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On the Coefficients of Multiple Walsh-Fourier Series with Small Gaps

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Abstract. For a Lebesgue integrable complex-valued function f defined over the m-dimensional torus $\mathbb{I}^m := [0,1)^m$, let $\hat{f}(\mathbf{n})$ denote the multiple Walsh-Fourier coefficient of f, where $\mathbf{n} = \left(n^{(1)}, \ldots, n^{(m)}\right) \in (\mathbb{Z}^+)^m$, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. The Riemann-Lebesgue lemma shows that $\hat{f}(\mathbf{n}) = o(1)$ as $|\mathbf{n}| \to \infty$ for any $f \in \mathrm{L}^1(\mathbb{I}^m)$. However, it is known that, these Fourier coefficients can tend to zero as slowly as we wish. The definitive result is due to Ghodadra Bhikha Lila for functions of bounded p-variation. We shall prove that this is just a matter only of local bounded p-variation for functions with multiple Walsh-Fourier series lacunary with small gaps. Our results, as in the case of trigonometric Fourier series due to J.R. Patadia and R.G. Vyas, illustrate the interconnection between 'localness' of the hypothesis and 'type of lacunarity' and allow us to interpolate the results.

1. Introduction

In 1949, N. J. Fine [3] proved using the second mean value theorem that if f is of bounded variation on $[0,1]$ and if $\hat{f}(n)$ denotes its (one dimensional) Walsh-Fourier coefficient, then $\hat{f}(n) = O(\frac{1}{n})$, for all $n \neq 0$. In [11] we have studied the order of magnitude of Walsh-Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes. Further, in [6] we have studied the order of magnitude of Walsh-Fourier coefficients of functions of various classes of generalized bounded variation and given lacunary analogues of our results in [11]. For a function of two variables several definitions of bounded variation are given and various properties are studied (see, for example, [12, 1]). In 2002 F. Móricz [13] studied the order of magnitude of double Fourier coefficients with the help of Riemann-Stieltjes integral of functions of two variables

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and in 2004 V. Fülöp and F. Móricz [4] studied the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy (see [2]) in a straightforward way without using Riemann-Stieltjes integral. In [8], we have defined the notion of bounded p-variation ($p \geq 1$) for a function from a rectangle $[a_1, b_1] \times \cdots \times [a_m, b_m]$ to $\mathbb{C}$ and studied the order of magnitude of trigonometric Fourier coefficients of such functions from $[0, 2\pi]^m$ to $\mathbb{C}$. We have also studied the order of magnitude of trigonometric Fourier coefficients of functions from $[0, 2\pi]^m$ to $\mathbb{C}$ having lacunary Fourier series with certain gaps and are of bounded p-variation only locally [9]. Then in [10] we have studied the order of magnitude of Walsh-Fourier coefficients for a function of bounded p-variation from $[0, 1]^m$ to $\mathbb{C}$ having non-lacunary multiple Walsh-Fourier series. Here we study the order of magnitude of multiple Walsh-Fourier coefficients of functions from $[0, 1]^m$ to $\mathbb{C}$ which are of bounded p-variation locally and having lacunary multiple Walsh-Fourier series having small gaps. Our results, as in the case of trigonometric Fourier series [14] and for a single Walsh and Vilenkin-Fourier series [6, 7], illustrate the interconnection between 'localness' of the hypothesis and 'type of lacunarity' and allow us to interpolate the results. Thus our results of this paper generalizes and gives lacunary analogue of our earlier results [10, Theorems 3 and 4]. For $n = 1$, our new results give our earlier results [6]. Also, for $p = 1$, our results give the lacunary analogue for multiple Walsh-Fourier coefficients of the results of Móricz [13] and Fülöp and Móricz [4], except possibly for the exact constant in their case.

2. Notations and Definitions

In [8] we have defined two concepts of bounded p-variation for functions of several variables that generalize the definitions of bounded variation for functions of several variables given by Vitali and by Hardy. For the sake of completeness, here we rewrite those definitions.

