frac{1}{2}}\left\|\langle\xi\rangle^{2}\partial_{\xi}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}^{\frac{1}{2}} \\
&\leq Ct^{-\frac{\nu}{2}}\left\|\langle\xi\rangle^{3}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}^{\frac{1}{2}}\left\|\langle\xi\rangle^{2}\partial_{\xi}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}^{\frac{1}{2}} \leq C\varepsilon
\end{aligned}$$

if $\nu > 2\gamma$. Next we use equation (1.7) for $\widehat{\varphi} = \mathcal{FU}\left(-t\right)u\left(t\right)$ in the domain $|\xi| \leq t^{\nu}$. Applying Lemma 5.1 we get

$$\begin{aligned}
\partial_t\widehat{\varphi}\left(t,\xi\right) &= \frac{i\lambda\xi^{2}}{2\left|t\right|}\left(\frac{1}{2}\left|\widehat{\varphi}\left(t,\xi\right)\right|^{2}+\left|\widehat{\varphi}\left(t,-\xi\right)\right|^{2}\right)\widehat{\varphi}\left(\xi\right) \\
&\quad +\frac{i\lambda\xi^{2}}{18\left|t\right|}e^{\frac{8}{9}it\Lambda(\xi)}\mathcal{D}_{3}\overline{\widehat{\varphi}\left(t,-\xi\right)}\widehat{\varphi}^{2}\left(t,\xi\right)+O\left(\left|t\right|^{-\frac{13}{12}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{3}\right),
\end{aligned}$$

for $\xi > 0$ and

$$\begin{aligned}
\partial_t\widehat{\varphi}\left(t,\xi\right) &= -\frac{i\lambda\xi^{2}}{2\left|t\right|}\left(\frac{1}{2}\left|\widehat{\varphi}\left(t,\xi\right)\right|^{2}+\left|\widehat{\varphi}\left(t,-\xi\right)\right|^{2}\right)\widehat{\varphi}\left(\xi\right) \\
&\quad -\frac{i\lambda\xi^{2}}{18\left|t\right|}e^{\frac{8}{9}it\Lambda(\xi)}\mathcal{D}_{3}\overline{\widehat{\varphi}\left(t,-\xi\right)}\widehat{\varphi}^{2}\left(t,\xi\right)+O\left(\left|t\right|^{-\frac{13}{12}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{3}\right),
\end{aligned}$$

for $\xi \leq 0$. Choosing

$$\Psi_{\pm}\left(t,\xi\right) = \exp\left(\pm\frac{i\lambda\xi^{2}}{2}\int_{1}^{t}\left(\frac{1}{2}\left|\widehat{\varphi}\left(\tau,\xi\right)\right|^{2}+\left|\widehat{\varphi}\left(\tau,-\xi\right)\right|^{2}\right)\frac{d\tau}{\tau}\right),$$

we get

$$\begin{aligned}
&\partial_t\left(\xi^{j}\widehat{\varphi}\left(t,\xi\right)\Psi_{\pm}\left(t,\xi\right)\right) \\
&= \pm\frac{i\lambda\xi^{2}}{18\left|t\right|}e^{\frac{8}{9}it\Lambda(\xi)}\mathcal{D}_{3}\overline{\widehat{\varphi}\left(t,-\xi\right)}\widehat{\varphi}^{2}\left(t,\xi\right)\xi^{j}\Psi_{\pm}\left(t,\xi\right) \\
&\quad +O\left(\left|t\right|^{2\nu-\frac{13}{12}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{3}\right),
\end{aligned}$$

for $j = 1,2$, $|\xi| \leq t^{\nu}$, $\nu < \frac{1}{24}$. Integrating in time, we obtain

$$\begin{aligned}
&\left|\xi^{j}\widehat{\varphi}\left(t,\xi\right)\Psi_{\pm}\left(t,\xi\right)\right| \\
&\leq \left|\widehat{\varphi}\left(1,\xi\right)\right|+C\left|\int_{1}^{t}e^{\frac{8}{9}i\tau\Lambda(\xi)}\mathcal{D}_{3}\overline{\widehat{\varphi}\left(\tau,-\xi\right)}\widehat{\varphi}^{2}\left(\tau,\xi\right)\xi^{j+2}\Psi_{\pm}\left(\tau,\xi\right)\frac{d\tau}{\tau}\right|+C\varepsilon^{3}.
\end{aligned}$$

Integrating by parts we get $\left|\xi^j \widehat{\varphi}(t,\xi)\right| < C\varepsilon$.

