

Horocycle Flows without Minimal Sets

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Abstract. We show that the horocycle flows of open tight hyperbolic surfaces do not admit minimal sets.

1. Introduction

Let $\{\phi^t\}$ be a flow of a metric space X . A subset of X is called a *minimal set* of $\{\phi^t\}$ if it is closed and invariant by ϕ^t , and is minimal among them with respect to the inclusion. If X is compact, then any flow on X admits a minimal set. But if X is not compact, this is not always the case. The first example of a flow without minimal set is constructed on an open surface by T. Inaba [8]. Later various examples are piled up by many authors including [2]. See also [12] for examples of Anzai skew products on an open annulus.

M. Kulikov [9] constructed an example of the horocycle flow of an open hyperbolic surface with this property. This is interesting since horocycle flows have long been studied by various mathematicians; function analysts, topologists, dynamical people and ergodic theoretists. Moreover an example in [9] is the first one constructed algebraically on an homogeneous space. However the example is constructed in a specific and elaborate way. The purpose of this paper is to show that more general Fuchsian groups also satisfy this property.

DEFINITION 1.1. A Fuchsian group Γ is called *tight* if it satisfies the following conditions.

- (1) Γ is purely hyperbolic.
- (2) $\Sigma = \Gamma \backslash \mathbb{H}^2$ is noncompact and admits an increasing and exhausting sequence of compact subsurfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ with geodesic boundaries such that there is a bound C on the length of components of $\partial \Sigma_n$.

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Tight Fuchsian groups are infinitely generated and of the first kind. The main result of this notes is the following.

THEOREM 1.2. *If Γ is a tight Fuchsian group, the horocycle flow $\{h^s\}_{s \in \mathbb{R}}$ on $\Gamma \backslash PSL(2, \mathbb{R})$ admits no minimal sets.*

COROLLARY 1.3. *Almost all orbits of the horocycle flow of a tight Fuchsian group are dense.*

REMARK 1.4. For some tight Fuchsian groups, the horocycle flow is ergodic, while for others it is not.

In Section 2, we explain conventions used in this paper. In Section 3, we prepare fundamental facts about the horocyclic limit points. Section 4 is devoted to the proof of Theorem 1.2. Finally in Section 5, we raise examples of tight Fuchsian groups and discuss Corollary 1.3 and Remark 1.4.

2. Conventions

The right coset space $PSL(2, \mathbb{R})/PSO(2)$ is identified with the upper half plane \mathbb{H} by sending a matrix $\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to a point $\frac{ai+b}{ci+d}$. The group $PSL(2, \mathbb{R})$ acts on \mathbb{H} as linear fractional transformations, and is identified with the unit tangent space $T^1\mathbb{H}$ by sending $M \in PSL(2, \mathbb{R})$ to $M_*(i, \vec{e})$, where (i, \vec{e}) is the upward unit tangent vector at i . The canonical projection is denoted by

$$\pi_1 : PSL(2, \mathbb{R}) = T^1\mathbb{H} \rightarrow \mathbb{H}.$$

The geodesic flow $\{\tilde{g}^t\}$ (resp. the horocycle flow $\{\tilde{h}^s\}$) on $PSL(2, \mathbb{R})$ is given by the right multiplication of the matrices $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$). The quotient space $PSL(2, \mathbb{R})/\langle \tilde{h}^s \rangle$ is identified with the annulus $\mathbb{A} = (\mathbb{R}^2 \setminus \{0\})/\langle \pm 1 \rangle$, by sending $M \in PSL(2, \mathbb{R})$ to a point $M(1, 0)^t \in \mathbb{A}$. The canonical projection is denoted by

$$\pi_2 : T^1\mathbb{H} = PSL(2, \mathbb{R}) \rightarrow \mathbb{A}.$$

The geodesic flow \tilde{g}^t induces a flow on \mathbb{A} , which is just the scalar multiplication by $e^{t/2}$. The further quotient space \mathbb{A}/\mathbb{R}_+ is equal to the circle at infinity $\partial_\infty\mathbb{H}$, both being defined as the right coset space of $PSL(2, \mathbb{R})$ by the subgroup of the upper triangular matrices.