Let R be the rectangle $R = [a_1, b_1] \times \cdots \times [a_m, b_m]$. By a (finite) partition $\mathcal{P}$ of R we mean the set $\mathcal{P} = \{R_1, \ldots, R_k\}$, in which R_i's are pairwise disjoint (no two have common interior) subrectangles of R having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is R. Let $f = f(x_1, \ldots, x_m)$ be a real or complex-valued function on R. For any subrectangle $R' = [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$ of R with $a_i \leq \alpha_i < \beta_i \leq b_i$ for all $i = 1, 2, \ldots, m$, we define $\Delta f(R')$ as follows: when $m = 2$ we put

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) \\ &= f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2);\end{aligned}$$

for $m = 3$

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]) \\ &= [f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \alpha_2, \beta_3) - f(\alpha_1, \beta_2, \beta_3) + f(\alpha_1, \alpha_2, \beta_3)] \\ &\quad - [f(\beta_1, \beta_2, \alpha_3) - f(\beta_1, \alpha_2, \alpha_3) - f(\alpha_1, \beta_2, \alpha_3) + f(\alpha_1, \alpha_2, \alpha_3)] \\ &= \Delta_{[\alpha_3, \beta_3]} \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]), \text{say;}\end{aligned}$$

and successively for any $m \geq 3$

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]) \\ &= \Delta_{[\alpha_m, \beta_m]} \Delta f([\alpha_1, \beta_1] \times \cdots \times [\alpha_{m-1}, \beta_{m-1}]).\end{aligned}$$

Definition V. For $p \geq 1$ we say that f is of *bounded p-variation over R* (written as $f \in \mathrm{BV_V}^{(p)}(R)$) if $V_p(f; R)$, the *total p-variation* of f over R, is finite, where

$$V_p(f; R) := \sup \left\{ \sum_{i=1}^{k} |\Delta f(R_i)|^p \right\}^{1/p},$$

in which the supremum is taken over all partitions $\{R_1, \ldots, R_k\}$ of R.

Remark 1. As noted in [8], for $p = 1$ our Definition V is equivalent to that Vitali (see, for example, [2], [4]). Also, the class $\mathrm{BV_V}^{(p)}(R)$ contains functions for which the m-dimensional Lebesgue integral over R fails to exist. The following notion of bounded p-variation is motivated by this fact.

Definition H. In case $m = 2$, we say that a function $f = f(x_1, x_2)$ is of bounded p-variation over $R := [a_1, b_1] \times [a_2, b_2]$, in symbol: $f \in \mathrm{BV_H}^{(p)}(R)$, if it is in the class $\mathrm{BV_V}^{(p)}(R)$ and if the marginal functions $f(x_1, a_2)$ and $f(a_1, x_2)$ are of bounded p-variation on the intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$, respectively in the sense of Wiener [17].

In case $m \geq 3$, the notion of bounded p-variation over a rectangle R can naturally be defined by the following recurrence: $f \in \mathrm{BV_H}^{(p)}(R)$ if $f \in \mathrm{BV_V}^{(p)}(R)$ and each of the marginal functions $f(x_1, \dots, a_k, \dots, x_m)$ is in the class $\mathrm{BV_H}^{(p)}(R(a_k))$, where $k = 1, \dots, m$ and

$$R(a_k) = \{(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m) \in \mathbb{R}^{m-1} : \; a_j \leq x_j \leq b_j \\ \text{for } j = 1, \dots, k-1, k+1, \dots, m\}.$$

This definition can be equivalently reformulated as follows: $f \in \mathrm{BV_H}^{(p)}(R)$ if and only if $f \in \mathrm{BV_V}^{(p)}(R)$ and for any choice of $(1 \leq) j_1 < \cdots < j_n (\leq n)$, $1 \leq n < m$, the function $f(x_1, \dots, a_{j_1}, \dots, a_{j_n}, \dots, x_m)$ is in the class $\mathrm{BV_V}^{(p)}(R(a_{j_1}, \dots, a_{j_n}))$, where

$$R(a_{j_1}, \dots, a_{j_n}) := \{(x_{\ell_1}, \dots, x_{\ell_{m-n}}) \in \mathbb{R}^{m-n} : a_j \leq x_j \leq b_j \\ \text{for } j = \ell_1, \dots, \ell_{m-n}\}$$

and $\{\ell_1, \dots, \ell_{m-n}\}$ is the complementary set of $\{j_1, \dots, j_n\}$ with respect to $\{1, \dots, m\}$.

REMARK 2. When $p = 1$ our Definition H is equivalent to the definition given by Hardy (see, for example, [2, 4]).

Let $\{r_n\}$, $n = 0, 1, 2, \dots$, denote the class of Rademacher functions defined by

$$r_0(x) = 1 \;\; (0 \leq x < 1/2), \;\; r_0(x) = -1 \;\; (1/2 \leq x < 1),$$

$$r_0(x+1) = r_0(x), \;\; r_n(x) = r_0(2^n x) \quad (n = 1, 2, 3, \dots).$$

The complete orthonormal Walsh system [16], say $\{\varphi_n\}$, $n = 0, 1, 2, \dots$, as ordered by Paley [15], is then given by

$$\varphi_0(x) \equiv 1, \quad \varphi_n(x) = r_{n_1}(x) r_{n_2}(x) \cdots r_{n_k}(x)$$