Consider a priori estimate of $\|\mathcal{J}u(t)\|_{\mathbf{H}^2}$. We have

$$\|\mathcal{J}u(t)\|_{\mathbf{H}^2} \le C\left\|\partial_x^2 \mathcal{J}u\right\|_{\mathbf{L}^2} + C\left\|\langle i\partial_x\rangle^{-3}\mathcal{J}u\right\|_{\mathbf{L}^2}.$$

Using the identity

$$\mathcal{P}u - \partial_x \mathcal{J}u = -3t\mathcal{L}u$$

with $\mathcal{J} = x - t\partial_x^2$, $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$, $\mathcal{P} = \partial_x x + 3t\partial_t$, we get

$$\left\|\partial_x^2\mathcal{J}u\right\|_{\mathbf{L}^2} \le C\|\partial_x\mathcal{P}u\|_{\mathbf{L}^2} + Ct\|u_x\|_{\mathbf{L}^\infty}^2\|u_{xx}\|_{\mathbf{L}^2} \le C\|\partial_x\mathcal{P}u\|_{\mathbf{L}^2} + C\varepsilon^3 t^\gamma.$$

We apply the operator $\partial_x\mathcal{P}$ to equation (1.3). In view of the commutators $[\mathcal{L},\mathcal{P}] = 3\mathcal{L}$, $[\mathcal{P},\partial_x] = -\partial_x$, we get

$$\mathcal{L}\partial_x\mathcal{P}u = \partial_x(\mathcal{P}+3)\mathcal{L}u = \lambda\partial_x(\mathcal{P}+3)\left(|u_x|^2 u_x\right).$$

Then by the energy method we obtain

$$\begin{aligned} \frac{d}{dt}\|\partial_x\mathcal{P}u\|_{\mathbf{L}^2} &\le C\|u_x\|_{\mathbf{L}^\infty}\|u_{xx}\|_{\mathbf{L}^\infty}\|\partial_x\mathcal{P}u\|_{\mathbf{L}^2} \\ +\|u_x\|_{\mathbf{L}^\infty}^2\|u_{xx}\|_{\mathbf{L}^2} &\le C\varepsilon^2 t^{-1}\|\partial_x\mathcal{P}u\|_{\mathbf{L}^2} + C\varepsilon^2 t^{\gamma-1} \end{aligned}$$

from which it follows $\|\mathcal{P}u\|_{\mathbf{L}^2} \le C\varepsilon t^\gamma$. Therefore $\left\|\partial_x^2\mathcal{J}u\right\|_{\mathbf{L}^2} \le C\varepsilon t^\gamma$ for all $t \in [1,T]$.

Finally we need to estimate the norm $\left\|\langle i\partial_x\rangle^{-3}\mathcal{J}u\right\|_{\mathbf{L}^2} = \left\|\langle\xi\rangle^{-3}\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}$. Differentiating equation (1.7) we get

$$\begin{aligned} &\partial_t\langle\xi\rangle^{-3}\widehat{\varphi_\xi} = \lambda t^{-1}\langle\xi\rangle^{-3}\partial_\xi(N_1+N_2) \\ &+\lambda t^{-1}e^{\frac{8}{9}it\Lambda(\xi)}\langle\xi\rangle^{-3}\partial_\xi(N_3+N_4) \\ &+\lambda t^{-1}\langle\xi\rangle^{-3}\partial_\xi N_5 + i\lambda\frac{8}{9}\xi^2 e^{\frac{8}{9}it\Lambda(\xi)}\langle\xi\rangle^{-3}(N_3+N_4). \end{aligned} \tag{6.2}$$

We denote $\psi_{j+} = \mathcal{V}(t)\, i\xi^j\widehat{\varphi}$, $\psi_{j-} = \mathcal{V}(-t)\,\mathcal{D}_{-1}i\xi^j\widehat{\varphi}$. Applying Lemma 4.1 we get

$$\begin{aligned} &\left\|\langle\xi\rangle^{-3}\partial_\xi N_k\right\|_{\mathbf{L}^2} \le \left\|\langle\xi\rangle^{-3}\partial_\xi\mathcal{V}^*\psi_{1\pm}^3\right\|_{\mathbf{L}^2} \\ \le\ & C|t|^{\frac{1}{2}}\left(|\psi_{1\pm}(0)|^3 + |\psi_{1\pm}(0)|^2|\psi_{2\pm}(0)|\right) + C\left\|\psi_{1\pm}^2\partial_x\mathcal{V}(t)\,\xi^2\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\|\psi_{1\pm}\psi_{2\pm}\partial_x\mathcal{V}(t)\,\xi\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\psi_{1\pm}^2\partial_x\mathcal{V}(t)\,\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \end{aligned}$$

for $k = 1, 2, 3, 4$ and

$$\begin{aligned} \left\| \langle \xi \rangle^{-3} \partial_\xi N_5 \right\|_{\mathbf{L}^2} &\leq \left\| \langle \xi \rangle^{-3} \partial_\xi \mathcal{W}^* |r|^2 r \right\|_{\mathbf{L}^2} \\ \leq \quad C \left\| |\mathcal{W}(t) \xi \widehat{\varphi}|^2 \partial_x \mathcal{W}(t) \xi \widehat{\varphi} \right\|_{\mathbf{L}^2} &+ C |t| \left\| x |\mathcal{W}(t) \xi \widehat{\varphi}|^3 \right\|_{\mathbf{L}^2}. \end{aligned}$$

By Lemma 2.1 with $\alpha = 0$, $j = 1, 2$ we find $|\psi_{j\pm}(0)| \leq C |t|^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{W}}$. Hence

$$|t|^{\frac{1}{2}} \left(|\psi_{1\pm}(0)|^3 + |\psi_{1\pm}(0)|^2 |\psi_{2\pm}(0)| \right) \leq C \|\widehat{\varphi}\|_{\mathbf{W}}^3.$$