For any $\xi \in \partial_\infty\mathbb{H}$, the preimage of ξ by the canonical projection $\mathbb{A} \rightarrow \partial_\infty\mathbb{H}$ is denoted by $\mathbb{A}(\xi)$. It is a ray of \mathbb{A} . For any point $p \in \mathbb{A}(\xi)$, $H(p) = \pi_1(\pi_2^{-1}(p))$ is a horocycle in \mathbb{H} tangent to $\partial_\infty\mathbb{H}$ at ξ . The open horodisk encircled by $H(p)$ is denoted by $D(p)$. The signed distance from $i \in \mathbb{H}^2$ to the horocycle $H(p)$ (positive if i is outside $D(p)$ and negative if inside) is $2 \log|p|$, where $|p|$ denotes the Euclidian norm of \mathbb{A} . Thus $|p| < 1$ if and only if $i \in D(p)$.

Given a Fuchsian group Γ , the flows $\{\tilde{g}^t\}$ and $\{\tilde{h}^s\}$ induce flows on $\Gamma \backslash PSL(2, \mathbb{R})$, denoted by $\{g^t\}$ and $\{h^s\}$. The right \mathbb{R} -action $\{h^s\}$ on $\Gamma \backslash PSL(2, \mathbb{R})$ is Morita equivalent to the left Γ -action on \mathbb{A} . Thus a dense $\{h^s\}$ -orbit in $\Gamma \backslash PSL(2, \mathbb{R})$ corresponds to a dense Γ -orbit in \mathbb{A} . Likewise a minimal set of the flow $\{h^s\}$ in $\Gamma \backslash PSL(2, \mathbb{R})$ corresponds to a minimal set for the Γ -action on \mathbb{A} .

For any $\tilde{v} \in T^1\mathbb{H}$, $t \mapsto \pi_1\tilde{g}^t(\tilde{v})$ is the unit speed geodesic in \mathbb{H} with initial vector \tilde{v} . Its positive endpoint in $\partial_\infty\mathbb{H}$ is denoted by $\tilde{v}(\infty)$. If Γ is purely hyperbolic, the quotient space $\Sigma = \Gamma \backslash \mathbb{H}$ is a hyperbolic surface, and its unit tangent bundle $T^1\Sigma$ is identified with $\Gamma \backslash PSL(2, \mathbb{R})$. The canonical projection is denoted by

$$\pi : T^1\Sigma \rightarrow \Sigma.$$

For any $v \in T^1\Sigma$,

$$v[0, \infty) = \{\pi g^t(v) \mid 0 \leq t < \infty\}$$

is the geodesic ray in Σ with ininitial vector v .

3. Horocyclic Limit Points

In this section we assume that Γ is a purely hyperbolic Fuchsian group of the first kind. As before, we denote $\Sigma = \Gamma \backslash \mathbb{H}$. Many of the contents in this section are taken from [13].

DEFINITION 3.1. A geodesic ray $v[0, \infty)$, $v \in T^1\Sigma$, is called a *quasi-minimizer* if there is $k > 0$ such that $d(\pi g^t(v), \pi(v)) \geq t - k$ for any $t \geq 0$.

See Figure 1.

DEFINITION 3.2. A point at infinity $\xi \in \partial_\infty \mathbb{H}$ is called a *horocyclic limit point* of Γ if any horodisk at ξ intersects the orbit Γi . Otherwise it is called *nonhorocyclic*.

See Figure 2. If ξ is a horocyclic limit point, then any horodisk at ξ intersects any orbit Γz .

LEMMA 3.3. For any lift \tilde{v} of $v \in T^1\Sigma$, the geodesic ray $v[0, \infty)$ in Σ is a quasi-minimizer if and only if $\xi = \tilde{v}(\infty)$ is a nonhorocyclic limit point.

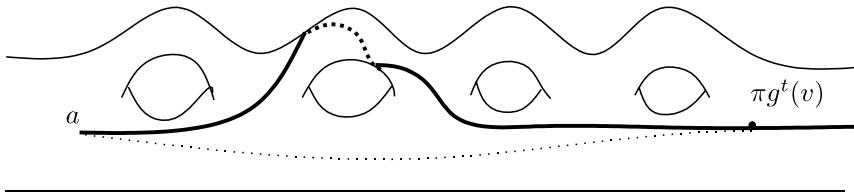


Fig. 1. A quasi-minimizer. It starts at a point a and after one turn goes straight to the right. Any point $\pi g^t(v)$ on the curve satisfies $t - d(a, \pi g^t(v)) \leq k$ for some k , where t is the length of the curve between the two points.