if $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$, in which $n_1 > n_2 > \cdots > n_k \geq 0$. For the functions φ_n, $\deg \varphi_n$ denotes the degree of φ_n defined by : $\deg \varphi_0 = 0$ and $\deg \varphi_n = n_1 + 1$, if φ_n is represented as the product of Rademacher characters as in preceding lines. Accordingly, for each $j \in \mathbb{N}$ we have $\varphi_{2^{j-1}} = r_{j-1}$ and $\deg \varphi_{2^{j-1}} = \deg r_{j-1} = j$. The degree of any real linear combination of finitely many elements φ_n $(n = 0, 1, \dots)$ (that is, a polynomial in functions φ_n on $[0, 1]$) is the maximum of the degree of the elements φ_n appearing in it.

For a periodic $f = f(x_1, \dots, x_m)$ with period 1 in each variable and Lebesgue integrable over the m-dimensional torus $\mathbb{I}^m := [0, 1)^m$, in symbol $f \in \mathrm{L}^1(\mathbb{I}^m)$, its formal multiple Walsh-Fourier series is given by

$$
\begin{aligned}
(2.1) \qquad & f(x_1, \dots, x_m) \\
& \sim \sum_{n^{(1)}=0}^{\infty} \cdots \sum_{n^{(m)}=0}^{\infty} \hat{f}\left(n^{(1)}, \dots, n^{(m)}\right) \varphi_{n^{(1)}}(x_1) \dots \varphi_{n^{(m)}}(x_m)
\end{aligned}
$$

where $\hat{f}\left(n^{(1)}, \dots, n^{(m)}\right) \equiv \hat{f}(\mathbf{n})$ is the $\mathbf{n}^{th}$ multiple Walsh-Fourier coefficient (see, for example, [5]) of f defined by

$$
\hat{f}(\mathbf{n}) = \int_{\mathbb{I}^m} f(x_1, \dots, x_m) \varphi_{n^{(1)}}(x_1) \dots \varphi_{n^{(m)}}(x_m) dx_1 \dots dx_m.
$$

Given a subset $E \subset (\mathbb{Z}^+)^m$, a function $f \in \mathrm{L}^1(\mathbb{I}^m)$ is said to be E-*spectral* (or, said to have spectrum E) if and only if $\hat{f}(\mathbf{n}) = 0$ for all $\mathbf{n}$ in $(\mathbb{Z}^+)^m \setminus E$. In what follows, we consider a set $E \subset (\mathbb{Z}^+)^m$ described in the following way: for each $j = 1, 2, \dots, m$ consider sets $E^{(j)} = \{n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, \dots\} \subset \mathbb{Z}^+$ with $\{n_k^{(j)}\}_{k=1}^{\infty}$ strictly increasing for each j and satisfying the small gap conditions

$$
(2.2) \qquad \left(n_{k+1}^{(j)} - n_k^{(j)}\right) \geq q_j \geq 1 \quad (k = 1, 2, \dots;\ j = 1, 2, \dots, m);
$$

and then put $E = \prod_{j=1}^{m} E^{(j)}$. Now $\mathbf{n_s} = (n_{s_1}^{(1)}, n_{s_2}^{(2)}, \dots, n_{s_m}^{(m)})$ denotes the typical element of E. When $m = 1$, E will be taken to be $E^{(1)}$ with the superscript in $n_k^{(1)}$'s omitted.

3. Results

We need the following lemmas which are proved in [8].

Lemma 3. *Let* $f \in \mathrm{BV_V}^{(p)}(R)$, *where* $R = [a_1, b_1] \times \cdots \times [a_m, b_m]$. *Let* $\{R_1, \dots, R_k\}$ *be a partition of* R. *Then* $f \in \mathrm{BV_V}^{(p)}(R_i)$ *for each* $i = 1, \dots, k$, *and that*

$$
\sum_{i=1}^{k} (V_p(f; R_i))^p \leq (V_p(f; R))^p.
$$

LEMMA 4. *Let $f \in \mathrm{BV_H}^{(p)}(R)$, where $R = [a_1, b_1] \times \cdots \times [a_m, b_m]$. Then the discontinuities of f are located on a countable number of $(m-1)$-dimensional hyperplanes parallel to some of the coordinate hyperplanes.*

Here we prove the following theorem.