Also by Lemma 2.1 with $\alpha = 0$, $j = 1, 2$

$$\langle x \rangle^{\frac{1}{2}} |\psi_{j\pm}| \leq C \left(x^{\frac{1}{4}} + |t|^{-\frac{1}{6}} \right) \|\widehat{\varphi}\|_{\mathbf{W}}.$$

By Lemma 3.4 with $j = 1, 2$, $\delta = 0$, $\alpha = \frac{3}{4}$ and $\alpha = 0$, we get

$$\begin{aligned} \left\| \left(x^{\frac{3}{8}} + |t|^{-\frac{1}{3}} \right) \partial_x \mathcal{V}(t) \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty} \\ + C \left\| \langle \xi \rangle^2 \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} &\leq C \langle t \rangle^\gamma \|\widehat{\varphi}\|_{\mathbf{W}}. \end{aligned}$$

Hence

$$\begin{aligned} &\left\| \psi_{1\pm}^2 \partial_x \mathcal{V}(t) \xi^2 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C \left\| \psi_{1\pm} \psi_{2\pm} \partial_x \mathcal{V}(t) \xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C \left\| x^{\frac{1}{2}} \psi_{1\pm}^2 \partial_x \mathcal{V}(t) \xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq \quad & C \|\widehat{\varphi}\|_{\mathbf{W}}^2 \left\| \left(x^{\frac{3}{8}} + |t|^{-\frac{1}{3}} \right) \partial_x \mathcal{V}(t) \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \langle t \rangle^\gamma \|\widehat{\varphi}\|_{\mathbf{W}}^3. \end{aligned}$$

Therefore

$$\left\| \langle \xi \rangle^{-3} \partial_\xi N_k \right\|_{\mathbf{L}^2} \leq C \langle t \rangle^\gamma \|\widehat{\varphi}\|_{\mathbf{W}}^3$$

for $k = 1, 2, 3, 4$. Next by Lemma 2.1 with $j = 1$, $\alpha = 0$, $\alpha = \frac{2}{3}$ and $\alpha = \frac{4}{3}$

$$\|\mathcal{W}(t) \xi \widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \leq C |t|^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{W}}$$

and

$$\left\| |x|^{\frac{1}{3}} \langle x \rangle^{\frac{1}{3}} \mathcal{W}(t) \xi \widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \leq C |t|^{-\frac{1}{3}} \|\widehat{\varphi}\|_{\mathbf{W}}$$

By Lemma 3.6 with $\alpha = 0$ we obtain

$$\|\partial_x \mathcal{W}(t)\,\xi\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\,|t|^{\frac{1}{6}}\,\|\widehat{\varphi}\|_{\mathbf{W}}.$$

Hence

$$\begin{aligned}
&\left\|\langle\xi\rangle^{-3}\,\partial_\xi N_5\right\|_{\mathbf{L}^2}\\
\le\ & C\left\||\mathcal{W}(t)\,\xi\widehat{\varphi}|^2\,\partial_x\mathcal{W}(t)\,\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + C\,|t|\left\|x\,|\mathcal{W}(t)\,\xi\widehat{\varphi}|^3\right\|_{\mathbf{L}^2}\\
\le\ & C\,|t|^{-\frac{1}{6}}\,\|\widehat{\varphi}\|_{\mathbf{W}}^3 + C\,\|\widehat{\varphi}\|_{\mathbf{W}}^3\left\|\langle x\rangle^{-1}\right\|_{\mathbf{L}^2(\mathbf{R})} \le C\,\|\widehat{\varphi}\|_{\mathbf{W}}^3.
\end{aligned}$$

Thus we get from (6.2)

$$\partial_t\,\langle\xi\rangle^{-3}\,\widehat{\varphi_\xi} = O\left(\langle t\rangle^{\gamma-1}\,\|\widehat{\varphi}\|_{\mathbf{W}}^3\right) + i\lambda\frac{8}{9}\xi^2 e^{\frac{8}{9}it\Lambda(\xi)}\,\langle\xi\rangle^{-3}\,(N_3+N_4)\,. \tag{6.3}$$

We need to estimate the last term in the right-hand side (6.3). We represent

$$\langle\xi\rangle^{-3}\,\xi^2\mathcal{V}^*(3t)\,\overline{\psi_{1-}}\psi_{1+}^2 = \langle\xi\rangle^{-3}\,\mathcal{V}^*(3t)\,x\overline{\psi_{1-}}\psi_{1+}^2 + R_1,$$

where

$$R_1 = \langle\xi\rangle^{-3}\left[\xi^2\mathcal{V}^*(3t) - \mathcal{V}^*(3t)\,x\right]\overline{\psi_{1-}}\psi_{1+}^2.$$

