Stability of two soliton collision for nonintegrable gKdV equations *

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Abstract

We continue our study of the collision of two solitons for the subcritical generalized KdV equations

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0. \tag{0.1}$$

Solitons are solutions of the type $u(t,x) = Q_{c_0}(x - x_0 - c_0 t)$ where $c_0 > 0$. In [21], mainly devoted to the case $f(u) = u^4$, we have introduced a new framework to understand the collision of two solitons Q_{c_1}, Q_{c_2} for (0.1) in the case $c_2 \ll c_1$ (or equivalently, $\|Q_{c_2}\|_{H^1} \ll \|Q_{c_1}\|_{H^1}$). In this paper, we consider the case of a general nonlinearity $f(u)$ for which Q_{c_1}, Q_{c_2} are nonlinearly stable. In particular, since f is general and c_1 can be large, the results are not pertubations of the ones for the power case in [21].

First, we prove that the two solitons survive the collision up to a shift in their trajectory and up to a small perturbation term whose size is explicitely controlled from above: after the collision, $u(t) \sim Q_{c_1^+} + Q_{c_2^+}$ where c_j^+ is close to c_j ($j = 1, 2$). Then, we exhibit new exceptional solutions similar to multi-soliton solutions: for all c_1, $c_2 > 0$, $c_2 \ll c_1$, there exists a solution $\varphi(t)$ such that

$$\varphi(t,x) = Q_{c_1}(x - \rho_1(t)) + Q_{c_2}(x - \rho_2(t)) + \eta(t,x), \text{ for } t \ll -1,$$

$$\varphi(t,x) = Q_{c_1}(x - \rho_1(t)) + Q_{c_2}(x - \rho_2(t)) + \eta(t,x), \text{ for } t \gg 1,$$

where $\rho_j(t) \to c_j$ ($j = 1, 2$) and $\eta(t)$ converges to 0 in a neighborhood of the solitons as $t \to \pm\infty$.

The analysis is splitted in two distinct parts. For the interaction region, we extend the algebraic tools developed in [21] for the power case, by expanding $f(u)$ as a sum of powers plus a perturbation term. To study the solutions in large time, we rely on previous tools on asymptotic stability in [17], [22] and [18], refined in [19], [20].

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1 Introduction

We consider the generalized Korteweg-de Vries (gKdV) equations:

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad u(0) = u_0 \in H^1(\mathbb{R}), \tag{1.1}$$

for general C^s nonlinearity f for which small solitons are stable. We assume that for $p = 2$, 3 or 4,

$$f(u) = u^p + f_1(u) \quad \text{where } f_1 \text{ is } C^{p+4} \text{ and} \quad \lim_{u\to 0}\left|\frac{f_1(u)}{u^p}\right| = 0. \tag{1.2}$$

Remark that if the nonlinearity is of the form $f(u) = au^p + f_1(u)$, $a > 0$, then we may assume $a = 1$ by considering $\widetilde{u}(t,x) = a^{\frac{1}{p-1}} u(t,x)$ instead of $u(t,x)$ and changing f_1 accordingly. We only consider the case where $p = 2$, 3 or 4 in (1.2) since otherwise solitons with small speed would not be stable, which is necessary in this paper. Denote $F(s) = \int_0^s f(s')ds'$.

The Cauchy problem for equation (1.1) is locally well-posed in $H^1(\mathbb{R})$ (see Kenig, Ponce and Vega [12]). All solutions considered in this paper are global in time. For H^1 solutions, the following quantities are conserved:

$$\begin{aligned} &\int u^2(t,x)dx = \int u_0^2(x)dx, \\ &E(u(t)) = \frac{1}{2}\int (\partial_x u)^2(t,x)dx - \int F(u(t,x))dx = \frac{1}{2}\int (\partial_x u_0)^2(x)dx - \int F(u_0(x))dx. \end{aligned} \tag{1.3}$$

Recall that equation (1.1) has soliton solutions, i.e. of the form $u(t,x) = Q_c(x - x_0 - ct)$ where $c > 0$, $x_0 \in \mathbb{R}$ and

$$Q_c'' + f(Q_c) = c\,Q_c, \quad Q_c \in H^1. \tag{1.4}$$

Note that, for all $c > 0$, if $p = 2$, 4 then there is at most one solution of (1.4) (up to translations), which is positive, whereas for $p = 3$, it might exist a positive and a negative solution of (1.4). For all $c > 0$, if a solution $Q_c > 0$ of (1.4) exists then it can be chosen even on $\mathbb{R}$ and decreasing on $\mathbb{R}^+$ (and similarly if $Q_c < 0$). We refer to section 6 of Berestycki and Lions [1] for these properties and a necessary and sufficient condition for existence.

