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Proof of Unsolvability of q-Bessel Equation Using Valuations

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Abstract. In this paper, we prove unsolvability of the q-Bessel equation associated with one of the q-Bessel functions, $J_\nu^{(3)}$, using the notion of the difference field extension of valuation ring type.

1. Introduction

The unsolvability of the Bessel equation with the value of the parameter ν satisfying $\nu - 1/2 \notin \mathbb{Z}$ is well-known. On the other hand, we know the unsolvability of the q-Bessel equation associated with one of the q-Bessel functions only when the value of ν is a rational number, for a transcendental number q. Here the q-Bessel function is written as

$$\begin{aligned} J_\nu^{(3)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \, {}_1\varphi_1(0;q^{\nu+1};q,qx^2) \\ &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n\geq 0} \frac{(-1)^n q^{n(n-1)/2}}{(q^{\nu+1},q;q)_n} (qx^2)^n, \end{aligned}$$

and satisfies the q-Bessel equation,

$$g_\nu(qx) + (x^2/4 - q^\nu - q^{-\nu}) g_\nu(x) + g_\nu(xq^{-1}) = 0, \quad g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2};q^2).$$

The q-Bessel functions are introduced in the book [5] by G. Gasper and M. Rahman, and they are q-difference counterparts of the Bessel function. In this paper, we study the unsolvability of the above q-Bessel equation with an arbitrary value of ν.

The unsolvability of a differential equation means that any non-trivial solution cannot be contained in a differential field extension over $\mathbb{C}(x)$ obtained by successive adjoining algebraic elements, primitive functions

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and/or exponential of primitive functions. This extension is called a Liouville extension (see [6, 12]). For the unsolvability of the Bessel equation, Galois-theoretical proof is well-known. E. R. Kolchin proved in his paper [7] that the Galois group of the Bessel equation is $\mathrm{SL}_2(\mathbb{C})$ iff. $\nu - 1/2 \notin \mathbb{Z}$. In the case $\nu - 1/2 \in \mathbb{Z}$, we find a fundamental system of solutions which are exponential over $\mathbb{C}(x)$ (cf. [7]).

There is another approach to unsolvability, which uses valuation rings. This idea originated with M. Rosenlicht [13]. The proof of unsolvability of the Bessel equation which uses valuation rings will be seen in K. Nishioka's book [9] written in Japanese. In that proof, she defined a differential field extension of valuation ring type, which is a generalization of the Liouville extension, and proved that there is no non-trivial solution contained in such an extension. By the general results on the extension of valuation ring type, we only have to prove that there is no algebraic solution to the Riccati equation associated with the Bessel equation.

For difference equations, C. H. Franke developed his Galois theory of linear difference equations and defined a difference counterpart of the Liouville extension in his papers [3, 4]. In this paper, the unsolvability of a difference equation means that any non-trivial solution cannot be contained in Franke's Liouvillian extension over $\mathbb{C}(x)$. The author proved in his paper [11] that the above q-Bessel equation is unsolvable when the value of ν is a rational number, for a transcendental number q. As in the case of differential equations, he developed a general result by using valuation rings and proved that there is no algebraic solution to the iterated difference Riccati equations associated with the q-Bessel equation. A difference Riccati equation is a equation of the form,

$$y(\tau(x)) = \frac{a(x)y(x) + b(x)}{c(x)y(x) + d(x)}, \quad \tau(x) = x + 1, qx, \text{etc.}$$

However, there was a technical problem to prove the non-existence of algebraic solutions for an arbitrary value of ν.

The solution we adopt here is to use the fact that $\alpha = q^\nu + q^{-\nu}$ has only finitely many zeros and poles when it is algebraic over the rational function field $\overline{\mathbb{Q}}(q)$. We will study the algebraic independence of $g_\nu(qx)$ and $g_\nu(x)$ over Franke's Liouvillian extension, and prove that the q-Bessel equation with an arbitrary value of the parameter ν is unsolvable, for a transcendental number q.

We use the notion of the difference field extension of valuation ring type, which was introduced in the author's paper [11] and advanced in the paper [10] by K. Nishioka and the author. In those papers, almost all the difference fields are assumed to be inversive. Since such requirements are not essential and they restrict the functions contained in the extension of valuation ring type, we will eliminate them.

Notation. Throughout the paper every field is of characteristic zero. When K is a field and τ is an isomorphism of K into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. We call τ the (transforming) operator and K the underlying field. For a difference field $\mathcal{K}$, K often denotes its underlying field. For $a \in K$, the element $\tau^n a \in K$ ($n \in \mathbb{Z}$), if it exists, is called the n-th transform of a and is sometimes denoted by a_n. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. For an algebraic closure $\overline{K}$ of K, the transforming operator τ is extended to an isomorphism $\overline{\tau}$ of $\overline{K}$ into itself, not necessarily in a unique way. We call the difference field $(\overline{K}, \overline{\tau})$ an algebraic closure of $\mathcal{K}$. For $p \in \mathbb{Z}_{>0}$, $\mathcal{K}^{(p)}$ denotes the difference field (K, τ^p). For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if K'/K is a field extension and $\tau'|_K = \tau$. In this case, we say that $\mathcal{K}'$ is a difference overfield of $\mathcal{K}$ and that $\mathcal{K}$ is a difference subfield of $\mathcal{K}'$. For brevity we sometimes use (K, τ') instead of $(K, \tau'|_K)$. We define a difference intermediate field in the proper way. Let $\mathcal{K}$ be a difference field, $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$ and B a subset of L. The difference subfield $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ of $\mathcal{L}$ is defined to be the difference field $(K(B, \tau B, \tau^2 B, \dots), \tau)$ and is denoted by $\mathcal{K}\langle B\rangle$ for brevity. A solution of a difference equation over $\mathcal{K}$ is defined to be an element of some difference overfield of $\mathcal{K}$ which satisfies the equation.

When R is a ring and τ is an isomorphism of R into itself, the pair $\mathcal{R} = (R, \tau)$ is called a difference ring. Let $\mathcal{R} = (R, \tau)$ and $\mathcal{R}' = (R', \tau')$ be difference rings. A homomorphism ϕ of R to R' is called a difference homomorphism of $\mathcal{R}$ to $\mathcal{R}'$ if $\phi\tau = \tau'\phi$ (cf. the books [2, 8]).

Let F/K be an algebraic function field of one variable. A place P of F/K is the maximal ideal of some valuation ring of F/K. The valuation ring and the normalized discrete valuation associated with P is denoted by $\mathcal{O}_P$ and v_P, respectively. A discrete valuation of F/K is a function $v\colon F \to \mathbb{Z} \cup \{\infty\}$ with the following properties.

(i) $v(x) = \infty \iff x = 0$.

(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in F$.

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$.

(iv) There exists an element $z \in F^\times$ with $v(z) \neq 0$ ($v(z) = 1$ for a normalized discrete valuation).

(v) $v(a) = 0$ for all $0 \neq a \in K$.

For a rational function field $K(x)/K$, P_α, $\alpha \in K$, denotes the place which has the prime element $x - \alpha$.

In Section 2, we define a notation representing difference Riccati equations. In Section 3 and 4, we define the refined difference field extension of valuation ring type and study a solution of a difference Riccati equation in it. In the final section, we study the unsolvability of the q-Bessel equation.

2. Difference Riccati Equation

For a second-order linear difference equation,

$$y_2 + ay_1 + by = 0,$$

by setting $z = y_1/y$, we obtain the following first-order difference equation,

$$z_1 = \frac{-az - b}{z}.$$

We call this the difference Riccati equation associated with the above equation. By iterations, we can express z_i in terms of z such as

$$z_2 = \frac{(a_1 a - b_1)z + a_1 b}{-az - b}.$$

Here, we introduce a notation about those iterations.