THEOREM 5. *Let $E \subset (\mathbb{Z}^+)^m$ be described as above and $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. Let $f \in \mathrm{BV_V}^{(p)}(I) \cap \mathrm{L}^p(I)$ $(p \geq 1)$, where I is the rectangle $I = [i_1 2^{-N_1}, (i_1+1)2^{-N_1}] \times \cdots \times [i_m 2^{-N_m}, (i_m+1)2^{-N_m}]$ in which $0 \leq i_j < 2^{N_j}$ and $2^{-N_j} \geq 1/q_j$ for each j. If f is E-spectral and $\mathbf{n_k} = (n^{(1)}_{k_1}, \ldots, n^{(m)}_{k_m}) \in (\mathbb{Z}^+)^m$ is such that $n^{(j)}_{k_j}$ is sufficiently large for each j, then*

$$\hat{f}(\mathbf{n_k}) = O\left(\frac{1}{\left(\prod_{j=1}^{m} n^{(j)}_{k_j}\right)^{1/p}}\right). \tag{3.1}$$

REMARK 6. This theorem gives a lacunary analogue of our earlier result [10, Theorem 3]. Since $\mathrm{BV_H}^{(p)}(I) \subset \mathrm{BV_V}^{(p)}(I) \cap \mathrm{L}^p(I)$ in view of Lemma 4 (see [8]), above theorem is true if we replace the assumption "$f \in \mathrm{BV_V}^{(p)}(I) \cap \mathrm{L}^p(I)$" by "$f \in \mathrm{BV_H}^{(p)}(I)$". In that case it gives lacunary analogue of our earlier result [10, Theorem 4] and simultaneously Walsh analogue of our earlier result [9].

REMARK 7. Observe that $n^{(j)}_k = k$ for all k and for each $j \Longrightarrow q_j = 1$ in (2.2) $\Longrightarrow N_j = 0$ for each j in above theorem and remark $\Longrightarrow I = [i_1, i_1+1) \times \cdots \times [i_m, i_m+1) \sim \mathbb{I}^m$; and one gets corresponding results for non-lacunary multiple Walsh-Fourier series [10, Theorem 3 and Theorem 4]. On the other hand, if the multiple Walsh-Fourier series (2.1) of an E-spectral $f \in \mathrm{L}^1(\mathbb{I}^m)$ is such that the sets $E^{(j)}$ satisfies more stringent small gap conditions

$$\left(n^{(j)}_{k+1} - n^{(j)}_k\right) \to \infty \ \text{ as } \ k \to \infty \quad (j = 1, 2, \ldots, m), \tag{3.2}$$

then above results hold if the rectangle I is just of positive measure and of the form as in Theorem 5. Because if $|I| > 0$, by the form of I, $I = [i_1 2^{-N_1}, (i_1+1)2^{-N_1}] \times \cdots \times [i_m 2^{-N_m}, (i_m+1)2^{-N_m}]$ where each $N_j \in \mathbb{N}$ can be taken as large as required. In view of (3.2), one gets $(n_{k+1}^{(j)} - n_k^{(j)}) \geq 2^{N_j}$ for each j and for all $k \geq k_0$, for a suitable positive integer k_0 depending on $N_1, \ldots, N_m$. Then adding to $f(x_1, \ldots, x_m)$ the multiple Walsh polynomial

$$\sum_{n^{(1)}=0}^{k_0} \cdots \sum_{n^{(m)}=0}^{k_0} \left(-\hat{f}\left(n^{(1)}, \ldots, n^{(m)}\right)\right) \varphi_{n^{(1)}}(x_1) \ldots \varphi_{n^{(m)}}(x_m)$$

one gets a function $g \in \mathrm{L}^1(\mathbb{I}^m)$ whose multiple Walsh-Fourier series is lacunary of the form (2.1) which is E-spectral with $E^{(j)}$ satisfying the small gap condition (2.2) with $q_j = 2^{N_j}$ and results are true for g. Since f and g differ by a polynomial, results are true for f as well. Our results thus interpolates lacunary and non-lacunary results concerning order of magnitude of multiple Walsh-Fourier coefficients—displaying beautiful interconnection between types of lacunarity (as determined by q_j in (2.2)) and localness of hypothesis to be satisfied by the generic function (as determined by the q_j-dependent lengths of sides of I).

In case $E^{(j)}$ satisfies the gap condition (2.2) with $q_j \geq 4$ for each j, then we have the following theorem.

Theorem 8. *Theorem 5 holds true if the rectangle I replaced by the rectangle $J = [y_1, y_1 + 4/q_1] \times \cdots \times [y_m, y_m + 4/q_m]$ where $0 \leq y_j \leq 1 - 4/q_j$ (j=1,2, ... ,m), if $E^{(j)}$ satisfies the gap condition*

$$\left(n_{k+1}^{(j)} - n_k^{(j)}\right) \geq q_j \geq 4 \quad (k = 1, 2, \ldots;\ j = 1, 2, \ldots, m). \tag{3.3}$$

Remark 9. We note that Theorem 8 is better than Theorem 5 as far as the location of the rectangle is concerned. But, unfortunately, it does not interpolate the results in both extreme cases like Theorem 5 (see, Remark 7). As a particular case, it does give the result when the small gap condition (3.2) is satisfied and I is just of positive measure (as in Remark 7), but it does not give the result as a particular case when the series is non-lacunary. This is because, when the series is non-lacunary one must take $q_j = 1$ for all j, which is not allowed in gap condition (3.3).