Also we denote $\psi_{j+} = \mathcal{V}(t)\,i\xi^j\widehat{\varphi}$, $\psi_{j-} = \mathcal{V}(-t)\,\mathcal{D}_{-1}i\xi^j\widehat{\varphi}$. By Lemma 4.1 we find

$$\begin{aligned}
&|t|\,\|R_1\|_{\mathbf{L}^2} = |t|\left\|\langle\xi\rangle^{-3}\left(\xi^2\mathcal{V}^*(3t) - \mathcal{V}^*(3t)\,x\right)\overline{\psi_{1-}}\psi_{1+}^2\right\|_{\mathbf{L}^2}\\
\le\ & C\,|t|^{\frac{1}{2}}\left(\left|\left(\overline{\psi_{1-}}\psi_{1+}^2\right)(0)\right| + \left|\left(\mathcal{A}\left(\overline{\psi_{1-}}\psi_{1+}^2\right)\right)(0)\right|\right)\\
&+C\left\|t\mathcal{A}_0\mathcal{A}\left(\overline{\psi_{1-}}\psi_{1+}^2\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}t\mathcal{A}_0\left(\overline{\psi_{1-}}\psi_{1+}^2\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

Then as above

$$\begin{aligned}
&|t|\,\|R_1\|_{\mathbf{L}^2} \le C\,|t|^{\frac{1}{2}}\left(|\psi_{1\pm}(0)|^3 + |\psi_{1\pm}(0)|^2\,|\psi_{2\pm}(0)|\right)\\
&+C\left\|\psi_{1\pm}^2\partial_x\mathcal{V}(t)\,\xi^2\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\,\|\psi_{1\pm}\psi_{2\pm}\partial_x\mathcal{V}(t)\,\xi\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
&+C\left\|x^{\frac{1}{2}}\psi_{1\pm}^2\partial_x\mathcal{V}(t)\,\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\,\langle t\rangle^{\gamma}\,\|\widehat{\varphi}\|_{\mathbf{W}}^3.
\end{aligned}$$

Next we represent

$$x\overline{\psi_{1-}}\psi_{1+}^2 = x^{\frac{5}{2}}\overline{\psi_{0-}}\psi_{0+}^2 + R_2,$$

where
$$R_2 = x\left(\overline{\psi_{1-}}\psi_{1+}^2 - x^{\frac{3}{2}}\overline{\psi_{0-}}\psi_{0+}^2\right).$$
By Lemma 3.4 with $j=0$, $\alpha=\frac{7}{4}$, $\delta=\frac{1}{2}$ we find
$$\left\| x^{\frac{7}{8}}\left\langle x\right\rangle^{-\frac{1}{2}}\left[x^{\frac{1}{2}}\psi_{0\pm}-\psi_{1\pm}\right]\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\left|t\right|^{\gamma-1}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}$$
and by Lemma 2.1
$$\left\|\left\langle x\right\rangle^{\frac{1}{2}}\psi_{1\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\|x^{\frac{1}{2}}\left\langle x\right\rangle^{\frac{1}{2}}\psi_{0\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}.$$
Hence
$$\begin{aligned}\left\|R_2\right\|_{\mathbf{L}^2} &\le C\left(\left\|\left\langle x\right\rangle^{\frac{1}{2}}\psi_{1\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\|x^{\frac{1}{2}}\left\langle x\right\rangle^{\frac{1}{2}}\psi_{0\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\right)^2 \\ &\times\left\| x^{\frac{7}{8}}\left\langle x\right\rangle^{-\frac{1}{2}}\left[x^{\frac{1}{2}}\psi_{0\pm}-\psi_{1\pm}\right]\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\left|t\right|^{\gamma-1}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3.\end{aligned}$$
Next we represent
$$\left\langle \xi\right\rangle^{-3}\mathcal{V}^*(3t)\,x^{\frac{5}{2}}\overline{\psi_{0-}}\psi_{0+}^2 = \xi^3\left\langle \xi\right\rangle^{-3}\mathcal{V}^*(3t)\,x\overline{\psi_{0-}}\psi_{0+}^2 + R_3,$$
where
$$R_3 = \left\langle \xi\right\rangle^{-3}\left[\xi^3\mathcal{V}^*(3t) - \mathcal{V}^*(3t)\,x^{\frac{3}{2}}\right]x\overline{\psi_{0-}}\psi_{0+}^2.$$
By Lemma 4.1 we find
$$\begin{aligned} &\left|t\right|\left\|R_3\right\|_{\mathbf{L}^2} = \left|t\right|\left\|\left\langle \xi\right\rangle^{-3}\left[\xi^3\mathcal{V}^*(3t) - \mathcal{V}^*(3t)\,x^{\frac{3}{2}}\right]x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2} \\ \le\ & C\left\|\partial_x\mathcal{A}^2(3t)\,x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\partial_x\mathcal{A}(3t)\,x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\left\|x^2\partial_x\left(\overline{\psi_{0-}}\psi_{0+}^2\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & C\left\|x\partial_x\mathcal{A}^2(3t)\,\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{3}{2}}\partial_x\mathcal{A}(3t)\,\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\left\|x^2\partial_x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|\mathcal{A}^2(3t)\,\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\left\|x^{\frac{1}{2}}\mathcal{A}(3t)\,\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\overline{\psi_{0-}}\psi_{0+}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)},\end{aligned}$$
since $\mathcal{A}(3t) = \overline{M}^3\frac{1}{3t}\partial_x M^3 = \frac{1}{3t}\partial_x + ix^{\frac{1}{2}}$, then $[\partial_x, x] = 1$, $[\mathcal{A}(3t), x] = \frac{1}{3t}$, and
$$x^{\frac{1}{2}}\partial_x\mathcal{A}(3t)\,x = x^{\frac{3}{2}}\partial_x\mathcal{A}(3t) + 2x^{\frac{1}{2}}\mathcal{A}(3t) - ix$$

and

$$\partial_x \mathcal{A}^2(3t)\, x = x\partial_x \mathcal{A}^2(3t) + 3\mathcal{A}^2(3t) - 2ix^{\frac{1}{2}}\mathcal{A}(3t).$$