Let $\mathcal{K} = (K, \tau)$ be a difference field, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau^{i-1}A)(\tau^{i-2}A) \cdots (\tau A)A \quad (i = 1, 2, \dots).$$

In this paper, $\mathrm{Eq}(A,i)/\mathcal{K}$ denotes the equation over $\mathcal{K}$,

$$y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}.$$

We easily see the following.

LEMMA 1. *If f is a solution of $Eq(A,k)/\mathcal{K}$ in a difference field extension $\mathcal{L}/\mathcal{K}$, $f \in \mathcal{L}$ is also a solution of $Eq(A,ki)/\mathcal{K}$ $(i = 1,2,\dots)$.*

LEMMA 2. *Let $B = A_k$ and $B_i = (\tau^{k(i-1)}B)(\tau^{k(i-2)}B)\cdots B$ $(i = 1,2,\dots)$. Then $B_i = A_{ki}$.*

LEMMA 3. *For any $k,l,m \in \mathbb{Z}_{>0}$,*

$$\begin{aligned} &f \in \mathcal{L} \text{ is a solution of } Eq(A_k, lm)/\mathcal{K}^{(k)} \\ \Longleftrightarrow &f \in \mathcal{L}^{(l)} \text{ is a solution of } Eq(A_{kl}, m)/\mathcal{K}^{(kl)}, \end{aligned}$$

where $\mathcal{L}$ is a difference overfield of $\mathcal{K}^{(k)}$.

3. Difference Field Extension of Valuation Ring Type

The following is the definition of the difference field extension of valuation ring type.

DEFINITION 4. Let $\mathcal{N}/\mathcal{K}$ be a difference field extension, where $\mathcal{N} = (N,\tau)$. We say that $\mathcal{N}/\mathcal{K}$ is of valuation ring type if there exists a chain of difference field extension,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N},$$

such that each $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the following.

(i) K_i/K_{i-1} is algebraic.

(ii) K_i/K_{i-1} is an algebraic function field of one variable, and there exists a place P of K_i/K_{i-1} such that $\tau^j P \subset P$ for some $j \in \mathbb{Z}_{>0}$.

REMARK. The above definition differs from the one in the author's former paper [11]. In that paper, the second condition is the following.

> $\mathcal{K}_i$ and $\mathcal{K}_{i-1}$ are inversive, K_i/K_{i-1} is an algebraic function field of one variable, and there exists a valuation ring $\mathcal{O}$ of K_i/K_{i-1} such that $\tau^j\mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$.

Since $\mathcal{K}_i$ and $\mathcal{K}_{i-1}$ are inversive, $\tau^j\mathcal{O} \subset \mathcal{O}$ implies $\tau^j\mathcal{O} = \mathcal{O}$. Hence the maximal ideal P of $\mathcal{O}$ satisfies $\tau^j P = P$. As a result, the former extension is of valuation ring type in the sense here.

In the following proposition and corollary, we introduce some elementary examples of difference field extensions of valuation ring type.

Proposition 5. *Let $\mathcal{K}$ be a difference field, and let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau^{i-1}A)(\tau^{i-2}A)\cdots A \quad (i = 1, 2, \dots).$$

Suppose $b^{(k)} = 0$ or $c^{(k)} = 0$ for some $k \in \mathbb{Z}_{>0}$. Let f be a solution of $Eq(A,1)/\mathcal{K}$ transcendental over K, and let $\mathcal{L} = (L, \tau) = \mathcal{K}\langle f\rangle$. Then we obtain the following.

(i) *L/K is an algebraic function field of one variable.*

(ii) *There is a place P of L/K such that $\tau^k P \subset P$.*

(iii) *$\mathcal{L}/\mathcal{K}$ is of valuation ring type.*

Proof. The proof is essentially the same as the proof of Proposition 5 in [11], except that we take

$$P = \{p/q \mid p, q \in K[g],\ \deg q - \deg p > 0\},$$

where $g = f$ if $c^{(k)} = 0$ or $g = 1/f$ if $c^{(k)} \neq 0$. $\square$

Corollary 6. *Let $\mathcal{K}$ be a difference field, and f a solution of $y_1 = ay + b$, $a, b \in K$, $a \neq 0$, transcendental over K. Then $\mathcal{K}\langle f\rangle/\mathcal{K}$ is of valuation ring type.*

Proof. Letting

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

we find the required by Proposition 5. □

We will show a theory of reduction.

Lemma 7. *Let $\mathcal{L}/\mathcal{K}$ be a difference field extension such that L/K is an algebraic function field of one variable and there exists a place P of L/K satisfying $\tau_L^j P \subset P$ for some $j \in \mathbb{Z}_{>0}$, where $\mathcal{L} = (L, \tau_L)$. Let $\overline{\mathcal{L}} = (\overline{L}, \tau)$ be an algebraic closure of $\mathcal{L}$ and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{L}}$. Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau^{i-1}A)(\tau^{i-2}A)\cdots A \quad (i = 1, 2, \dots).$$

Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for $i = 1, 2, \dots$. If Eq$(A,1)/\mathcal{K}$ has a solution $f \in \overline{\mathcal{L}}$, then Eq$(A,k)/\mathcal{K}$ has a solution in $\overline{\mathcal{K}}$ for some $k \in \mathbb{Z}_{>0}$.

Proof. The solution f satisfies $(\tau f)(cf+d) = af+b$. We may suppose $f \notin \overline{K}$. Then f is transcendental over $\overline{K}$. Since we supposed $c \neq 0$, we find $\tau f = (af+b)/(cf+d) \in K(f) \subset L(f)$. Let $\mathcal{M} = \mathcal{L}\langle f \rangle$. Since L/K is an algebraic function field of one variable and $M = L(f) \subset \overline{L}$, M/K and $M\overline{K}/\overline{K}$ are algebraic function fields of one variable. Choose a place P of L/K such that $\tau^j P \subset P$ for some $j \in \mathbb{Z}_{>0}$.

Step 1. We prove that there exists a place P' of $M\overline{K}/\overline{K}$ such that $P' \supset P$ and $\tau^k P' \subset P'$ for some $k \in \mathbb{Z}_{>0}$. Let $P_1, \dots, P_n$ $(n \geq 1)$ be all of the places of $M\overline{K}/\overline{K}$ such that $P_i \supset P$. Note that $\tau^j(M\overline{K})/\tau^j\overline{K}$ is an algebraic function field of one variable. Let $v_i = v_{P_i}|_{\tau^j(M\overline{K})}$, then v_i is a discrete valuation of $\tau^j(M\overline{K})/\tau^j\overline{K}$. In fact, for a prime element $t \in L$ of P, it follows that $\tau^j t \in \tau^j P \subset P \subset P_i$, which implies $v_i(\tau^j t) = v_{P_i}(\tau^j t) > 0$.