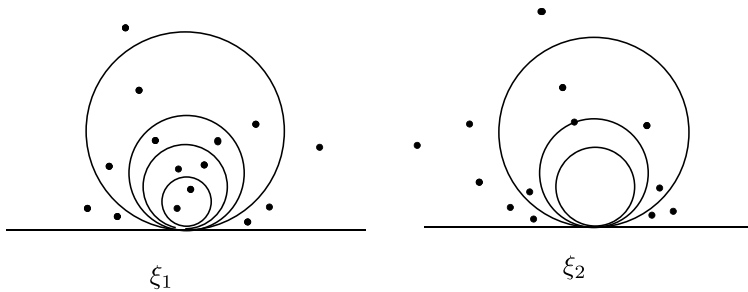


Fig. 2. ξ_1 is horocyclic and ξ_2 is nonhorocyclic.

DEFINITION 3.4. For any point $\xi \in \partial_\infty \mathbb{H}$, let $\tilde{v} \in T_i^1 \mathbb{H}$ be a tangent vector at $i \in \mathbb{H}$ such that $\tilde{v}(\infty) = \xi$. The *Busemann function* $B_\xi : \mathbb{H} \rightarrow \mathbb{R}$ is defined for $z \in \mathbb{H}$ by

$$(3.1) \quad B_\xi(z) = \lim_{t \rightarrow \infty} (d(z, \pi_1 \tilde{g}^t(\tilde{v})) - t).$$

Notice that for $k > 0$, the set $\{B_\xi < -k\}$ is a horodisk at ξ which is k -apart from i .

PROOF OF LEMMA 3.3. One may assume that \tilde{v} in the lemma is a unit tangent vector at $i \in \mathbb{H}$. Suppose that the point $\xi = \tilde{v}(\infty)$ is a nonhorocyclic limit point. Then there is $k > 0$ such that for any $\gamma \in \Gamma$, $B_\xi(\gamma i) \geq -k$. Since the limit in (3.1) is non increasing, this implies $d(\gamma i, \pi_1 \tilde{g}^t(\tilde{v})) \geq t - k$ for any $\gamma \in \Gamma$ and $t \geq 0$. On $\Sigma = \Gamma \setminus \mathbb{H}$, we get $d(\pi(v), \pi g^t(v)) \geq t - k$ for any $t \geq 0$. That is, $v[0, \infty)$ is a quasi-minimizer.

The converse can be shown by reversing the argument. \square

LEMMA 3.5. For any $\xi \in \partial_\infty \mathbb{H}$ and for any $p \in \mathbb{A}(\xi) \subset \mathbb{A}$, the following conditions are equivalent.

- (1) Γp is dense in \mathbb{A} .
- (2) $0 \in \overline{\Gamma p}$.
- (3) ξ is a horocyclic limit point.

PROOF. (3) \Rightarrow (2): By (3), for any $p \in \mathbb{A}(\xi)$, there is $\gamma \in \Gamma$ such that $\gamma^{-1}i \in D(p)$. That is, $i \in D(\gamma p)$, namely $|\gamma p| < 1$. Since p is an arbitray point of $\mathbb{A}(\xi)$ and since the Γ -action on \mathbb{A} commutes with the scalar multiplicaton, this implies (2).

(2) \Rightarrow (1): For any $\gamma \in \Gamma \setminus \{e\}$, let $W^u(\gamma)$ be the ray in \mathbb{A} corresponding to the eigenspace of γ associated to the eigenvalue whose absolute value is bigger than 1. In other words,

$$W^u(\gamma) = \{q \in \mathbb{A} \mid |\gamma^{-n}q| \rightarrow 0, \quad n \rightarrow \infty\}.$$

Assume p satisfies (2). Then we have $\overline{\Gamma p} \cap W^u(\gamma) \neq \emptyset$. See Figure 3.