Also we have

$$\begin{aligned} &\mathcal{A}(3t)\left(\overline{\psi_{0-}}\psi_{0+}^2\right) = \overline{M}^3 \frac{1}{3t}\partial_x\left(\left(\overline{\overline{M}\psi_{0-}}\right)(M\psi_{0+})^2\right) \\ = &\ \frac{1}{3}\overline{\mathcal{A}(t)\psi_{0-}}\psi_{0+}^2 + \frac{2}{3}\left(\overline{\psi_{0-}}(\psi_{0+})(\mathcal{A}(t)\psi_{0+})\right) \\ = &\ \frac{1}{3}\overline{\psi_{1-}}\psi_{0+}^2 + \frac{2}{3}\overline{\psi_{0-}}\psi_{0+}\psi_{1+}, \end{aligned}$$

similarly

$$\begin{aligned} \mathcal{A}^2(3t)\left(\overline{\psi_{0-}}\psi_{0+}^2\right) &= \frac{1}{3}\overline{\psi_{2-}}\psi_{0+}\psi_{0+} + \frac{4}{3}\overline{\psi_{1-}}\psi_{0+}\psi_{1+} \\ &\quad + \frac{2}{3}\overline{\psi_{0-}}\left(\psi_{1+}^2\right) + \frac{2}{3}\overline{\psi_{0-}}\psi_{0+}\psi_{2+}. \end{aligned}$$

Hence we get

$$\begin{aligned} &|t|\,\|R_3\|_{\mathbf{L}^2} \le C\left(\left\|x^{\frac{5}{4}}\langle x\rangle^{\frac{1}{2}}\psi_{0\pm}\psi_{0\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\|x^{\frac{3}{4}}\langle x\rangle^{\frac{1}{2}}\psi_{0\pm}\psi_{1\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\right. \\ &+ \left\|x^{\frac{1}{4}}\langle x\rangle^{\frac{1}{2}}\psi_{0\pm}\psi_{2\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\left.+ \left\|x^{\frac{1}{4}}\langle x\rangle^{\frac{1}{2}}\psi_{1\pm}\psi_{1\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\right)\left\|x^{\frac{3}{4}}\langle x\rangle^{-\frac{1}{2}}\partial_x\psi_{0\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C\left(\left\|x^{\frac{5}{8}}\psi_{0\pm}\psi_{1\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\|x^{\frac{9}{8}}\psi_{0\pm}\psi_{0\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\right)\left\|x^{\frac{3}{8}}\partial_x\psi_{1\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C\left\|x^{\frac{5}{8}}\psi_{0\pm}\psi_{0\pm}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\left\|x^{\frac{3}{8}}\partial_x\psi_{2\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|\psi_{2\pm}\psi_{0\pm}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C\left\|\psi_{1\pm}^2\psi_{0\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x^{\frac{1}{2}}\psi_{1\pm}\psi_{0\pm}^2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left\|x\psi_{0\pm}^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

By Lemma 3.4 with $\alpha = \frac{3}{2}$, $\delta = \frac{1}{2}$, $j = 0$, and $\alpha = \frac{3}{4}$, $\delta = 0$; $j = 1, 2$ we obtain

$$\begin{aligned} &\left\|x^{\frac{3}{4}}\langle x\rangle^{-\frac{1}{2}}\partial_x\psi_{0\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\|x^{\frac{3}{8}}\partial_x\psi_{1\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\|x^{\frac{3}{8}}\partial_x\psi_{2\pm}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & C\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty} + C\left\|\langle\xi\rangle^2\phi_\xi\right\|_{\mathbf{L}^2} \le C|t|^\gamma\|\widehat{\varphi}\|_{\mathbf{W}}. \end{aligned}$$

Also by Lemma 2.1

$$\left\| x^{\frac{1}{4}} \langle x\rangle^{\frac{3}{4}} \psi_{0\pm} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\| \langle x\rangle^{\frac{1}{2}} \psi_{1\pm} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\| \langle x\rangle^{\frac{1}{4}} \psi_{2\pm} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le C \left\|\widehat{\varphi}\right\|_{\mathbf{W}}.$$

Hence we find

$$\left\|R_3\right\|_{\mathbf{L}^2} \le C \left|t\right|^{\gamma-1} \left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3.$$