Let $\check{P}_i = P_i \cap \tau^j(M\overline{K})$, then $\check{P}_i$ is the place of $\tau^j(M\overline{K})/\tau^j\overline{K}$ associated with v_i. We find

$$\tau^j P \subset P \cap \tau^j L \subset P_i \cap \tau^j(M\overline{K}) = \check{P}_i.$$

Since $\tau^j P_1, \dots, \tau^j P_n$ are all of the places of $\tau^j(M\overline{K})/\tau^j\overline{K}$ such that $\tau^j P_i \supset \tau^j P$, we obtain the sequence,

$$\check{P}_1 = \tau^j P_{l_1},\ \check{P}_{l_1} = \tau^j P_{l_2},\ \dots \quad (1 \le l_i \le n).$$

Let $l_0 = 1$. For any $i \in \mathbb{Z}_{>0}$, $\tau^j P_{l_i} = \check{P}_{l_{i-1}} \subset P_{l_{i-1}}$. Choose $m, m' \in \mathbb{Z}_{\ge 0}$ such that $l_m = l_{m'}$ and $m < m'$. Then we find

$$\tau^{(m'-m)j} P_{l_{m'}} \subset \tau^{(m'-m-1)j} P_{l_{m'-1}} \subset \cdots \subset P_{l_m} = P_{l_{m'}}.$$

Step 2. Let P' be the place in the previous step. Then $\tau^k P' \subset P'$ for some $k \in \mathbb{Z}_{>0}$. We prove $\tau^k \mathcal{O}_{P'} \subset \mathcal{O}_{P'}$. Assume $\tau^k \mathcal{O}_{P'} \not\subset \mathcal{O}_{P'}$. There exists $x \in \tau^k \mathcal{O}_{P'} \setminus \mathcal{O}_{P'}$, which satisfies $v_{P'}(x) < 0$. Let s be a prime element for $\tau^k P'$. By $s \in \tau^k P' \subset P'$, we obtain $v_{P'}(s) = n \ge 1$, and so

$$v_{P'}(x^n s) = n v_{P'}(x) + v_{P'}(s) \le -n + n = 0.$$

This implies $x^n s \notin P'$. On the other hand, we have $x \in \tau^k \mathcal{O}_{P'}$ and $s \in \tau^k P'$, which imply $x^n s \in \tau^k P' \subset P'$. We obtained a contradiction.

Step 3. Let t be a prime element for P', and let $e = v_{P'}(\tau^k t) \ge 1$. We prove that for any $x \in M\overline{K}$, $v_{P'}(\tau^k x) = e v_{P'}(x)$. We may suppose $x \ne 0$. Let $x = t^n u$, $n \in \mathbb{Z}$, $u \in \mathcal{O}_{P'}^{\times}$. By $u, u^{-1} \in \mathcal{O}_{P'}$, we obtain $\tau^k u, (\tau^k u)^{-1} = \tau^k u^{-1} \in \tau^k \mathcal{O}_{P'} \subset \mathcal{O}_{P'}$, and so $\tau^k u \in \mathcal{O}_{P'}^{\times}$. Hence

$$\begin{aligned} v_{P'}(\tau^k x) &= v_{P'}((\tau^k t)^n \tau^k u) \\ &= n v_{P'}(\tau^k t) + v_{P'}(\tau^k u) \\ &= en + 0 \\ &= e v_{P'}(x). \end{aligned}$$

Step 4. Let $\phi \colon M\overline{K} \to \overline{K}((t))$ be the embedding, and let $\phi(\tau^k t) = \sum_{i=e}^{\infty} r_i t^i$, $r_i \in \overline{K}$, $r_e \ne 0$. Then $\phi \colon (M\overline{K}, \tau^k) \to (\overline{K}((t)), \sigma)$ is a difference isomorphism, where

$$\sigma\left(\sum_{i=0}^{\infty} \alpha_i t^i\right) = \sum_{i=0}^{\infty} \tau^k(\alpha_i) \left(\sum_{l=e}^{\infty} r_l t^l\right)^i.$$

Since $f \in \overline{\mathcal{L}}$ is a solution of $\mathrm{Eq}(A,1)/\mathcal{K}$, $f \in \overline{\mathcal{L}}$ is also a solution of $\mathrm{Eq}(A,k)/\mathcal{K}$. Hence f satisfies

$$f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}.$$

If we assume $v_{P'}(f) < 0$, we find $v_{P'}(f_k) = ev_{P'}(f) < 0$. On the other hand, the above equation implies

$$ev_{P'}(f) + v_{P'}(f) \geq v_{P'}(f),$$

a contradiction. Hence we conclude $v_{P'}(f) \geq 0$.

Let $\phi(f) = \sum_{i=0}^{\infty} h_i t^i$, $h_i \in \overline{K}$. Then

$$\phi(f_k) = \sigma(\phi(f)) = \sum_{i=0}^{\infty} \tau^k(h_i) \left(\sum_{l=e}^{\infty} r_l t^l \right)^i .$$

Comparing the coefficients of t^0 of the equation,

$$\phi(f_k)(c^{(k)}\phi(f) + d^{(k)}) = a^{(k)}\phi(f) + b^{(k)},$$

we obtain

$$\tau^k(h_0)(c^{(k)}h_0 + d^{(k)}) = a^{(k)}h_0 + b^{(k)}.$$

This implies $h_0 \in \overline{\mathcal{K}}$ is a solution of $\mathrm{Eq}(A,k)/\mathcal{K}$. $\square$

THEOREM 8. *Let $\mathcal{K} = (K, \tau_K)$ be a difference field, and let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau_K^{i-1}A)(\tau_K^{i-2}A)\cdots A \quad (i = 1, 2, \dots).$$

Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for $i = 1, 2, \dots$. Let $k \in \mathbb{Z}_{>0}$. Suppose $Eq(A,k)/\mathcal{K}$ has a solution in a certain difference field extension $\mathcal{N}/\mathcal{K}$ of valuation ring type. Then $Eq(A,ki)/\mathcal{K}$ has a solution in $\overline{\mathcal{K}}$ for some $i \in \mathbb{Z}_{>0}$, where $\overline{\mathcal{K}}$ is the algebraic closure of $\mathcal{K}$ in an algebraic closure $\overline{\mathcal{N}}$ of $\mathcal{N}$.

PROOF. The proof is essentially the same as the proof of Theorem 2 in [11]. The place P plays the same role as the valuation ring $\mathcal{O}$ did. $\square$

The following lemma is used in the next section. Afterwards, we introduce Franke's Liouvillian extension (cf. the papers [3, 4]). We recall that $\mathcal{K}^{(p)}$ denotes the difference field (K, τ^p) for $\mathcal{K} = (K, \tau)$.

Lemma 9. *Let $\mathcal{N}/\mathcal{K}$ be a difference field extension, and let $p \in \mathbb{Z}_{>0}$. If $\mathcal{N}/\mathcal{K}$ is of valuation ring type, then $\mathcal{N}^{(p)}/\mathcal{K}^{(p)}$ is also of valuation ring type.*

Proof. By definition, there exists the chain of difference field extensions,

$$\mathcal{K} = \mathcal{K}_0 \subset \cdots \subset \mathcal{K}_n = \mathcal{N}.$$

We think of the chain,

$$\mathcal{K}^{(p)} = \mathcal{K}_0^{(p)} \subset \cdots \subset \mathcal{K}_n^{(p)} = \mathcal{N}^{(p)}.$$

In the case that $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies the condition (i) in Definition 4, namely K_i/K_{i-1} is algebraic, $\mathcal{K}_i^{(p)}/\mathcal{K}_{i-1}^{(p)}$ satisfies the same condition. In the case that $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies the condition (ii), K_i/K_{i-1} is an algebraic function field of one variable, and there exists a place P of K_i/K_{i-1} such that $\tau^j P \subset P$ for some $j \in \mathbb{Z}_{>0}$. Then $(\tau^p)^j P = (\tau^j)^p P \subset P$, which implies that $\mathcal{K}_i^{(p)}/\mathcal{K}_{i-1}^{(p)}$ satisfies the condition (ii). Hence $\mathcal{N}^{(p)}/\mathcal{K}^{(p)}$ is of valuation ring type. $\square$

In the following definition, the symbol $*$ is used. For a inversive difference field $\mathcal{K}$ and an element e, $\mathcal{K}\langle e\rangle^*$ denotes the difference overfield of $\mathcal{K}$ whose underlying field is $K(\ldots, e_{-2}, e_{-1}, e, e_1, e_2, \ldots)$. It is called the inversive closure of $\mathcal{K}\langle e\rangle$.

Definition 10. Let $\mathcal{N}/\mathcal{K}$ be a difference field extension of inversive difference fields. We say that $\mathcal{N}/\mathcal{K}$ is a generalized Liouvillian extension (GLE) if there exists a chain of extensions of inversive difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \cdots \subset \mathcal{K}_n = \mathcal{N},$$

such that for each $i = 1, 2, \ldots, n$, $\mathcal{K}_i = \mathcal{K}_{i-1}\langle e^{(i)}\rangle^*$, where $e^{(i)}$ satisfies one of the following.