Choose $q \in \overline{\Gamma p} \cap W^u(\gamma)$. Let Γ_n be the fundamental group of the sub-surface Σ_n in Definition 1.1. The subgroups Γ_n are finitely generated with

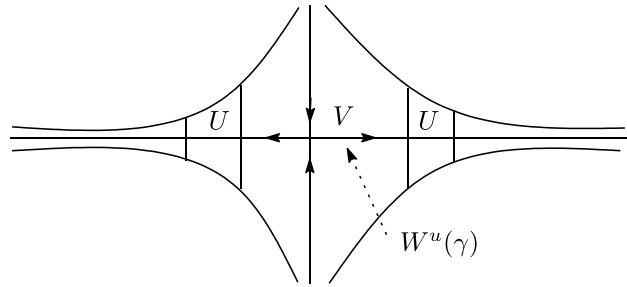


Fig. 3. U is a partial fundamental domain for the action of γ . If $\overline{\Gamma p}$ intersects V , it also intersects U . The four hyperbolae can be chosen arbitrarily near the axes.

Cantor limit sets Λ_n and form an exhausting sequence of subgroups of Γ . Moreover $\cup_n \Lambda_n$ is dense in $\partial_\infty \mathbb{H}$. Let \mathbb{A}_n be the inverse image of Λ_n by the canonical projection $\mathbb{A} \rightarrow \partial_\infty \mathbb{H}$. We have $\gamma \in \Gamma_n$ and $q \in \mathbb{A}_n$ for any large n . By the Hedlund theorem [5], the Γ_n actions on \mathbb{A}_n are minimal. In particular, $\overline{\Gamma q} \supset \mathbb{A}_n$. Since this holds for any large n and since $\overline{\cup_n \mathbb{A}_n} = \mathbb{A}$, we obtain $\overline{\Gamma q} = \mathbb{A}$. On the other hand, since $q \in \overline{\Gamma p}$, we have $\overline{\Gamma p} \supset \overline{\Gamma q}$, showing (1).

(1) \Rightarrow (3): For any $p \in \mathbb{A}(\xi)$, there is γ such that $|\gamma^{-1}p| < 1$. Then $i \in D(\gamma^{-1}p)$. We thus have $\Gamma i \cap D(p) \neq \emptyset$ for any horodisk $D(p)$ at ξ . \square

LEMMA 3.6. *There are horocyclic limit points and nonhorocyclic limit points.*

PROOF. Any point in $\partial_\infty \mathbb{H}$ which is fixed by any $\gamma \in \Gamma \setminus \{e\}$ is a horocyclic limit point. To show the second statement, let D_i be the Dirichlet fundamental domain of $i \in \mathbb{H}$. That is,

$$D_i = \{z \in \mathbb{H} \mid d(z, i) \leq d(z, \gamma i), \forall \gamma \in \Gamma\}.$$

Then any point ξ of $\overline{D_i} \cap \partial_\infty \mathbb{H}$ is a nonhorocyclic limit point. In fact, the horodisk $\{B_\xi < 0\}$ contains no point of Γi . See Figure 4. \square

4. Proof of Theorem 1.2

In this section Γ is to be a tight Fuchsian group.

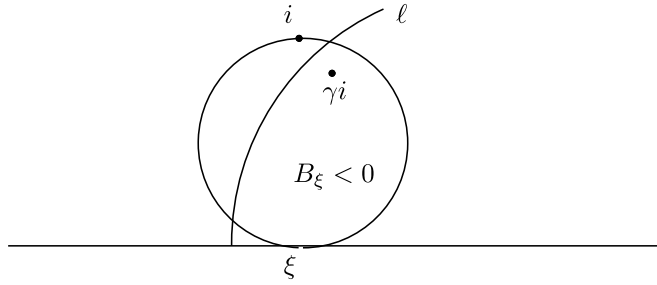


Fig. 4. If γi is contained in $\{B_\xi < 0\}$, then D_i must be contained in the region above the perpendicular bisector ℓ . A contradiction to the definition of ξ .

LEMMA 4.1. *Let $\xi \in \partial_\infty \mathbb{H}$ be a nonhorocyclic limit point. Then there is $r > 0$ such that for any $p \in \mathbb{A}(\xi)$, $e^{r/2}p \in \overline{\Gamma p}$.*

This lemma implies Theorem 1.2. In fact, if X is a minimal set for the Γ -action on \mathbb{A} . Then X must be a proper subset of \mathbb{A} by Lemmata 3.5 and 3.6. Choose $p \in X$ and let $p \in \mathbb{A}(\xi)$. Then ξ is a nonhorocyclic limit point by Lemma 3.5. The above lemma implies that there is $r > 0$ such that $X \cap e^{r/2}X \neq \emptyset$. Since X is minimal, this implies $X = e^{r/2}X$, showing that X contains 0 in its closure. This means that $0 \in \overline{\Gamma p}$, contrary to the fact that ξ is a nonhorocyclic limit point.