In this paper, we consider only nonlinearly stable solitons in the sense of Weinstein [29], i.e. such that

$$\frac{d}{dc'}\int Q_{c'}^2(x)dx_{|c'=c} > 0. \tag{1.5}$$

Note that since $p = 2$, 3 or 4 in (1.2), this condition is satisfied for $c > 0$ small enough. We recall the following stability result.

Stability result [29]. *Let $c > 0$ be such that (1.5) holds. Then, there exist $K, \alpha_0 > 0$ such that for any $u_0 \in H^1$, if $\|u_0 - Q_c\|_{H^1} \leq \alpha_0$, then the solution $u(t)$ of (1.1) is global and, for all $t \in \mathbb{R}$, $\inf_{y\in\mathbb{R}} \|u(t,.+y) - Q_c\|_{H^1} \leq K\alpha_0$.*

From [1] and (1.2), it follows that there exists $c_*(f) > 0$ (possibly $+\infty$) defined by

$$c_*(f) = \sup\{c > 0 \text{ such that } \forall c' \in (0,c), \exists Q_{c'} \text{ positive solution of (1.4)}\}.$$

In [19], we have proved that $0 < c < c_*(f)$ and (1.5) are sufficient conditions of asymptotic stability in the energy space H^1 around the soliton Q_c. Combining the stability result and the asymptotic stability result, we obtain the following.

Asymptotic stability [17], [19]. *Let* $0 < c < c_*(f)$ *be such that* (1.5) *holds. There exists* $\alpha_0 > 0$ *such that for any* $u_0 \in H^1$, *if* $\|u_0 - Q_c\|_{H^1} \leq \alpha_0$, *then the solution* $u(t)$ *of* (1.1) *is global and there exist* $c^+ \in (0, c_*(f))$, $t \mapsto \rho(t) \in \mathbb{R}$ *such that for all* $A > 0$,

$$u(t) - Q_{c^+}(. - \rho(t)) \to 0 \quad \text{in } H^1(x > \tfrac{c}{10}t) \text{ as } t \to +\infty. \tag{1.6}$$

We also recall from [15] the following result of existence and uniqueness of asymptotic N-soliton-like solutions (see Theorem 1 and Remark 2 in [15])

Asymptotic N-soliton-like solution [15]. *Let* $N \geq 1$ *and* $x_1, \ldots, x_N \in \mathbb{R}$. *Let* $0 < c_N < \ldots < c_1 < c_*(f)$ *be such that* (1.5) *holds for all* c_j, $j = 1, \ldots, N$. *Then, there exists a unique* H^1 *solution* $u(t)$ *of* (1.1) *such that*

$$\lim_{t \to -\infty} \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t) \right\|_{H^1} = 0.$$

Recall also that this behavior is in some sense stable in the energy space, see Martel, Merle and Tsai [22].

We are concerned with the problem of collision of two solitons. This is a classical problem in nonlinear wave propagation which we briefly review (see also the introduction of [21] and to the references therein). First, Fermi, Pasta and Ulam [6] and Zabusky and Kruskal [30] have exhibited from the numerical point of view remarkable phenomena related to soliton collision. Next, Lax [13] has developed a mathematical framework to study these problems, known now as complete integrability. The inverse scattering transform (for a review on this theory, we refer for example to Miura [23]) then provided explicit formulas for N-soliton solutions (Hirota [8]): let $f(u) = u^2$ or $f(u) = u^3$, and let $c_1 > \ldots > c_N > 0$, $\delta_1, \ldots, \delta_N \in \mathbb{R}$. There exists an explicit solution $U(t,x)$ of (1.1) which satisfies

$$\left\| U(t,x) - \sum_{j=1}^{N} Q_{c_j}(. - c_j t - \delta_j) \right\|_{H^1} \underset{t \to -\infty}{\longrightarrow} 0, \quad \left\| U(t,x) - \sum_{j=1}^{N} Q_{c_j}(. - c_j t - \delta'_j) \right\|_{H^1} \underset{t \to +\infty}{\longrightarrow} 0,$$

for some δ'_j such that the shifts $\Delta_j = \delta'_j - \delta_j$ depend on the (c_k). For example, the following function $U_{1,c}$, is a 2-soliton solution of (1.1) with $p = 2$, $(0 < c < 1)$:

$$U_{1,c}(t,x) = 6 \frac{\partial^2}{\partial x^2} \log\left(1 + e^{x-t} + e^{\sqrt{c}(x-ct)} + \alpha e^{x-t} e^{\sqrt{c}(x-ct)}\right) \quad \text{with} \quad \alpha = \left(\frac{1-\sqrt{c}}{1+\sqrt{c}}\right)^2. \tag{1.7}$$