Thus we get the representation

$$\langle\xi\rangle^{-3} \xi^2 \mathcal{V}^* (3t) \overline{\psi_{1-}} \psi_{1+}^2 = \xi^3 \langle\xi\rangle^{-3} \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+}^2 + O\left(\left|t\right|^{\gamma-1} \left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3 \right)$$

and by equation (6.3)

$$\partial_t \langle\xi\rangle^{-3} \widehat{\varphi_\xi} = O\left(\langle t\rangle^{\gamma-1} \left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3 \right) + \xi^3 e^{\frac{8}{9} it\Lambda(\xi)} I, \tag{6.4}$$

where

$$I = i\lambda \frac{8}{9} \langle\xi\rangle^{-3} \left(\mathcal{D}_{-3} \mathcal{V}^* (-3t) x \overline{\psi_{0+}} \psi_{0-}^2 + \mathcal{D}_3 \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+}^2 \right).$$

The last term on the right-hand side of (6.4) we rewrite in the form of full derivative

$$\xi^3 e^{\frac{8}{9} it\Lambda(\xi)} I = \partial_t \left(\frac{9\xi^3}{8i\Lambda(\xi)} e^{\frac{8}{9} it\Lambda(\xi)} I \right) - \frac{9\xi^3}{8i\Lambda(\xi)} e^{\frac{8}{9} it\Lambda(\xi)} \partial_t I.$$

Finally we estimate

$$\begin{aligned} & \left\| \langle\xi\rangle^{-3} \partial_t \left(\mathcal{D}_3 \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+}^2 \right) \right\|_{\mathbf{L}^2} \\ \le & \left\| \langle\xi\rangle^{-3} \partial_t \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2} \\ & + \left\| \langle\xi\rangle^{-3} \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+} \partial_t \psi_{0+} \right\|_{\mathbf{L}^2}. \end{aligned}$$

By Lemma 4.1 as above we get

$$\begin{aligned} & \left\| \langle\xi\rangle^{-3} \partial_t \mathcal{V}^* (3t) x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2} \le C \left\| \mathcal{A}_0 \mathcal{A}^2 x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C \left\| x^{\frac{1}{2}} \mathcal{A}_0 \mathcal{A} x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C \left\| x \mathcal{A}_0 x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C \left\| \mathcal{A}_0 x^2 \left(\overline{\psi_{0-}} \psi_{0+}^2 \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C \left|t\right|^{-\frac{1}{2}} \left\| x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le \; & C \langle t\rangle^{\gamma-1} \left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3. \end{aligned}$$

Finally by Lemma 3.4 with $\alpha = 1$, $\delta = \frac{1}{2}$, we have

$$\left\| x^{\frac{1}{2}} \langle x \rangle^{-1} \partial_t \psi_{0\pm} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C \langle t \rangle^{\gamma - 1} \|\widehat{\varphi}\|_{\mathbf{W}} .$$

Hence

$$\begin{aligned} & \left\| \langle \xi \rangle^{-3} \mathcal{V}^* (3t) \, x \overline{\psi_{0-}} \psi_{0+} \partial_t \psi_{0+} \right\|_{\mathbf{L}^2} \\ \le \;& C \left\| x^{\frac{1}{4}} \langle x \rangle^{\frac{1}{2}} \psi_{0\pm} \right\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| x^{\frac{1}{2}} \langle x \rangle^{-1} \partial_t \psi_{0\pm} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le \langle t \rangle^{\gamma-1} \|\widehat{\varphi}\|^3_{\mathbf{W}} . \end{aligned}$$

Thus from (6.4) we find

$$\partial_t \left(\langle \xi \rangle^{-3} \widehat{\varphi_\xi} - \frac{9\xi^3}{8i\Lambda(\xi)} e^{\frac{8}{9} it\Lambda(\xi)} I \right) = O\left(\langle t \rangle^{\gamma-1} \|\widehat{\varphi}\|^3_{\mathbf{W}} \right) .$$

Integrating we get

$$\left\| \langle \xi \rangle^{-3} \widehat{\varphi_\xi} - \frac{9\xi^3}{8i\Lambda(\xi)} e^{\frac{8}{9} it\Lambda(\xi)} I \right\|_{\mathbf{L}^2} \le C\varepsilon t^\gamma .$$

Since

$$\begin{aligned} \|I\|_{\mathbf{L}^2} &= C \left\| \mathcal{D}_{-3} \mathcal{V}^* (-3t) \, x \overline{\psi_{0+}} \psi_{0-}^2 + \mathcal{D}_3 \mathcal{V}^* (3t) \, x \overline{\psi_{0-}} \psi_{0+}^2 \right\|_{\mathbf{L}^2} \\ &\le C \left\| x \psi_{0\pm}^3 \right\|_{\mathbf{L}^2} \le C \|\widehat{\varphi}\|^3_{\mathbf{W}} , \end{aligned}$$

then we obtain

$$\left\| \langle \xi \rangle^{-3} \widehat{\varphi_\xi} \right\|_{\mathbf{L}^2} \le C \|I\|_{\mathbf{L}^2} + C\varepsilon t^\gamma < C\varepsilon t^\gamma .$$

Thus we get $\|u\|_{\mathbf{X}_T} < C\varepsilon$. Lemma 6.2 is proved. □

7. Proof of Theorem 1.2

By Lemma 6.2 we see that a priori estimate $\|u\|_{\mathbf{X}_T} \le C\varepsilon$ is true for all $T > 0$. Therefore the global existence of solutions of the Cauchy problem (1.3) satisfying the estimate

$$\|u\|_{\mathbf{X}_\infty} \le C\varepsilon$$

follows by a standard continuation argument from the local existence Theorem 6.1.