4. Proof Of Results

Proof of Theorem 5. For the sake of simplicity in writing, we carry out the proof for $m=2$. For each $j=1,2$, consider the polynomial $P_{N_j}^{(j)}(x_j)$ defined as follows: if $N_j=0$, put $P_{N_j}^{(j)} \equiv 1$ and if $N_j \in \mathbb{N}$ then put

$$P_{N_j}^{(j)}(x_j) = \prod_{i=0}^{N_j-1} \left(1 + r_i\left(i_j 2^{-N_j}\right) r_i(x_j)\right).$$

Then, by definition of r_i, we have

$$\begin{aligned}
x_j \in & \left[i_j 2^{-N_j}, (i_j+1)2^{-N_j}\right) \\
& \implies r_i(x_j) = r_i\left(i_j 2^{-N_j}\right) \quad \text{for all } \ i = 0,1,\dots,N_j-1 \\
& \implies 1 + r_i\left(i_j 2^{-N_j}\right) r_i(x_j) = 1 + \left(r_i\left(i_j 2^{-N_j}\right)\right)^2 = 1 + (\pm 1)^2 = 2 \\
& \qquad\qquad \text{for all } \ i = 0,1,\dots,N_j-1 \\
& \implies P_{N_j}^{(j)}(x_j) = \prod_{i=0}^{N_j-1} 2 = 2^{N_j};
\end{aligned}$$

$$\begin{aligned}
x_j \notin & \left[i_j 2^{-N_j}, (i_j+1)2^{-N_j}\right) \\
& \implies r_i(x_j) \neq r_i\left(i_j 2^{-N_j}\right) \quad \text{for at least one } \ i \in \{0,1,\dots,N_j-1\} \\
& \implies 1 + r_i\left(i_j 2^{-N_j}\right) r_i(x_j) = 1 + (\pm 1)(\mp 1) = 1 - 1 = 0 \\
& \qquad\qquad \text{for at least one } \ i \in \{0,1,\dots,N_j-1\} \\
& \implies P_{N_j}^{(j)}(x_j) = 0.
\end{aligned}$$

Thus, we have

$$P_{N_j}^{(j)}(x_j) = \begin{cases} 2^{N_j} & \text{if } x_j \in \left[i_j 2^{-N_j}, (i_j+1)2^{-N_j}\right), \\ 0 & \text{if } x_j \in [0,1) \setminus \left[i_j 2^{-N_j}, (i_j+1)2^{-N_j}\right). \end{cases}$$

Consider $\mathbf{N} = (N_1, N_2)$ and put $P_{\mathbf{N}}(x_1,x_2) = P_{N_1}^{(1)}(x_1) P_{N_2}^{(2)}(x_2)$. Then by the above property of $P_{N_j}^{(i)}$ $(j=1,2)$, we have

$$P_{\mathbf{N}}(x_1,x_2) = \begin{cases} 2^{N_1+N_2} & \text{if } (x_1,x_2) \in \left[\frac{i_1}{2^{N_1}}, \frac{i_1+1}{2^{N_1}}\right) \times \left[\frac{i_2}{2^{N_2}}, \frac{i_2+1}{2^{N_2}}\right) \\ 0 & \text{if } (x_1,x_2) \in \mathbb{I}^2 \setminus \left[\frac{i_1}{2^{N_1}}, \frac{i_1+1}{2^{N_1}}\right) \times \left[\frac{i_2}{2^{N_2}}, \frac{i_2+1}{2^{N_2}}\right). \end{cases} \tag{4.1}$$