As pointed out in [21], the problem of describing the collision of two traveling waves is a general problem for nonlinear PDEs, which is completely open, except in the integrable case described above. This kind of problems have been studied since the 60's from both experimental and numerical points of view.

We recall some numerical works for equations of gKdV type. Bona et al. [2], and Kalisch and Bona [11], studied numerically the problem of collision of two solitary waves for the Benjamin and the BBM equations. Shih [26] studied the case of the gKdV equation (1.1) with some half-integer values of p. Li and Sattinger [14] investigated the collision problem for the Ion Acoustic Plasma equation, and Craig et al. [5] report on numerics for the Euler equation with free surface. In all these works, the numerics match the experiments and show that for these models, unlike for the pure solitons of the integrable case, the collision of two solitary waves fails to be elastic by a very small but non zero dispersion.

Finally, the multi-soliton solutions of the NLS (nonlinear Schrödinger) model, with special nonlinearity and under spectral assumptions (ruling out the existence of small solitary waves) have been studied by Perelman [24] and Rodnianski, Schlag and Soffer [25] (in a special case where the collision has a negligeable effect on the solitary waves due to a very small time of interaction). See also Cao and Malomed [3], Holmer, Marzuola and Zworski [10], and Holmer and Zworski [9] for the case of the collision of a soliton of the NLS equation with a Dirac potential.

In [21], we present a complete rigorous description of the collision of two solitons of (1.1) for the nonlinearity $f(u) = u^4$ in the case where one soliton is small with respect to the other. First, we prove that the collision is not completely elastic in this case i.e. there does not exist pure 2-soliton solution (Theorem 1.1 in [21]). Note that this is the first rigorous result related to inelastic (but close to elastic) collision, and that a precise measurement of the defect follows from the analysis (see Theorems 1.1 and 1.2 in [21]). We also prove that for any solution behaving as $t \to -\infty$ approximately as the sum of two solitons of different sizes, the two solitons are preserved after the collision, with a residual term very small compared to the sizes of the two solitons. Moreover, we give a detailled description of the collision such that explicit formulas for the main orders of the shifts on the trajectories of the solitons (see Theorems 1.2 and 1.3 in [21]).

In this paper, we consider the same questions as in [21] for (1.1) with a general nonlinearity $f(u)$ satisfying (1.2). The results will apply in particular to $f(u) = u^2 + \lambda u^q$ or $f(u) = u^3 + \lambda u^q$, for $q > 2$, $\lambda \in \mathbb{R}$.

We consider two solitons Q_{c_1}, Q_{c_2}, with $0 < c_2 \ll c_1 < c_*(f)$. Note that the condition on c_1, i.e. $0 < c_1 < c_*(f)$ is not a smallness condition. Indeed, for many nonlinearities $c_*(f) = +\infty$. Thus, we do not simply perturb the power case.

Our first result describes the collision for the asymptotic 2-soliton like solutions.

Theorem 1.1 (Behavior after collision of the asymptotic 2-soliton-like solution) *Let $p = 2$, 3 or 4. Assume that f satisfies (1.2). Let $0 < c_1 < c_*(f)$ be such that the positive solution Q_{c_1} of (1.4) with $c = c_1$ satisfies (1.5). There exist $c_0 = c_0(c_1) \in (0, c_1)$ and $K = K(c_1) > 0$ such that for any $0 < c_2 < c_0$, if Q_{c_2} is a solution of (1.4) with $c = c_2$, then the following holds. Let $u(t)$ be the solution of (1.1) satisfying*

$$\lim_{t \to -\infty} \|u(t) - Q_{c_1}(. - c_1 t) - Q_{c_2}(. - c_2 t)\|_{H^1} = 0. \tag{1.8}$$