(i) $e^{(i)}$ is algebraic over K_{i-1},

(ii) $e_1^{(i)} = e^{(i)} + \beta$ for some $\beta \in K_{i-1}$,

(iii) $e_1^{(i)} = \alpha e^{(i)}$ for some $\alpha \in K_{i-1}$.

Let $p \in \mathbb{Z}_{>0}$. We say that $\mathcal{N}/\mathcal{K}$ is a pLE if $\mathcal{N}^{(p)}/\mathcal{K}^{(p)}$ is a GLE.

LEMMA 11. *Let $\mathcal{N}/\mathcal{K}$ be a difference field extension of inversive difference fields. If $\mathcal{N}/\mathcal{K}$ is a GLE, then $\mathcal{N}/\mathcal{K}$ is of valuation ring type.*

PROOF. This is proved in the same way as Lemma 6 in [11]. □

4. Algebraic Independence

In this section, we will show a theory of algebraic independence for solutions of a system of linear difference equations.

Let $\mathcal{K} = (K, \tau_K)$ be a difference field, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau_K^{i-1} A)(\tau_K^{i-2} A) \cdots A \quad (i = 1, 2, \dots).$$

Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for $i = 1, 2, \dots$.

DEFINITION 12. Let $\mathcal{M} = (M, \tau_M)$ be a difference overfield of $\mathcal{K}$, and $R = M[Y, Z]$ a polynomial ring. We define the homomorphism $T_{\mathcal{M}} \colon R \to R$ by $T_{\mathcal{M}}|_M = \tau_M$ and

$$\begin{pmatrix} T_{\mathcal{M}} Y \\ T_{\mathcal{M}} Z \end{pmatrix} = A \begin{pmatrix} Y \\ Z \end{pmatrix}.$$

Since we supposed $\det A \neq 0$, it follows that $M[T_{\mathcal{M}} Y, T_{\mathcal{M}} Z] = M[Y, Z]$. Hence $T_{\mathcal{M}} Y$ and $T_{\mathcal{M}} Z$ are algebraically independent over M, which implies that $T_{\mathcal{M}}$ is injective. By

$$\begin{pmatrix} T_{\mathcal{M}}^i Y \\ T_{\mathcal{M}}^i Z \end{pmatrix} = A_i \begin{pmatrix} Y \\ Z \end{pmatrix} \quad (i = 1, 2, \dots) \tag{1}$$

and $\det A_i \neq 0$, we find that $T^i_{\mathcal{M}} Y$ and $T^i_{\mathcal{M}} Z$ are algebraically independent over M.

LEMMA 13. *Let $\mathcal{M} = (M, \tau_M)$ be a difference overfield of $\mathcal{K}$, $R = M[Y, Z]$ a polynomial ring, and let $P \in R \setminus M$ satisfy $T_{\mathcal{M}} P = \omega P$, $\omega \in M$. Then there exist $i \in \mathbb{Z}_{>0}$ and a solution $(f, g) \neq 0$ of the equation over $\mathcal{M}^{(i)}$,*

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = A_i \begin{pmatrix} y \\ z \end{pmatrix},$$

such that $P(f, g) = 0$.

PROOF. Let T denote $T_{\mathcal{M}}$ for brevity, and let $P = P_1^{r_1} \dots P_n^{r_n}$ be an irreducible decomposition of P. Then

$$\omega P = TP = (TP_1)^{r_1} \cdots (TP_n)^{r_n}.$$

Hence $TP_1, \dots, TP_n$ are the irreducible components of P. This implies that there exists $i \in \mathbb{Z}_{>0}$ such that $T^i P_1 = \omega_1 P_1$, $\omega_1 \in M^{\times}$. Since (P_1) is a prime ideal, $R/(P_1)$ is an integral domain. Let L be its quotient field. Note that L/M is a field extension. Let $\tau \colon R/(P_1) \to R/(P_1)$ be the homomorphism such that $\overline{Q} \mapsto \overline{T^i Q}$. We will show that τ is injective. Suppose $\overline{T^i Q} = 0$, namely $T^i Q \in (P_1)$. There exists $D \in R$ such that $T^i Q = DP_1$. By $T^i P_1 = \omega_1 P_1$, we find

$$\omega_1 T^i Q = D T^i P_1,$$

and so

$$\omega_1^{-1} D = \frac{T^i Q}{T^i P_1} \in (\tau_M^i M)(T^i Y, T^i Z).$$

We also find

$$\omega_1^{-1} D \in R = M[Y, Z] \subset M[T^i Y, T^i Z].$$

Hence it follows that $\omega_1^{-1} D \in (\tau_M^i M)[T^i Y, T^i Z]$, which implies that $T^i E = \omega_1^{-1} D$ for some $E \in R$. From the above equations, we obtain

$$T^i Q = \omega_1^{-1} D T^i P_1 = (T^i E)(T^i P_1) = T^i (E P_1),$$
$$Q = E P_1 \in (P_1).$$

This implies $\overline{Q} = 0$, and that τ is injective.

Extend τ to the quotient field L. Then τ is an isomorphism of L into itself. Let $\mathcal{L} = (L, \tau)$, which is a difference overfield of $\mathcal{M}^{(i)}$. By $\overline{P}_1 = 0$, we find

$$P(\overline{Y}, \overline{Z}) = \overline{P} = \overline{P}_1^{r_1} \cdots \overline{P}_n^{r_n} = 0.$$

From the equation (1), it follows that

$$\begin{pmatrix} \overline{T^i Y} \\ \overline{T^i Z} \end{pmatrix} = A_i \begin{pmatrix} \overline{Y} \\ \overline{Z} \end{pmatrix},$$

which yields

$$\begin{pmatrix} \tau\overline{Y} \\ \tau\overline{Z} \end{pmatrix} = A_i \begin{pmatrix} \overline{Y} \\ \overline{Z} \end{pmatrix}.$$

Hence $(\overline{Y}, \overline{Z}) \in \mathcal{L}^2$ is a solution of the equation over $\mathcal{M}^{(i)}$,

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = A_i \begin{pmatrix} y \\ z \end{pmatrix}$$

Finally, we will show $(\overline{Y}, \overline{Z}) \neq 0$. Assume $(\overline{Y}, \overline{Z}) = 0$. Then we obtain $P_1 \mid Y$ and $P_1 \mid Z$, which imply $P_1 \in M$, a contradiction. □

THEOREM 14. *Suppose that for any $i \in \mathbb{Z}_{>0}$, $Eq(A_i, 1)/\mathcal{K}^{(i)}$ has no solution algebraic over K. Let $\mathcal{U} = (U, \tau)$ be a difference overfield of $\mathcal{K}$, and $(f, g) \neq 0$ a solution in $\mathcal{U}$ of the equation over $\mathcal{K}$,*

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}.$$

Let $\mathcal{N}/\mathcal{K}$ be a difference field extension in $\mathcal{U}$ of valuation ring type. Then f and g are algebraically independent over N.

PROOF. Assume that f and g are algebraically dependent over N. Since they satisfy

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \quad b \neq 0,\, c \neq 0,$$

and $(f,g)\neq 0$, we find $f\neq 0$ and $g\neq 0$.

(i) In the case that both f and g are algebraic over N, let $h=f/g$. Then

$$h_1=\frac{f_1}{g_1}=\frac{af+bg}{cf+dg}=\frac{ah+b}{ch+d}.$$

Hence $h\in\mathcal{U}$ is a solution of $\mathrm{Eq}(A,1)/\mathcal{K}$. Since h is algebraic over N, the extension $\mathcal{N}\langle h\rangle/\mathcal{N}$ is algebraic, which implies that $\mathcal{N}\langle h\rangle/\mathcal{K}$ is of valuation ring type. By Theorem 8, there exists $i\in\mathbb{Z}_{>0}$ such that $\mathrm{Eq}(A,i)/\mathcal{K}$ has a solution algebraic over K. Hence $\mathrm{Eq}(A_i,1)/\mathcal{K}^{(i)}$ has a solution algebraic over K (see §2). This contradicts the assumption of this theorem.