Now we turn to the proof of asymptotic formula (1.6) for the solutions u of the Cauchy problem (1.3). By the representation for $j = 1, 2$

$$\partial_x^j u(t) = \mathcal{D}_t \left(M\psi_{+j} + \overline{M}\psi_{-j} + r_j \right),$$

where $\psi_{+j} = \mathcal{V}(t)(i\xi)^j \widehat{\varphi}$, $\psi_{-j} = \mathcal{V}(-t)\mathcal{D}_{-1}(i\xi)^j \widehat{\varphi}$, $r_j = \mathcal{W}(t)(i\xi)^j \widehat{\varphi}$, and Lemma 2.1 we have

$$\psi_{\pm j} = \pm A(\pm t) x^{\frac{j}{2}} \widehat{\varphi}\left(\pm x^{\frac{1}{2}}\right) + O\left(t^{\frac{1-2j}{6}}\right).$$

Therefore

$$\partial_x^j u(t) = \mathcal{D}_t \left(MA(t) x^{\frac{j}{2}} \widehat{\varphi}\left(x^{\frac{1}{2}}\right) + \overline{M}A(-t) x^{\frac{j}{2}} \widehat{\varphi}\left(-x^{\frac{1}{2}}\right)\right) + O\left(t^{-\frac{1+j}{3}}\right).$$

We need to compute the asymptotics of the function $\widehat{\varphi}(t,\xi)$. As in the proof of Lemma 6.2 we get

$$\begin{aligned} & \partial_t \left(\xi^j \widehat{\varphi}(t,\xi) \Psi_{\pm}(t,\xi) \right) \\ = \ & -\frac{\lambda \xi^2}{18|t|} e^{\frac{8}{9}it\Lambda(\xi)} \mathcal{D}_3 \overline{\widehat{\varphi}(t,-\xi)} \widehat{\varphi}^2(t,\xi) \xi^j \Psi_{\pm}(t,\xi) \\ & + O\left(|t|^{2\nu - \frac{13}{12}} \|\widehat{\varphi}\|_{\mathbf{W}}^3 \right), \end{aligned}$$

for $j = 1, 2$, $|\xi| \le t^{\nu}$, $\nu < \frac{1}{24}$. For

$$y_j(t,\xi) = \xi^j \widehat{\varphi}(t,\xi) \left(\theta_+ \Psi_+(t,\xi) + \theta_- \Psi_-(t,\xi) \right),$$

integrating in time, we obtain

$$\begin{aligned} & y_j(t,\xi) - y_j(s,\xi) \\ = \ & C\xi^2 \int_s^t e^{\frac{8}{9}i\tau\Lambda(\xi)} \mathcal{D}_3 \overline{\widehat{\varphi}(\tau,-\xi)} \widehat{\varphi}^2(\tau,\xi) \xi^j \Psi_{\pm}(\tau,\xi) \frac{d\tau}{\tau} \\ = \ & C \left| \int_s^t e^{\frac{8}{9}i\tau\Lambda(\xi)} \partial_\tau \left(\tau^{-1} \mathcal{D}_3 \overline{\widehat{\varphi}(\tau,-\xi)} \widehat{\varphi}^2(\tau,\xi) \xi^{j-1} \Psi_{\pm}(\tau,\xi) \right) d\tau \right|. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & \| y_j(t) - y_j(s) \|_{\mathbf{L}^\infty} \\ \le \ & C\varepsilon^3 \int_s^t \tau^{-\frac{13}{12} + 2\nu + \gamma} + \tau^{-1-2\nu+\gamma} d\tau \le C\varepsilon^3 s^{-\frac{1}{12} + 2\nu + \gamma} \end{aligned}$$

for any $t > s > 0$, $\frac{1}{48} > \nu > \frac{\gamma}{2}$. Therefore there exists a unique final state $y_{j,F} \in \mathbf{L}^\infty$ such that

$$\left\| y_j(t) - y_{j,F} \right\|_{\mathbf{L}^\infty} \le C\varepsilon^3 t^{-\frac{1}{12}+2\nu+\gamma} \tag{7.1}$$

for all $t > 0$. Since

$$\Psi_\pm(t,\xi) = \exp\left(\pm \frac{i\lambda\xi^2}{2} \int_1^t \left(\frac{1}{2} \left| \widehat{\varphi}(\tau,\xi) \right|^2 + \left| \widehat{\varphi}(\tau,-\xi) \right|^2 \right) \frac{d\tau}{\tau} \right)$$

for

$$\begin{aligned} \int_1^t \left(\frac{1}{2} \left| \widehat{\varphi}(\tau,\xi) \right|^2 + \left| \widehat{\varphi}(\tau,-\xi) \right|^2 \right) \frac{d\tau}{\tau} &= \int_1^t Y(\tau,\xi) \frac{d\tau}{\tau} \\ &= Y_{0,F}(\xi) \log t + \Phi(t). \end{aligned} \tag{7.2}$$

where

$$Y(\tau,\xi) = \frac{1}{2} \left| y_0(\tau,\xi) \right|^2 + \left| y_0(\tau,-\xi) \right|^2$$

and $Y_{0,F}(\xi) = \frac{1}{2} \left| y_{0,F}(\xi) \right|^2 + \left| y_{0,F}(-\xi) \right|^2$. We study the asymptotics in time of the remainder term $\Phi(t)$. We have

$$\begin{aligned} \Phi(t) - \Phi(s) &= \int_s^t \left(\left| Y(\tau) \right|^2 - \left| Y(t) \right|^2 \right) \frac{d\tau}{\tau} \\ &\quad + \left(\left| Y(t) \right|^2 - \left| Y_{0,F} \right|^2 \right) \log \frac{t}{s}. \end{aligned}$$