Lemma 4.1 reduces to the following lemma about the geodesic flow on $T^1\Sigma$.

LEMMA 4.2. *Let \tilde{v} be an arbitrary vector in $T^1\mathbb{H}$ such that $\xi = \tilde{v}(\infty)$ is a nonhorocyclic limit point, and let $v \in T_a^1\Sigma$ be the projected image of \tilde{v} ($a \in \Sigma$). Then there are sequences of vectors $v_n \in T_a^1\Sigma$ and positive numbers r_n such that $v_n \rightarrow v$, $r_n \rightarrow r > 0$ and $d(g^{t+r_n}(v_n), g^t(v)) \rightarrow 0$ as $t \rightarrow \infty$.*

Let us see that Lemma 4.2 implies Lemma 4.1. The last statement shows that $g^{r_n}(v_n)$ lies on the strong stable manifold of v . Thus we have $g^{r_n}(v_n) = h^{s_n}(v)$ for some $s_n \in \mathbb{R}$. We assumed $v_n = g^{-r_n}h^{s_n}(v) \rightarrow v$. Now the family $\{g^{r_n}\}$ is equicontinuous at v , because $r_n \rightarrow r$. Therefore $d(g^{r_n}(v), h^{s_n}(v)) \rightarrow 0$. That is, $h^{s_n}(v) \rightarrow g^r(v)$. Up on $T^1\mathbb{H} = PSL(2, \mathbb{R})$, this means that there are $\gamma_n \in \Gamma$ such that $\gamma_n \tilde{h}^{s_n}(\tilde{v}) \rightarrow \tilde{g}^r(\tilde{v})$. Let p be

the projection of \tilde{v} to \mathbb{A} . Then down on \mathbb{A} , we have $\gamma_n p \rightarrow e^{r/2} p$, showing Lemma 4.1.

PROOF OF LEMMA 4.2. It is no loss of generality to assume that $a \in \Sigma_1$, where $v \in T_a^1 \Sigma$. In fact one can take the subsurface Σ_1 in Definition 1.1 as large as we want. By the assumption on \tilde{v} , the geodesic ray $v[0, \infty)$ is a quasi-minimizer and thus proper. Let t_n be the maximum time when $\pi g^{t_n}(v)$ hits $\partial \Sigma_n$. Let c_n be a closed curve on $\partial \Sigma_n$ starting and ending at $\pi g^{t_n}(v)$. We choose the direction of c_n in such a way that the tangent vectors of the curves $v[0, \infty)$ and c_n form an angle $\leq \pi/2$ at the point $\pi g^{t_n}(v)$. See Figure 5.

By Definition 1.1 (2), there are $0 < c < C$ such that $c \leq |c_n| \leq C$ for any n . (If the boundary curve of $\partial \Sigma_n$ is too short, we choose c_n as its multiple.) Form a concatenation β_n^T of three curves $\pi g^t(v)$ ($0 \leq t \leq t_n$), c_n and $\pi g^t(v)$ ($t_n \leq t \leq T$), where T is some big number. Let α_n^T be the geodesic joining a and $\pi g^T(v)$ in the homotopy class of β_n^T . If $T \rightarrow \infty$, this curve converges to a geodesic ray $\pi g^t(v_n)$ for some $v_n \in T_a^1 \Sigma$. Moreover the two geodesic rays $v[0, \infty)$ and $v_n[0, \infty)$ are asymptotic. See Figure 6. We have $v_n \rightarrow v$ by virtue of the bound C in Definition 1.1. See Figure 7.

Since $v[0, \infty)$ and $v_n[0, \infty)$ are asymptotic, there is $r_n \in \mathbb{R}$ such that

$$d(\pi g^t(v), \pi g^{t+r_n}(v_n)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The directions are also asymptotic, and therefore

$$d(g^t(v), g^{t+r_n}(v_n)) \rightarrow 0 \text{ in } T^1 \Sigma.$$

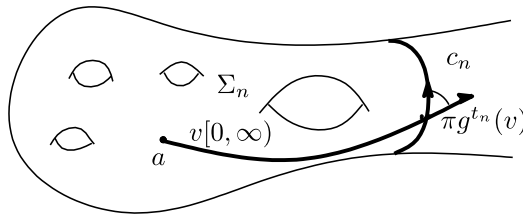


Fig. 5. The curves $v[0, \infty)$ and c_n .