Then, there exist $\rho_1(t), \rho_2(t)$, $c_1^+ > c_2^+ > 0$ and $K > 0$ such that

$$w^+(t,x) = u(t,x) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t))$$

satisfies $\sup_{t \in \mathbb{R}} \|w^+(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}}$ *and*

$$\lim_{t \to +\infty} \|w^+(t)\|_{H^1(x > \frac{1}{10} c_2 t)} = 0, \quad \limsup_{t \to +\infty} \|w^+(t)\|_{H^1} \leq K c_2^{\frac{2}{p-1} - \frac{1}{4} - \frac{1}{100}}. \tag{1.9}$$

$$\lim_{t \to +\infty} |\rho_1'(t) - c_1^+| + |\rho_2'(t) - c_2^+| = 0. \tag{1.10}$$

Moreover, $\lim_{t\to+\infty} E(w^+(t)) = E^+$ *and* $\lim_{t\to+\infty}\int (w^+)^2(t) = M^+$ *exist and*

$$\frac{1}{2}\limsup_{t\to+\infty}\int((w_x^+)^2 + c_2(w^+)^2)(t) \leq 2E^+ + c_2M^+ \leq \liminf_{t\to+\infty}\int((w_x^+)^2 + 2c_2(w^+)^2)(t), \quad (1.11)$$

$$\frac{1}{K}(2E^+ + c_2M^+) \leq \frac{c_1^+}{c_1} - 1 \leq K(2E^+ + c_2M^+), \quad (1.12)$$

$$\frac{1}{K}c_2^{\frac{2}{p-1}-\frac{1}{2}}(2E^+ + c_1M^+) \leq 1 - \frac{c_2^+}{c_2} \leq Kc_2^{\frac{2}{p-1}-\frac{1}{2}}(2E^+ + c_1M^+). \quad (1.13)$$

By time and space translation invariances, the conclusions of Theorem 1.1 hold for any asymptotic 2−soliton solution. If $p = 2$ or 4, Q_{c_2} is necessarily a positive solution. If $p = 3$, Q_{c_2} can be positive or negative. By considering $-f(-u)$ instead of $f(u)$ one can also consider the case $Q_{c_1} < 0$ for $p = 3$.

Remark 1

1. Note that there exists $K > 0$ such that for c small,

$$\forall x \in \mathbb{R}, \quad \frac{1}{K}c^{\frac{1}{p-1}}e^{-\sqrt{c}|x|} \leq |Q_c(x)| \leq Kc^{\frac{1}{p-1}}e^{-\sqrt{c}|x|}, \quad (1.14)$$

so that, for c small,

$$\|Q_c\|_{H^1} \sim K_1c^{\frac{1}{p-1}-\frac{1}{4}}, \quad \|Q_c\|_{L^\infty} \sim K_2c^{\frac{1}{p-1}}. \quad (1.15)$$

A main information provided by Theorem 1.1 is that the 2-soliton structure is preserved for all time at the main order. Indeed, we observe that for $p = 2, 3$ or 4, $\frac{2}{p-1} - \frac{1}{4} - \frac{1}{100} < \frac{1}{p-1}$, thus from (1.9) and (1.15), the two soliton structure is recovered asymptotically in large time. Moreover, since $\sup_{t\in\mathbb{R}}\|w^+(t)\|_{H^1} \leq Kc_2^{\frac{1}{p-1}} \ll \|Q_{c_2}\|_{H^1}$, the 2 soliton structure is preserved also during the collision. Note that this estimate is optimal, the perturbation due to the collision being exactly of size $c_2^{\frac{1}{p-1}}$ in H^1 during the collision region.

Theorems 1.2 and 1.3 below give other illustrations of the stability of the two soliton dynamics through the collision.

2. Estimate (1.12) means that the speed of the soliton Q_{c_1} can only increase through the interaction, and that if $c_1^+ = c_1$ then $u(t)$ is a pure 2-soliton solution both at $+\infty$ and $-\infty$. Similarly, c_2 can only decrease. Remarkably, for $p = 3$, the property does not depend on the sign of Q_{c_2}.

Note that it is well-known for the case $f(u) = u^2$ or u^3 that the solution $u(t)$ considered in Theorem 1.1 is pure at $\pm\infty$ ($u(t)$ is explicit in the integrable cases). In contrast, in the case $f(u) = u^4$ it was proved in Theorem 1.1 of [21] that there exists no pure 2-soliton solution at both $+\infty$ and $-\infty$. In the general case $f(u)$, whether or not the collision is elastic is an open question. A natural question related to Theorem 1.1 is thus to try to understand, in the case of a general nonlinearity $f(u)$ in which situation the collision is elastic or inelastic, and what is the size of the defect.