(ii) In the case $\mathrm{tr.deg}\,N(f,g)/N=1$, there exists an irreducible polynomial $P\in N[Y,Z]\setminus\{0\}$ such that $P(f,g)=0$. It follows that

$$\begin{aligned}(T_{\mathcal{N}}P)(f,g)&=(P^\tau(aY+bZ,cY+dZ))(f,g)\\&=P^\tau(af+bg,cf+dg)\\&=P^\tau(f_1,g_1)=\tau(P(f,g))\\&=0,\end{aligned}$$

where P^τ denotes the polynomial whose coefficients are the first transforms of corresponding coefficients of P. Hence we find $P\mid T_{\mathcal{N}}P$. By the definition of $T_{\mathcal{N}}$, we obtain $\deg T_{\mathcal{N}}P\leq\deg P$, and so $T_{\mathcal{N}}P=\omega P$, $\omega\in N$. Let $m=\deg P(\geq 1)$, and let F be the sum of the terms of degree m of P. It follows that $T_{\mathcal{N}}F=\omega F$. By Lemma 13, there exist $i\in\mathbb{Z}_{>0}$ and a solution $(\hat{f},\hat{g})\neq 0$ of the equation over $\mathcal{N}^{(i)}$,

$$\begin{pmatrix}y_1\\z_1\end{pmatrix}=A_i\begin{pmatrix}y\\z\end{pmatrix},$$

satisfying $F(\hat{f},\hat{g})=0$. Since F is homogeneous, we find

$$F(\hat{f}/\hat{g},1)=0, \tag{2}$$

where we note that $\hat{g}\neq 0$ is obtained from $c^{(i)}\neq 0$. Let $h=\hat{f}/\hat{g}$. The above equation (2) implies that h is algebraic over N. Since h satisfies

$$h_1=\frac{\hat{f}_1}{\hat{g}_1}=\frac{a^{(i)}\hat{f}+b^{(i)}\hat{g}}{c^{(i)}\hat{f}+d^{(i)}\hat{g}}=\frac{a^{(i)}h+b^{(i)}}{c^{(i)}h+d^{(i)}},$$

h is a solution of $\mathrm{Eq}(A_i, 1)/\mathcal{N}^{(i)}$ algebraic over N. By Lemma 9, we find that $\mathcal{N}^{(i)}/\mathcal{K}^{(i)}$ is of valuation ring type, and so $\mathcal{N}^{(i)}\langle h\rangle/\mathcal{K}^{(i)}$ is of valuation ring type. This implies that $\mathrm{Eq}(A_i, 1)/\mathcal{K}^{(i)}$ has a solution in $\mathcal{N}^{(i)}\langle h\rangle$. Hence by Theorem 8, there exists $j \in \mathbb{Z}_{>0}$ such that $\mathrm{Eq}(A_i, j)/\mathcal{K}^{(i)}$ has a solution algebraic over K. Therefore we conclude that $\mathrm{Eq}(A_{ij}, 1)/\mathcal{K}^{(ij)}$ has a solution algebraic over K, a contradiction.

In any case, we obtained a contradiction. Thus f and g are algebraically independent over N. □

Corollary 15. *Under the same conditions as in Theorem 14, let $\mathcal{N}/\mathcal{K}$ be a difference field extension in $\mathcal{U}$ such that $\mathcal{N}$ is inversive, and that $\mathcal{N}/\mathcal{K}^*$ is a pLE, where $\mathcal{K}^*$ is the inversive closure in $\mathcal{N}$. The underlying field of $\mathcal{K}^*$ is*

$$\{x \in N \mid \tau^i x \in K \textit{ for some } i\}.$$

Then f and g are algebraically independent over N.

Proof. Since $\mathcal{N}/\mathcal{K}^*$ is a pLE, $\mathcal{N}^{(p)}/\mathcal{K}^{*(p)}$ is a GLE, and is of valuation ring type. Let $B = A_p$ and

$$B_i = (\tau^{p(i-1)}B)(\tau^{p(i-2)}B)\cdots B \quad (i = 1, 2, \dots).$$

Then we obtain $B_i = A_{pi}$ (see §2). Note that for any $i \in \mathbb{Z}_{>0}$, $\mathrm{Eq}(B_i, 1)/\mathcal{K}^{(pi)}$ has no solution algebraic over K.

We will show that for any $i \in \mathbb{Z}_{>0}$, $\mathrm{Eq}(B_i, 1)/\mathcal{K}^{*(pi)}$ has no solution algebraic over K^*. Assume that there exists $i \in \mathbb{Z}_{>0}$ such that $\mathrm{Eq}(B_i, 1)/\mathcal{K}^{*(pi)}$ has a solution h algebraic over K^*. Since h is algebraic over K^*, there exists $P \in K^*[X] \setminus \{0\}$ such that $P(h) = 0$. The polynomial P satisfies $P^{\tau^{pij}} \in K[X] \setminus \{0\}$ for some $j \in \mathbb{Z}_{>0}$. By $P^{\tau^{pij}}(h_j) = 0$, we find that h_j is algebraic over K. Since h satisfies

$$h_1(c^{(pi)}h + d^{(pi)}) = a^{(pi)}h + b^{(pi)},$$

it follows that

$$h = \frac{-d^{(pi)}h_1 + b^{(pi)}}{c^{(pi)}h_1 - a^{(pi)}} \in K(h_1),$$

which implies that $h_{j-1} \in K(h_j)$ is algebraic over K. In the same way, we find that $h_j, h_{j-1}, \ldots, h$ are algebraic over K inductively. Hence $\mathrm{Eq}(B_i, 1)/\mathcal{K}^{(pi)}$ has a solution h algebraic over K. We obtained a contradiction.

Finally, we note that $(f, g) \neq 0$ is a solution in $\mathcal{U}^{(p)}$ of the equation over $\mathcal{K}^{*(p)}$,

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = B \begin{pmatrix} y \\ z \end{pmatrix}.$$

In fact, we obtain

$$\begin{pmatrix} \tau^p f \\ \tau^p g \end{pmatrix} = A_p \begin{pmatrix} f \\ g \end{pmatrix} = B \begin{pmatrix} f \\ g \end{pmatrix}$$

from

$$\begin{pmatrix} \tau f \\ \tau g \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix}.$$

Hence by Theorem 14, we conclude that f and g are algebraically independent over N. $\square$

5. q-Bessel Equation

In the book [5] by G. Gasper and M. Rahman, we find one of the q-Bessel functions, $J_\nu^{(3)}(x; q)$, and the equation,

$$g_\nu(qx) + (x^2/4 - q^\nu - q^{-\nu}) g_\nu(x) + g_\nu(xq^{-1}) = 0,$$

where $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2}; q^2)$. In this section, we study the algebraic independence of solutions. Note that the above equation can be rewritten as follows,

$$\begin{pmatrix} g_\nu(qx) \\ g_\nu(x) \end{pmatrix} = \begin{pmatrix} -x^2/4 + q^\nu + q^{-\nu} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_\nu(x) \\ g_\nu(xq^{-1}) \end{pmatrix}.$$

Let C be an algebraically closed field, and t a transcendental element over C. Let $q \in C^{\times}$, $\mathcal{K} = (C(t), \tau_q \colon t \mapsto qt)$, and

$$a = -\frac{t^2}{4} + \alpha, \quad \alpha \in C,$$

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(C(t)),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau_q^{i-1} A)(\tau_q^{i-2} A) \cdots A \quad (i = 1, 2, \dots).$$