We claim that if $\mathbf{n_k} = (n^{(1)}_{k_1}, n^{(2)}_{k_2}) \in (\mathbb{Z}^+)^2$ is such that $\hat{f}(\mathbf{n_k}) \neq 0$ then $(fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) = \hat{f}(\mathbf{n_k})$. In fact, writing (x, y) in place of (x_1, x_2), we have

$$
\begin{aligned}
(4.2) \quad (fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) &= \int_{\mathbb{I}^2} f(x,y) P^{(1)}_{N_1}(x) P^{(2)}_{N_2}(y) \varphi_{n^{(1)}_{k_1}}(x) \varphi_{n^{(2)}_{k_2}}(y)\, dx dy \\
&= \int_{\mathbb{I}^2} f(x,y) \left(\prod_{i=0}^{N_1-1} \left(1 + r_i\left(i_1 2^{-N_1}\right) r_i(x)\right) \right) \\
&\qquad \times \left(\prod_{j=0}^{N_2-1} \left(1 + r_j\left(i_2 2^{-N_2}\right) r_j(y)\right) \right) \\
&\qquad \times \varphi_{n^{(1)}_{k_1}}(x) \varphi_{n^{(2)}_{k_2}}(y)\, dx dy \\
&= \hat{f}(\mathbf{n_k}) + \sum_{i=0}^{N_1-1} r_i\left(i_1 2^{-N_1}\right) \hat{f}(r_i \varphi_{\mathbf{n_k}}) + \sum_{j=0}^{N_2-1} r_j\left(i_2 2^{-N_2}\right) \hat{f}(r_j \varphi_{\mathbf{n_k}}) \\
&\quad + \sum_{i,j=0}^{N_1-1} r_i\left(i_1 2^{-N_1}\right) r_j\left(i_1 2^{-N_1}\right) \hat{f}(r_i r_j \varphi_{\mathbf{n_k}}) \\
&\quad + \sum_{i,j=0}^{N_2-1} r_i\left(i_2 2^{-N_2}\right) r_j\left(i_2 2^{-N_2}\right) \hat{f}(r_i r_j \varphi_{\mathbf{n_k}}) \\
&\quad + \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} r_i\left(i_1 2^{-N_1}\right) r_j\left(i_2 2^{-N_2}\right) \hat{f}(r_i r_j \varphi_{\mathbf{n_k}}) \\
&\quad + \cdots + \left(\prod_{i=0}^{N_1-1} r_i\left(i_1 2^{-N_1}\right) \right) \left(\prod_{j=0}^{N_2-1} r_j\left(i_2 2^{-N_2}\right) \right) \\
&\qquad \times \hat{f}(r_0 \ldots r_{N_1-1} r_0 \ldots r_{N_2-1} \varphi_{\mathbf{n_k}}).
\end{aligned}
$$

By our assumption the first term in the right hand side of (4.2) is nonzero. The characters appearing in the other terms in the right hand side of (4.2) are of the form $(\varphi\varphi_{n^{(1)}_{k_1}})(\psi\varphi_{n^{(2)}_{k_2}})$ where φ is (a function of x alone) such that $\deg \varphi \le N_1$ and ψ is (a function of y alone) such that $\deg \psi \le N_2$ and the degree of at least one of φ and ψ is nonzero. In view of the Paley ordering of Walsh characters, for each $j \in \mathbb{N}$ there are totally 2^{j-1} characters of degree j, namely $\varphi_{2^{j-1}} \equiv r_{j-1}$, $\varphi_{2^{j-1}+1} \equiv r_{j-1}\varphi_1$, $\varphi_{2^{j-1}+2} \equiv r_{j-1}\varphi_2$, $\ldots$, $\varphi_{2^j-1} \equiv r_{j-1}\varphi_{2^{j-1}-1} \equiv r_{j-1}r_{j-2}\cdots r_1 r_0$. Consequently, total number of characters of positive degree $\leq N$ is given by $2^0 + 2^1 + 2^2 + \cdots + 2^{N-1} = 2^N - 1$; they are from φ_1 to φ_{2^N-1}. It follows that when $\varphi_{n^{(j)}_{k_j}}$ is multiplied by any character of positive degree $\leq N_j$ the resulting character φ_{m_j} is such that

$$n^{(j)}_{k_j} < m_j \leq n^{(j)}_{k_j} + 2^{N_j} - 1 < n^{(j)}_{k_j} + 2^{N_j} \leq n^{(j)}_{k_j} + q_j \leq n^{(j)}_{k_j+1},$$

in view of (2.2) and the fact that $q_j \geq 2^{N_j}$. Since either deg $\varphi > 0$ or deg $\psi > 0$, either $m_1 \notin E_1$ or $m_2 \notin E_2$. Therefore $(m_1, m_2) \notin E$. Since f is E-spectral, $\hat{f}\left((\varphi\varphi_{n^{(1)}_{k_1}})(\psi\varphi_{n^{(2)}_{k_2}})\right) = \hat{f}(\varphi_{m_1}\varphi_{m_2}) \equiv \hat{f}(m_1, m_2) = 0$. Thus all the terms of the right hand side of (4.2) vanish except the first. This means that

$$(fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) = \hat{f}(\mathbf{n_k}) \text{ if } \hat{f}(\mathbf{n_k}) \neq 0. \tag{4.3}$$