Our second result is related to the construction of an object similar to the 2-soliton solutions with a perturbation term, such that the speeds as $t \to \pm\infty$ are the same. We also obtain an explicit formula for the first order of the resulting shift on the first soliton. The formula is related to the functions $c \mapsto \int Q_c$ and $c \mapsto \int Q_c^2$ for c close to c_1.

Theorem 1.2 (Existence of 2-soliton like solutions) *Let $p = 2$, 3 or 4. Assume that f satisfies (1.2). Let $0 < c_1 < c_*(f)$ be such that the positive solution Q_{c_1} of (1.4) with $c = c_1$ satisfies (1.5). There exist $c_0 = c_0(c_1) \in (0, c_1)$ and $K = K(c_1) > 0$ such that if $0 < c_2 < c_0$, and Q_{c_2} is solution of (1.4) with $c = c_2$, then there exist a global H^1 solution $\varphi(t) = \varphi_{c_1,c_2}(t)$ of (1.1) and Δ_1, $\Delta_2 \in \mathbb{R}$, $\rho_1(t)$, $\rho_2(t)$ satisfying, for all $t, x \in \mathbb{R}$,*

$$\varphi(-t, -x) = \varphi(t, x), \tag{1.16}$$

and such that the following holds for $w^{\pm}(t)$ where

$$w^-(t, x) = \varphi(t, x) - Q_{c_1}(x + \rho_1(-t)) - Q_{c_2}(x + \rho_2(-t)),$$

$$w^+(t, x) = \varphi(t, x) - Q_{c_1}(x - \rho_1(t)) - Q_{c_2}(x - \rho_2(t)),$$

1. *Asymptotic behavior at $\pm\infty$:*

$$\lim_{t\to-\infty} \|w^-(t)\|_{H^1(x<\frac{c_2 t}{10})} = 0, \quad \lim_{t\to+\infty} \|w^+(t)\|_{H^1(x>\frac{c_2 t}{10})} = 0, \tag{1.17}$$

$$\lim_{t\to+\infty} |\rho_1'(t) - c_1| + |\rho_2'(t) - c_2| = 0. \tag{1.18}$$

2. *Distance to the sum of two solitons: there exists $t_0 > 0$ such that*

$$\|w^-(-t)\|_{H^1} + \|w^+(t)\|_{H^1} \leq K c_2^{\frac{2}{p-1} - \frac{1}{4} - \frac{1}{100}}, \quad \text{for all } t > t_0. \tag{1.19}$$

3. *Shift property: there exist $\delta_1(c_1)$, $\delta_2(c_1) \in \mathbb{R}$ such that for $T_{c_1,c_2} = c_1^{\frac{3}{2}} \left(\frac{c_2}{c_1}\right)^{-\frac{1}{2} - \frac{1}{100}}$,*

$$|\rho_1(T_{c_1,c_2}) - (c_1 T_{c_1,c_2} + \tfrac{1}{2}\Delta_1)| \leq K c_2^{\frac{2}{p-1} - \frac{1}{2}}, \quad |\rho_2(T_{c_1,c_2}) - (c_2 T_{c_1,c_2} + \tfrac{1}{2}\Delta_2)| \leq K c_2^{\frac{1}{12}}, \tag{1.20}$$

$$\left| \Delta_1 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-1} - \frac{1}{2}} \delta_1(c_1) \right| \leq K c_2^{\frac{2}{p-1} - \frac{1}{2}}, \quad |\Delta_2 - \delta_2(c_1)| \leq K c_2^{\frac{1}{12}}. \tag{1.21}$$

Moreover,

$$\delta_1(c_1) = 2\,\mathrm{sgn}(Q_{c_2}(0)) \frac{\int Q_{c_1} \frac{d}{dc} \int Q_{c|c=c_1}}{\frac{d}{dc}\left(\int Q_c^2\right)_{|c=c_1}}. \tag{1.22}$$

Remark 2

1. By (1.2), assuming c_1 small is sufficient to ensure the assumptions of Theorem 1.2. However, Theorem 1.2 holds for any (c_1, c_2) such that $0 < c_1 < c_*$, $0 < c_2 < c_0(c_1)$ and (1.5) holds for c_1.