We think of the equation over $\mathcal{K}$,

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}.$$

First of all, we will investigate relations between $a^{(i)}, b^{(i)}, c^{(i)}, d^{(i)}$. We obtain

$$A_2 = (\tau_q A) A = \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a - 1 & -a_1 \\ a & -1 \end{pmatrix},$$

and for $i \geq 2$,

$$\begin{aligned} A_i = (\tau_q A_{i-1}) A &= \begin{pmatrix} a_1^{(i-1)} & b_1^{(i-1)} \\ c_1^{(i-1)} & d_1^{(i-1)} \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a a_1^{(i-1)} + b_1^{(i-1)} & -a_1^{(i-1)} \\ a c_1^{(i-1)} + d_1^{(i-1)} & -c_1^{(i-1)} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} A_i = (\tau_q^{i-1} A) A_{i-1} &= \begin{pmatrix} a_{i-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{(i-1)} & b^{(i-1)} \\ c^{(i-1)} & d^{(i-1)} \end{pmatrix} \\ &= \begin{pmatrix} a_{i-1} a^{(i-1)} - c^{(i-1)} & a_{i-1} b^{(i-1)} - d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix}. \end{aligned}$$

Hence we obtain the following relations,

$$\begin{cases} a^{(i)} = a_{i-1} a^{(i-1)} - c^{(i-1)}, \\ b^{(i)} = -a_1^{(i-1)}, \\ c^{(i)} = a^{(i-1)}, \\ d^{(i)} = b^{(i-1)}, \end{cases} \qquad (i \geq 2)$$

$$\begin{cases} a^{(i)} = a_{i-1}a^{(i-1)} - a^{(i-2)}, \\ d^{(i)} = -a_1^{(i-2)}. \end{cases} \qquad (i \geq 3)$$

We will prove the following by induction,

$$a^{(i)} = (-1)^i \frac{q^{(i-1)i}}{4^i} t^{2i} + (\text{a polynomial of deg} \leq 2i-2).$$

Since $a^{(1)} = a = -t^2/4 + \alpha$ and

$$\begin{aligned} a^{(2)} = a_1 a - 1 &= \left(-\frac{q^2}{4}t^2 + \alpha\right)\left(-\frac{t^2}{4} + \alpha\right) - 1 \\ &= \frac{q^2}{4^2}t^4 + (\deg \leq 2), \end{aligned}$$

the result is true for $i = 1, 2$. Suppose $i \geq 3$ and that the result is true for smaller numbers. Then it follows that

$$\begin{aligned} a^{(i)} &= a_{i-1}a^{(i-1)} - a^{(i-2)} \\ &= \left(-\frac{q^{2(i-1)}}{4}t^2 + \alpha\right)\left((-1)^{i-1}\frac{q^{(i-2)(i-1)}}{4^{i-1}}t^{2i-2} + (\deg \leq 2i-4)\right) \\ &\quad - (\deg \leq 2i-4) \\ &= (-1)^i \frac{q^{i(i-1)}}{4^i}t^{2i} + (\deg \leq 2i-2), \end{aligned}$$

the required. Hence we find $a^{(i)} \neq 0$ and $\deg a^{(i)} = 2i$. By the above relations, we also find $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for $i = 1, 2, \ldots$.

Proposition 16. *Suppose that q is transcendental over $\mathbb{Q}$. Then for any $i \in \mathbb{Z}_{>0}$, $Eq(A_i, 1)/\mathcal{K}^{(i)}$ has no solution algebraic over K.*

Proof. Assume that there exists $i_0 \in \mathbb{Z}_{>0}$ such that $\mathrm{Eq}(A_{i_0}, 1)/\mathcal{K}^{(i_0)}$ has a solution f algebraic over K. Let $\mathcal{L} = \mathcal{K}^{(i_0)}\langle f \rangle = (L, \tau)$. Note that $\tau|_K = \tau_q^{i_0}$ and $\tau t = \tau_q^{i_0} t = q^{i_0} t$. We choose $i_1 \in \mathbb{Z}_{>0}$ in the following way. When q is transcendental over $\mathbb{Q}(\alpha)$, let $i_1 = 1$. When q is algebraic over $\mathbb{Q}(\alpha)$, $\mathbb{Q}(q, \alpha)/\mathbb{Q}$ is an algebraic function field of one variable. We find that $\overline{\mathbb{Q}}(q, \alpha)/\overline{\mathbb{Q}}$ is also an algebraic function field of one variable and an algebraic extension of $\overline{\mathbb{Q}}(q)/\overline{\mathbb{Q}}$. Since α is non-zero in this case, α has only finitely many zeros and poles $P^{(1)}, \dots, P^{(\nu)}$ in $\overline{\mathbb{Q}}(q, \alpha)/\overline{\mathbb{Q}}$. There exists $i_1 \in \mathbb{Z}_{>0}$ such that for any $i \geq i_1$,

$$P_{1^{1/i}} \neq P^{(1)} \cap \overline{\mathbb{Q}}(q), \dots, P^{(\nu)} \cap \overline{\mathbb{Q}}(q),$$

where $1^{1/i}$ is the primitive i-th root of unity.

Let $k = 3i_0 i_1$ and $k' = 3i_1$. Since $f \in \mathcal{L}$ is a solution of $\mathrm{Eq}(A_{i_0}, 1)/\mathcal{K}^{(i_0)}$, $f \in \mathcal{L}$ is a solution of $\mathrm{Eq}(A_{i_0}, k')/\mathcal{K}^{(i_0)}$. Hence $f \in \mathcal{L}^{(k')}$ is a solution of $\mathrm{Eq}(A_k, 1)/\mathcal{K}^{(k)}$, which yields the equation,

$$(\tau^{k'} f)(c^{(k)} f + d^{(k)}) = a^{(k)} f + b^{(k)}. \tag{3}$$

We obtain $\det A_k = 1$ from $\det A = 1$, and so $c^{(k)} f + d^{(k)} \neq 0$ from the above equation. It implies $L = C(t, f, \tau f, \dots, \tau^{k'-1} f)$, where we note that f is algebraic over K. Let $n = [L : C(t)] < \infty$. By Lemma 8 in the paper [11] by the author, we find $L = C(x)$, $x^n = t$. It follows that x is transcendental over C. By the following calculation,

$$\left(\frac{\tau x}{x}\right)^n = \frac{\tau(x^n)}{x^n} = \frac{\tau t}{t} = q^{i_0},$$

we find $\tau x/x \in C$. Let $r = \tau x/x \in C^\times$, which yields $\tau x = rx$.

We have $f \in C(x)^\times$ and $A_k \in M_2(C[x^n])$. Let $f = P/Q$, where $P, Q \in C[x]$ are relatively prime. From the equation (3), we obtain

$$\frac{P_{k'}}{Q_{k'}} = \frac{a^{(k)} \frac{P}{Q} + b^{(k)}}{c^{(k)} \frac{P}{Q} + d^{(k)}} = \frac{a^{(k)} P + b^{(k)} Q}{c^{(k)} P + d^{(k)} Q}.$$

Since $P_{k'}$ and $Q_{k'}$ are relatively prime, the following system of equations is obtained,

$$\begin{cases} RP_{k'} = a^{(k)} P + b^{(k)} Q, \\ RQ_{k'} = c^{(k)} P + d^{(k)} Q, \end{cases} \quad R \in C[x]. \tag{4}$$