Now, let $\mathbf{n_k} = (n^{(1)}_{k_1}, n^{(2)}_{k_2})$ be such that $n^{(j)}_{k_j}$ are large enough with $\hat{f}(\mathbf{n_k}) \neq 0$ and let $m_j \in \mathbb{N}$ be such that $2^{m_j} \leq n^{(j)}_{k_j} < 2^{m_j+1}$ with $m_j > N_j$ for each $j = 1, 2$. For simplicity in notation, let us now write k, ℓ, s, and t for $n^{(1)}_{k_1}$, $n^{(2)}_{k_2}$, m_1, and m_2 respectively. Then $2^s \leq k < 2^{s+1}$, $2^t \leq \ell < 2^{t+1}$ and in view of (4.3) and (4.1) we have

$$\begin{aligned} \hat{f}(\mathbf{n_k}) &= (fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) \\ &= 2^{N_1+N_2} \int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}} \int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} f(x, y)\varphi_k(x)\varphi_\ell(y)dxdy. \end{aligned} \tag{4.4}$$

For each $i = 0, 1, 2, 3, \ldots, 2^s$ and $j = 0, 1, 2, 3, \ldots, 2^t$ put $a_i = (i/2^s)$, $b_j = (j/2^t)$. Then, as $2^s \leq k < 2^{s+1}$, by definition of Walsh functions, φ_k takes the value 1 on one half of each of the intervals $(a_{i-1},\ a_i)$ and the value -1 on the other half, and hence

$$\int_{a_{i-1}}^{a_i} \varphi_k(x)dx = 0, \qquad (i = 1, 2, 3, \ldots, 2^s). \tag{4.5}$$

Similarly, as $2^t \leq \ell < 2^{t+1}$, the function φ_ℓ takes the value 1 on one half of each of the intervals $(b_{j-1},\ b_j)$ and the value -1 on the other half, and hence

$$\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy = 0, \qquad (j = 1, 2, 3, \ldots, 2^t). \tag{4.6}$$

Next, define three functions f_1, f_2, f_3 on $I = \big[i_1 2^{-N_1}, (i_1+1)2^{-N_1}\big) \times \big[i_2 2^{-N_2}, (i_2+1)2^{-N_2}\big)$ by setting

$$f_1(x,y) = f(a_{i-1}, y) \quad \big(a_{i-1} \le x < a_i; \quad i_2 2^{-N_2} \le y < (i_2+1)2^{-N_2}\big)$$

for $i = i_1 2^{s-N_1} + 1, i_1 2^{s-N_1} + 2, \ldots, (i_1+1)2^{s-N_1}$;

$$f_2(x,y) = f(x, b_{j-1}) \quad \big(i_1 2^{-N_1} \le x < (i_1+1)2^{-N_1}; \quad b_{j-1} \le y < b_j\big)$$

for $j = i_2 2^{t-N_2} + 1, i_2 2^{t-N_2} + 2, \ldots, (i_2+1)2^{t-N_2}$; and

$$f_3(x,y) = f(a_{i-1}, b_{j-1}) \ \ (a_{i-1} \le x < a_i; \ b_{j-1} \le y < b_j)$$

for $i = i_1 2^{s-N_1} + 1, i_1 2^{s-N_1} + 2, \ldots, (i_1+1)2^{s-N_1}$; $j = i_2 2^{t-N_2} + 1, i_2 2^{t-N_2} + 2, \ldots, (i_2+1)2^{t-N_2}$. Then in view of Fubini's theorem and relations (4.5) and (4.6) we have

$$\begin{aligned}
&\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}} \int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} f_1(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \\
&\quad = \int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}} \left[\sum_{i=i_1 2^{s-N_1}+1}^{(i_1+1)2^{s-N_1}} f(a_{i-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x)dx\right] \varphi_\ell(y)dy = 0,
\end{aligned}$$

$$\begin{aligned}
&\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}} \int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} f_2(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \\
&\quad = \int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} \left[\sum_{j=i_2 2^{t-N_2}+1}^{(i_2+1)2^{t-N_2}} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy\right] \varphi_k(x)dx = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}} \int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} f_3(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \\
&= \sum_{i=i_1 2^{s-N_1}+1}^{(i_1+1)2^{s-N_1}} \sum_{j=i_2 2^{t-N_2}+1}^{(i_2+1)2^{t-N_2}} f(a_{i-1}, b_{j-1}) \\
&\qquad \times \left[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx\right] \left[\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy\right] = 0.
\end{aligned}$$