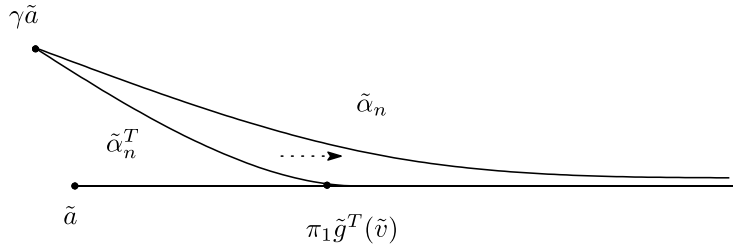


Fig. 6. A lift $\tilde{v}[0, \infty)$ of the curve $v[0, \infty)$ and a lift $\tilde{\alpha}_n^T$ of α_n^T to \mathbb{H} . For any T , the curves $\tilde{\alpha}_n^T$ start at the same point, say $\gamma_{\tilde{a}}$, and converges to $\tilde{\alpha}_n$. The projection of $\tilde{\alpha}_n$ to Σ is the curve $v_n[0, \infty)$.

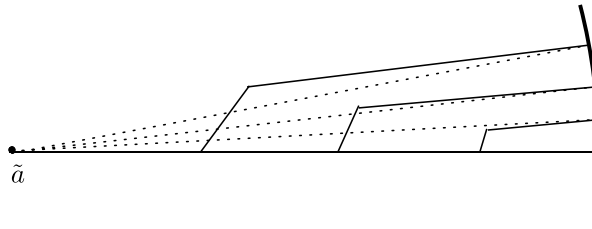


Fig. 7. The dotted lines are the lifts of curves $v_n[0, \infty)$ which starts at the same point \tilde{a} . This shows that $v_n \rightarrow v$.

Finally we have $r_n \in (b, C]$, where $b > 0$ is defined as follows: for any point z in a horocycle H , let $[z, w]$ be the geodesic segment of length $c/2$ tangent to H at z . Define b by $b = B_\xi(w) - B_\xi(z)$. For details see Figure 8. This shows Lemma 4.2. \square

The proof of Theorem 1.2 is now complete.

5. Examples and Remarks

Let Γ be a tight Fuchsian group. The bi-invariant Haar measure of $PSL(2, \mathbb{R})$ induces a measure m on $T^1\Sigma = \Gamma \backslash PSL(2, \mathbb{R})$ invariant both by the geodesic and horocycle flows. The spaces $\partial_\infty\mathbb{H}$ and \mathbb{A} are equipped

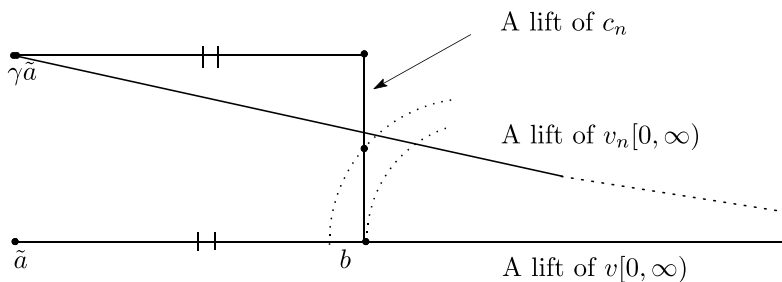


Fig. 8. The figure depicts the case where the length of c_n takes the minimal value c and $v[0, \infty)$ intersects c_n perpendicularly. The dotted circles are horocycles. Note that the lift of $v_n[0, \infty)$ intersects the lift of c_n at a point above the midpoint. Thus we have $B_\xi(\gamma\tilde{a}) - B_\xi(\tilde{a}) > b$.

with the standard Lebesgue measures. In what follows, all the statements concerning the measures are to be with respect to these measures. The left Γ action on \mathbb{A} is Morita equivalent to the horocycle flow on $T^1\Sigma$ in the measure theoretic sense. Especially the former is ergodic if and only if the latter is ([15], 2.2.3).