Hence

$$R \begin{pmatrix} P_{k'} \\ Q_{k'} \end{pmatrix} = \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix},$$

$$R \begin{pmatrix} d^{(k)} & -b^{(k)} \\ -c^{(k)} & a^{(k)} \end{pmatrix} \begin{pmatrix} P_{k'} \\ Q_{k'} \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Since P and Q are relatively prime, we find $R \in C^{\times}$. From the equation (4), we obtain

$$\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x RP_{k'} = \deg_x P.$$

Note that $\deg_x a^{(k)} = 2kn > 0$, and we find

$$\deg_x a^{(k)}P = \deg_x b^{(k)}Q,$$

and so

$$\begin{aligned}\deg_x Q - \deg_x P &= \deg_x a^{(k)} - \deg_x b^{(k)}\\ &= 2kn - 2(k-1)n\\ &= 2n.\end{aligned}$$

Let

$$f = \sum_{i=2n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\ e_{2n} \neq 0.$$

be the formal power series representation of f. We will show $f \in C(t)$. Assume that there exists $i \geq 2n$ such that $n \nmid i$ and $e_i \neq 0$. Let $ln + m$ $(0 < m < n)$ be the minimum of such numbers. We will derive a contradiction. The following degrees are needed,

$$\deg_x a^{(k)} = 2kn, \quad \deg_x b^{(k)} = 2(k-1)n,$$
$$\deg_x c^{(k)} = 2(k-1)n, \quad \deg_x d^{(k)} = 2(k-2)n.$$

The first term of

$$\begin{aligned}&a^{(k)}f + b^{(k)}\\ &= a^{(k)}\left(e_{2n}\left(\frac{1}{x}\right)^{2n} + \cdots + e_{ln}\left(\frac{1}{x}\right)^{ln} + e_{ln+m}\left(\frac{1}{x}\right)^{ln+m} + \cdots\right) + b^{(k)}\end{aligned}$$

whose exponent is not divisible by n has the exponent

$$-2kn + (ln + m).$$

On the other hand, the first term of

$$\begin{aligned}&f_{k'}(c^{(k)}f+d^{(k)})\\&=\left\{\frac{e_{2n}}{r^{2nk'}}\left(\frac{1}{x}\right)^{2n}+\cdots+\frac{e_{ln}}{r^{lnk'}}\left(\frac{1}{x}\right)^{ln}+\frac{e_{ln+m}}{r^{(ln+m)k'}}\left(\frac{1}{x}\right)^{ln+m}+\cdots\right\}\\&\times\left\{c^{(k)}\left(e_{2n}\left(\frac{1}{x}\right)^{2n}+\cdots+e_{ln}\left(\frac{1}{x}\right)^{ln}+e_{ln+m}\left(\frac{1}{x}\right)^{ln+m}+\cdots\right)+d^{(k)}\right\}\end{aligned}$$

whose exponent is not divisible by n has the exponent

$$\geq -2(k-2)n+(ln+m).$$

Hence we obtain

$$-2kn+(ln+m)\geq -2(k-2)n+(ln+m),$$

and so $0\geq 4n$, a contradiction. We have proved that for any $i\geq 2n$, $n\nmid i$ implies $e_i=0$. Hence

$$f\in C((1/x^n))\cap C(1/x)=C(1/x^n)=C(1/t)=C(t).$$

By $\mathcal{L}=\mathcal{K}^{(i_0)}\langle f\rangle$ and $\mathcal{K}^{(i_0)}=(C(t),\tau)$, we find $\mathcal{L}=\mathcal{K}^{(i_0)}$ and $L=C(t)$, the latter of which implies $n=1$ and $x=t$. Hence the operators satisfy the following relations,

$$\tau=\tau|_K=\tau_q^{i_0},\quad \tau^{k'}=\tau_q^{i_0k'}=\tau_q^{k}.$$

We will show $e_j\in\mathbb{Q}[q,1/q,\alpha]$ for $j\in\mathbb{Z}_{\geq 2}$. Note that the degrees of $a^{(i)},b^{(i)},c^{(i)},d^{(i)}\in\mathbb{Q}[q,\alpha][t]$ are as follows,

$$\begin{aligned}\deg a^{(k)}&=2k,\quad \deg b^{(k)}=2(k-1),\\ \deg c^{(k)}&=2(k-1),\quad \deg d^{(k)}=2(k-2).\end{aligned}$$

We have

$$(5)\qquad (\tau_q^k f)(c^{(k)}f+d^{(k)})=\left(\sum_{i=2}^{\infty}\frac{e_i}{q^{ki}}\left(\frac{1}{t}\right)^{i}\right)\left(c^{(k)}\sum_{i=2}^{\infty}e_i\left(\frac{1}{t}\right)^{i}+d^{(k)}\right)$$

and

$$a^{(k)}f + b^{(k)} = a^{(k)} \sum_{i=2}^{\infty} e_i \left(\frac{1}{t}\right)^i + b^{(k)}. \tag{6}$$

Comparing the coefficients of $(1/t)^{-2k+2}$, we obtain

$$0 = (-1)^k \frac{q^{(k-1)k}}{4^k} e_2 + (-1)^k \frac{q^{(k-1)k}}{4^{k-1}},$$

and so $e_2 = -4$. Hence the result is true for $j = 2$. Suppose $j \geq 3$ and that the result is true for smaller numbers. The coefficient of $(1/t)^{-2k+j}$ of the formula (6) is

$$(-1)^k \frac{q^{(k-1)k}}{4^k} e_j + (\text{an element of } \mathbb{Q}[q, 1/q, \alpha]),$$

and the coefficient of $(1/t)^{-2k+j}$ of the formula (5) coincides with that of

$$\left(\sum_{i=2}^{j-1} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=2}^{j-1} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right),$$

which belongs to $\mathbb{Q}[q, 1/q, \alpha]$. Hence we find $e_j \in \mathbb{Q}[q, 1/q, \alpha]$, the required.

We define a homomorphism

$$\begin{aligned} \phi\colon \mathbb{Q}[q, \alpha] &\to \mathbb{Q}[1^{1/k}, \beta], \\ q &\mapsto 1^{1/k}, \\ \alpha &\mapsto \beta \in C, \end{aligned}$$

in the following way. In the case that q is transcendental over $\mathbb{Q}(\alpha)$, let ϕ be the homomorphism of the polynomial ring $\mathbb{Q}[\alpha][q]$ to $\mathbb{Q}[\alpha][1^{1/k}]$ substituting $1^{1/k}$ for q. In the case that q is algebraic over $\mathbb{Q}(\alpha)$, we use the fact that $k = 3i_0 i_1 \geq i_1$. By the definition of i_1, we find

$$P_{1^{1/k}} \neq P^{(1)} \cap \overline{\mathbb{Q}}(q), \ldots, P^{(\nu)} \cap \overline{\mathbb{Q}}(q).$$

Let P' be the place of $\overline{\mathbb{Q}}(q, \alpha)/\overline{\mathbb{Q}}$ such that $P' \supset P_{1^{1/k}}$, and s a prime element of P'. Let $\phi_0\colon \overline{\mathbb{Q}}[[s]] \to \overline{\mathbb{Q}}$ be the homomorphism sending $\sum_{i=0}^{\infty} h_i s^i$

to h_0, $\iota\colon \mathcal{O}_{P'} \to \overline{\mathbb{Q}}[[s]]$ the embedding, and ϕ the homomorphism defined by $\phi = \phi_0 \circ \iota$. Since the values of $q - 1^{1/k}$, q and $1^{1/k}$ are calculated as follows,

$$v_{P'}(q - 1^{1/k}) = e(P'|P_{1^{1/k}})v_{P_{1^{1/k}}}(q - 1^{1/k}) = e(P'|P_{1^{1/k}}) \geq 1,$$
$$v_{P'}(q) = e(P'|P_{1^{1/k}})v_{P_{1^{1/k}}}(q) = 0,$$
$$v_{P'}(1^{1/k}) = 0,$$

we find

$$\phi(q) = \phi(q - 1^{1/k}) + \phi(1^{1/k}) = 0 + 1^{1/k} = 1^{1/k}.$$

If we assume $v_{P'}(\alpha) \neq 0$, then $P' = P^{(i)}$ for some $1 \leq i \leq \nu$, which implies

$$P_{1^{1/k}} = P' \cap \overline{\mathbb{Q}}(q) = P^{(i)} \cap \overline{\mathbb{Q}}(q),$$

a contradiction. Hence we find $v_{P'}(\alpha) = 0$, and so $\phi(\alpha) \in \overline{\mathbb{Q}} \subset C$. The restriction $\phi|_{\mathbb{Q}[q,\alpha]}$ is the required.