Using these equations in (4.4) we get

$$
\begin{aligned}
|\hat{f}(\mathbf{n_k})| &= 2^{N_1+N_2}\left|\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}}\int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} f(x,y)\varphi_k(x)\varphi_\ell(y)dxdy\right| \\
&= 2^{N_1+N_2}\left|\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}}\int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} (f-f_1-f_2+f_3)(x,y)\varphi_k(x)\varphi_\ell(y)dxdy\right| \\
&\le 2^{N_1+N_2}\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}}\int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} |(f-f_1-f_2+f_3)(x,y)|dxdy \\
&\le 2^{N_1+N_2}\left(\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}}\int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} |(f-f_1-f_2+f_3)(x,y)|^p dxdy\right)^{1/p} \\
&\qquad \times\left(2^{-(N_1+N_2)}\right)^{1/q},
\end{aligned}
$$

in view of the Hölder's inequality (when $p>1$) since $f-f_1-f_2+f_3\in \mathrm{L}^p(I)$, where q is such that $1/p+1/q=1$. Observe that when $p=1$, we don't use Hölder's inequality and in that case we consider the inequality except last step. In any case, it follows that

$$
\begin{aligned}
|\hat{f}(\mathbf{n_k})|^p &\le 2^{N_1+N_2}\int_{i_2 2^{-N_2}}^{(i_2+1)2^{-N_2}}\int_{i_1 2^{-N_1}}^{(i_1+1)2^{-N_1}} |(f-f_1-f_2+f_3)(x,y)|^p dxdy \\
&= 2^{N_1+N_2}\sum_{i=i_1 2^{s-N_1}+1}^{(i_1+1)2^{s-N_1}}\ \sum_{j=i_2 2^{t-N_2}+1}^{(i_2+1)2^{t-N_2}}\int_{b_{j-1}}^{b_j}\int_{a_{i-1}}^{a_i} |(f-f_1-f_2+f_3)(x,y)|^p dxdy \\
&= 2^{N_1+N_2}\sum_{i=i_1 2^{s-N_1}+1}^{(i_1+1)2^{s-N_1}}\ \sum_{j=i_2 2^{t-N_2}+1}^{(i_2+1)2^{t-N_2}}\int_{b_{j-1}}^{b_j}\int_{a_{i-1}}^{a_i} |f(x,y)-f(a_{i-1},y) \\
&\qquad\qquad - f(x,b_{j-1})+f(a_{i-1},b_{j-1})|^p dxdy \\
&\le 2^{N_1+N_2}\sum_{i=i_1 2^{s-N_1}+1}^{(i_1+1)2^{s-N_1}}\ \sum_{j=i_2 2^{t-N_2}+1}^{(i_2+1)2^{t-N_2}} (V_p(f;[a_{i-1},a_i]\times[b_{j-1},b_j]))^p \\
&\qquad\qquad \times (a_i-a_{i-1})(b_j-b_{j-1}) \\
&\le \frac{2^{N_1+N_2}}{2^s 2^t}(V_p(f;I))^p \le \frac{2^{N_1+N_2+2}}{k\ell}(V_p(f;I))^p,
\end{aligned}
$$

in view of Lemma 3. Thus we get

$$|\hat{f}(\mathbf{n_k})| \leq \frac{2^{(N_1+N_2+2)/p} \cdot V_p(f;I)}{(k\ell)^{1/p}}.$$

This completes the proof of Theorem 5. □

Proof of Theorem 8. For each $j = 1, 2, \ldots, m$, if we take N_j as the largest integer satisfying $2^{-N_j} \geq 1/q_j$, i.e., $2^{-N_j} \geq 1/q_j > 2^{-N_j-1}$, we have $4/q_j > 2 \times 2^{-N_j}$. Since $[y_j, y_j + 4/q_j]$ is a subinterval of $[0, 1]$ of length $4/q_j$ there exists an integer i_j such that $0 \leq i_j < 2^{-N_j}$ and $[i_j 2^{-N_j}, (i_j + 1)2^{-N_j}] \subset [y_j, y_j + 4/q_j]$ for each j. But then

$$\begin{aligned} I &= [i_1 2^{-N_1}, (i_1+1)2^{-N_1}] \times \cdots \times [i_m 2^{-N_m}, (i_m+1)2^{-N_m}] \\ &\subset [y_1, y_1 + 4/q_1] \times \cdots \times [y_m, y_m + 4/q_m] = J. \end{aligned}$$

Since $f \in \mathrm{BV_V}^{(p)}(J) \cap \mathrm{L}^p(J)$, we have $f \in \mathrm{BV_V}^{(p)}(I) \cap \mathrm{L}^p(I)$. So, by Theorem 5, (3.1) holds. This completes the proof of Theorem 8. □

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