By (7.1) we obtain $\left\| \Phi(t) - \Phi(s) \right\|_{\mathbf{L}^\infty} \le C\varepsilon^3 s^{-\delta}$ for any $t > s > 0$, where $\delta = \frac{1}{12} + \gamma$. Hence there exists a unique real-valued function $\Phi_F \in \mathbf{L}^\infty$ such that

$$\left\| \Phi(t) - \Phi_F \right\|_{\mathbf{L}^\infty} \le C\varepsilon^3 t^{-\delta} \tag{7.3}$$

for all $t > 0$. Representation (7.2) and estimate (7.3) yield

$$\left\| \Psi_\pm(t,\xi) - \exp\left(\pm \frac{i\lambda\xi^2}{2} \left| Y_{0,F} \right|^2 \log t \pm \frac{i\lambda\xi^2}{2} \Phi_F \right) \right\|_{\mathbf{L}^\infty} \le C t^{-\delta}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\left\|\widehat{\varphi}\left(t,\xi\right)-y_{0,F}\theta_{\pm}\Psi_{\mp}\left(t,\xi\right)\right\|_{\mathbf{L}^{\infty}}=\left\|y_{0}\left(t\right)-y_{0,F}\theta_{\pm}\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\delta}$$

and

$$\left\|y_{0,F}\theta_{\pm}\Psi_{\mp}-y_{0,F}\theta_{\pm}e^{\left(\mp\frac{i\lambda\xi^{2}}{2}\left|Y_{F}\right|^{2}\log t\mp\frac{i\lambda\xi^{2}}{2}\Phi_{F}\right)}\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\delta}.$$

Therefore we obtain the estimate

$$\left\|\widehat{\varphi}\left(t,\xi\right)-y_{0,F}\theta_{\pm}e^{\left(\mp\frac{i\lambda\xi^{2}}{2}\left(\frac{1}{2}\left|y_{0,F}(\xi)\right|^{2}+\left|y_{0,F}(-\xi)\right|^{2}\right)\log t\mp\frac{i\lambda\xi^{2}}{2}\Phi_{F}\right)}\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\delta}.$$

Using the factorization of $\mathcal{U}\left(t\right)$ we have

$$\begin{aligned}
&\Bigg\|\partial_{x}^{j}u\left(t\right)-\mathcal{D}_{t}M^{+}A\left(t\right)x^{\frac{j}{2}}\\
&\quad\times\left(y_{0,F}\left(x^{\frac{1}{2}}\right)e^{-\frac{i\lambda x}{2}\Phi_{F}+\left(-\frac{i\lambda x}{2}\left(\frac{1}{2}\left|y_{0,F}\left(x^{\frac{1}{2}}\right)\right|^{2}+\left|y_{0,F}\left(-x^{\frac{1}{2}}\right)\right|^{2}\right)\log t\right)}\right)\\
&\quad-\mathcal{D}_{t}M^{-}A\left(-t\right)x^{\frac{j}{2}}\\
&\quad\times\left(y_{0,F}\left(-x^{\frac{1}{2}}\right)e^{\frac{i\lambda x}{2}\Phi_{F}+\left(\frac{i\lambda x}{2}\left(\frac{1}{2}\left|y_{0,F}\left(-x^{\frac{1}{2}}\right)\right|^{2}+\left|y_{0,F}\left(x^{\frac{1}{2}}\right)\right|^{2}\right)\log t\right)}\right)\Bigg\|_{\mathbf{L}^{\infty}}\\
&\leq Ct^{-\frac{1}{2}}\\
&\quad\times\left\|\xi^{j}\left(\widehat{\varphi}\left(t,\xi\right)-y_{0,F}\theta_{\pm}e^{\left(\mp\frac{i\lambda\xi^{2}}{2}\left(\frac{1}{2}\left|y_{0,F}(\xi)\right|^{2}+\left|y_{0,F}(-\xi)\right|^{2}\right)\log t\mp\frac{i\lambda\xi^{2}}{2}\Phi_{F}\right)}\right)\right\|_{\mathbf{L}^{\infty}}\\
&\leq Ct^{-\frac{1}{2}-\frac{1}{12}+\gamma}.
\end{aligned}$$

This completes the proof of asymptotics (1.6). Theorem 1.2 is proved.

Acknowledgments. The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers JP15H03630, JP25220702. The work of P.I.N. is partially supported by CONACYT and PAPIIT project IN100113.

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(Received December 7, 2015)

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