Denote by Λ_h the set of the horocyclic limit points. For a point $z \in \mathbb{H}$, denote the Dirichlet fundamental domain of z by D_z , i.e.

$$D_z = \{w \in \mathbb{H} \mid d(w, z) \leq d(w, \gamma z), \forall \gamma \in \Gamma\}.$$

Let $F_z = \overline{D}_z \cap \partial_\infty\mathbb{H}$ and $E_z = \Gamma F_z$. Sullivan [14] showed that $E_z \cup \Lambda_h$ is a full measure set of $\partial_\infty\mathbb{H}$ (for any Fuchsian group).

PROOF OF CORLLARY 1.3. By virtue of the Sullivan theorem and Lemma 3.5, we only need to show that F_z is a null set. Assume the contrary. For any $\gamma \in \Gamma \setminus \{e\}$, $\gamma F_z \cap F_z$ is at most two points, lying on the bisector of z and γz . Let

$$F'_z = F_z \setminus \bigcup_{\gamma \in \Gamma \setminus \{e\}} \gamma F_z,$$

and let B be the inverse image of F'_z by the canonincal projection $\mathbb{A} \rightarrow \partial_\infty\mathbb{H}$. Then B is a partial measurable fundamental domain for the Γ action on \mathbb{A} , i.e. B is positive measured and $B \cap \gamma B = \emptyset$ for any $\gamma \in \Gamma \setminus \{e\}$. By

the Morita equivalence, the horocycle flow also has a partial fundamental domain: there is a positive measured set $A \subset T^1\Sigma$ such that $A \cap h^n A = \emptyset$ for any $n \in \mathbb{Z} \setminus \{0\}$. To see this, consider the inverse image $\pi_2^{-1}(B)$ by the canonical projection $\pi_2 : PSL(2, \mathbb{R}) \rightarrow \mathbb{A}$. Clearly the \mathbb{Z} action h^n restricted to $\pi_2^{-1}(B)$ admits a fundamental domain A . On the other hand, $\pi_2^{-1}(B)$ can be embedded in $T^1\Sigma = \Gamma \backslash PSL(2, \mathbb{R})$ since $B \cap \gamma B = \emptyset$ for any $\gamma \in \Gamma \setminus \{e\}$.

Let us show that almost all points in A has a proper horocycle orbit. Choose an arbitrary compact set K of $T^1\Sigma$ and let

$$a_n = m(A \cap h^{-n}(K)) = m(h^n(A) \cap K).$$

Then we have

$$\sum_{n \in \mathbb{Z}} a_n \leq m(K) < \infty.$$

For any $n_0 \in \mathbb{N}$, we have

$$m(A \cap \bigcup_{|n| \geq n_0} h^{-n}(K)) \leq \sum_{|n| \geq n_0} a_n.$$

Therefore

$$m(A \cap \bigcap_{n_0 \in \mathbb{N}} \bigcup_{|n| \geq n_0} h^{-n}(K)) = 0.$$

Since K is an arbitrary compact set, this shows that almost all points in A admits a proper horocyclic orbit. But a proper orbit is a minimal set. This is against Theorem 1.2, completing the proof of Corollary 1.3. \square

Let us discuss Remark 1.4 by examples.

Example 5.1. Let Γ_0 be a cocompact purely hyperbolic Fuchsian group (a surface group), and let Γ be a nontrivial normal subgroup of Γ_0 of infinite index. Then Γ is a tight Fuchsian group.

When $G = \Gamma_0/\Gamma$ is free abelian, then the horocycle flow on $\Gamma \backslash PSL(2, \mathbb{R})$ is known to be ergodic [1], [10]. On the other hand, if G is nonamenable, there is a nonconstant bounded harmonic function on the surface $\Sigma = \Gamma \backslash \mathbb{H}$ [11]. That is, there is a nonconstant bounded Γ invariant measurable function on $\partial_\infty \mathbb{H}$, and therefore the Γ action on $\partial_\infty \mathbb{H}$ is not ergodic. This

implies that the Γ action on \mathbb{A} is not ergodic. By the Morita equivalence, the horocycle flow on $\Gamma \backslash PSL(2, \mathbb{R})$ is not ergodic.

Let \mathcal{F} be a surface foliation on a compact manifold. If \mathcal{F} admits no transverse invariant measures, then there is a continuous leafwise Riemannian metric of curvature -1 [3]. One may ask the following question.

QUESTION 5.2. Are generic leaves of \mathcal{F} either compact, planar, annular or tight?

See [4] for related topics. This is true for the Hirsch foliation [7] and Lie G foliations. For the latter, see [6] for the idea of the proof.

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