We will extend the homomorphism $\phi\colon \mathbb{Q}[q, \alpha] \to \mathbb{Q}[1^{1/k}, \beta]$. First, extend it to the homomorphism of $\mathbb{Q}[q, 1/q, \alpha]$ to $\mathbb{Q}[1^{1/k}, (1^{1/k})^{-1}, \beta] \subset C$, and second, to the homomorphism of $\mathbb{Q}[q, 1/q, \alpha]((1/t))$ to $C((1/t))$ sending $\sum_{i=m}^{\infty} h_i(1/t)^i$ to $\sum_{i=m}^{\infty} \phi(h_i)(1/t)^i$. Then ϕ is a difference homomorphism of $(\mathbb{Q}[q, 1/q, \alpha]((1/t)), t \mapsto qt)$ to $(C((1/t)), t \mapsto 1^{1/k}t)$.

Let

$$\hat{f} = \phi(f), \quad \hat{a} = \phi(a^{(k)}), \quad \hat{b} = \phi(b^{(k)}),$$
$$\hat{c} = \phi(c^{(k)}), \quad \hat{d} = \phi(d^{(k)}).$$

By $\phi(\tau_q^k f) = \phi(f)$, we find the equation,

$$\hat{f}(\hat{c}\hat{f} + \hat{d}) = \hat{a}\hat{f} + \hat{b}. \tag{7}$$

We will show $\hat{f} \in C(t)$. By $f \in C(1/t)$, there exists $s \in \mathbb{Z}_{\geq 0}$ and $m_0 \in \mathbb{Z}_{\geq 0}$ such that

$$m \geq m_0 \Rightarrow F_f(m, s) = 0,$$

where $F_f(m, s) = \det(e_{m+i+j})_{0 \leq i,j \leq s}$ is the Hankel determinant of f. Refer to the book [1] by J. W. S. Cassels for the Hankel determinant. For all

$m \geq m_0$,

$$\begin{aligned} F_{\hat{f}}(m,s) &= \det(\phi(e_{m+i+j}))_{0\leq i,j\leq s} \\ &= \phi(\det(e_{m+i+j})_{0\leq i,j\leq s}) \\ &= \phi(F_f(m,s)) \\ &= \phi(0) = 0. \end{aligned}$$

Hence $\hat{f} \in C(1/t) = C(t)$.

Let $\hat{f} = \hat{P}/\hat{Q}$, where $\hat{P}, \hat{Q} \in C[t]$ are relatively prime. From

$$\phi(a^{(i)}) = (-1)^i \frac{(1^{1/k})^{(i-1)i}}{4^i} t^{2i} + (\deg \leq 2i-2),$$

we obtain

$$\begin{gathered} \deg \hat{a} = \deg \phi(a^{(k)}) = 2k, \\ \deg \hat{b} = \deg \phi(b^{(k)}) = \deg \phi(-a_1^{(k-1)}) = \deg \phi(a^{(k-1)}) = 2(k-1), \\ \deg \hat{c} = \deg \phi(c^{(k)}) = \deg \phi(a^{(k-1)}) = 2(k-1), \\ \deg \hat{d} = \deg \phi(d^{(k)}) = \deg \phi(-a_1^{(k-2)}) = \deg \phi(a^{(k-2)}) = 2(k-2), \end{gathered}$$

and from $\det A_k = a^{(k)}d^{(k)} - b^{(k)}c^{(k)} = 1$,

$$\hat{a}\hat{d} - \hat{b}\hat{c} = \phi(1) = 1,$$

which implies $\hat{c}\hat{f} + \hat{d} \neq 0$. Thus from the equation (7),

$$\frac{\hat{P}}{\hat{Q}} = \hat{f} = \frac{\hat{a}\hat{f} + \hat{b}}{\hat{c}\hat{f} + \hat{d}} = \frac{\hat{a}\hat{P} + \hat{b}\hat{Q}}{\hat{c}\hat{P} + \hat{d}\hat{Q}}.$$

Since $\hat{P}$ and $\hat{Q}$ are relatively prime, the following system of equations is obtained,

$$\begin{cases} \hat{R}\hat{P} = \hat{a}\hat{P} + \hat{b}\hat{Q}, \\ \hat{R}\hat{Q} = \hat{c}\hat{P} + \hat{d}\hat{Q}, \end{cases} \quad \hat{R} \in C[t].$$

This yields

$$\hat{R}\begin{pmatrix}\hat{P}\\ \hat{Q}\end{pmatrix} = \begin{pmatrix}\hat{a} & \hat{b}\\ \hat{c} & \hat{d}\end{pmatrix}\begin{pmatrix}\hat{P}\\ \hat{Q}\end{pmatrix}, \tag{8}$$

and so

$$\hat{R}\begin{pmatrix}\hat{d} & -\hat{b}\\ -\hat{c} & \hat{a}\end{pmatrix}\begin{pmatrix}\hat{P}\\ \hat{Q}\end{pmatrix}=\begin{pmatrix}\hat{P}\\ \hat{Q}\end{pmatrix}.$$

Since $\hat{P}$ and $\hat{Q}$ are relatively prime, we find $\hat{R}\in C^{\times}$. Hence the equation (8) yields

$$\begin{pmatrix}\hat{a}-\hat{R} & \hat{b}\\ \hat{c} & \hat{d}-\hat{R}\end{pmatrix}\begin{pmatrix}\hat{P}\\ \hat{Q}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

which implies

$$\begin{aligned}0&=\det\begin{pmatrix}\hat{a}-\hat{R} & \hat{b}\\ \hat{c} & \hat{d}-\hat{R}\end{pmatrix}\\ &=1-(\hat{a}+\hat{d})\hat{R}+\hat{R}^2.\end{aligned}$$

However, this contradicts $\deg\hat{a}=2k>2(k-2)=\deg\hat{d}$. $\square$

THEOREM 17. *Suppose that q is transcendental over $\mathbb{Q}$. Let $\mathcal{U}$ be a difference overfield of $\mathcal{K}$, and $(f,g)\neq 0$ a solution in $\mathcal{U}$ of the equation over $\mathcal{K}$,*

$$\begin{pmatrix}y_1\\ z_1\end{pmatrix}=A\begin{pmatrix}y\\ z\end{pmatrix}.$$

Let $\mathcal{N}/\mathcal{K}$ be a pLE in $\mathcal{U}$. Then f and g are algebraically independent over N.

PROOF. This is straightforwardly proved by Corollary 15. $\square$

COROLLARY 18. *Suppose $C=\mathbb{C}$ and that q is a transcendental number. The q-Bessel equation over $\mathcal{K}=(\mathbb{C}(t),\tau_q\colon t\mapsto qt)$,*

$$y_2+(t^2/4-q^{\nu}-q^{-\nu})y_1+y=0,$$

with an arbitrary value of the parameter $\nu\in\mathbb{C}$ is unsolvable.

Proof. If g is a non-trivial solution contained in a pLE $\mathcal{N}/\mathcal{K}$, $(f = g_1, g) \neq 0$ is a solution in $\mathcal{N}$ of the equation over $\mathcal{K}$,

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} -t^2/4 + \alpha & -1 \\ 1 & 0 \end{pmatrix}, \ \alpha = q^{\nu} + q^{-\nu} \in \mathbb{C}.$$

However, Theorem 17 implies $f, g \notin N$, a contradiction. $\square$

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