2. Recall that $\|Q_{c_2}\|_{L^2} \sim K c_2^{\frac{1}{p-1} - \frac{1}{4}}$. This is to be compared with the size of $w^{\pm}(t)$ in (1.19). Note that in estimate (1.19), $\frac{1}{100}$ has no particular meanning. By the technique of the present paper, one can get $\|w^+(t)\|_{H^1} \leq K(\epsilon_0) c_2^{\frac{2}{p-1} - \frac{1}{4} - \epsilon_0}$, for any $\epsilon_0 > 0$, which is sharp, see a lower bound on $w^+(t)$ for the case $f(u) = u^4$, in Theorem 1.2 in [21].

3. If there exists a Viriel property for $f(u)$ and Q_{c_1}, as it is the case for $f(u) = u^p$ ($p = 2, 3, 4$, see [21], [20]), then $\rho_j(t) - c_j t \to x_j^+$ as $t \to +\infty$, for some x_j^+ ($j = 1, 2$). In

particular, it is the case if c_1 is small since then the problem is a pertubation of $f(u) = u^p$ and the Viriel argument still works for $f(u)$.

Note also that at $t = T_{c_1,c_2}$, the two solitons are already decoupled, by exponential decay. Thus, (1.20) means that through the collision, the two solitons are shifted by Δ_1, respectively, Δ_2 at the first order. In (1.21), we see that the main part of Δ_1 (if $\delta_1 \neq 0$) is the product of a power of $\frac{c_2}{c_1}$ (depending only on p) by $\delta(c_1)$ which depends on Q_{c_1} and thus on the nonlinearity $f(s)$ on the interval $s \in [0, Q_{c_1}(0)]$. By the stability assumption, we have $\frac{d}{dc} \int Q^2_{c|_{c=c_1}} > 0$, but the other term in (1.22) $\frac{d}{dc} \int Q_{c|_{c=c_1}}$ may have any sign (for example, for $f(u) = u^p$, $p = 2$, 3 and 4 this term is respectively positive, zero and negative, see [21]). Note that the shift on Q_{c_1} depends on the sign of Q_{c_2}.

Similarly, we observe that $\delta_2(c_1)$ depends only on c_1. Thus, if $\delta_2 \neq 0$, it follows that the main order of the shift on Q_{c_2} is independent of c_2. In [21], we have computed δ_2 for $f(u) = u^4$ and there are well-known formulas for the case $p = 2$, 3 (see e.g. Miura [23]).

Theorem 1.3 (Stability of the 2-soliton structure) *Let $\varphi(t) = \varphi_{c_1,c_2}(t)$ be the solution constructed in Theorem 1.2, under the same assumptions. There exists $c_0 = c_0(c_1) \in (0, c_1)$ and $K = K(c_1) > 0$ such that if $0 < c_2 < c_0$ then the following holds. Assume that*

$$\|u_0 - \varphi(0)\|_{H^1} \leq c_2^{\frac{1}{p-1}+\frac{1}{2}}, \tag{1.23}$$

and let $u(t)$ be the H^1 solution of (1.1). Then, there exist $\rho_1(t)$, $\rho_2(t) \in \mathbb{R}$ and $c_1^{\pm}$, $c_2^{\pm} > 0$ such that

1. *Global in time stability:*

$$w(t,x) = u(t,x) - Q_{c_1}(x - \rho_1(t)) - Q_{c_2}(x - \rho_2(t)) \quad \textit{satisfies}$$

$$\|w(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}}, \quad \textit{for all } t \in \mathbb{R}. \tag{1.24}$$

2. *Asymptotic stability:*

$$\lim_{t\to-\infty} \|u(t) - Q_{c_1^-}(. - \rho_1(t)) - Q_{c_2^-}(. - \rho_2(t))\|_{H^1(x<\frac{c_2 t}{10})} = 0,$$

$$\lim_{t\to+\infty} \|u(t) - Q_{c_1^+}(. - \rho_1(t)) - Q_{c_2^+}(. - \rho_2(t))\|_{H^1(x>\frac{c_2 t}{10})} = 0,$$

$$\left|\frac{c_1^{\pm}}{c_1} - 1\right| \leq K c_2^{\frac{1}{p-1}+\frac{1}{2}}, \quad \left|\frac{c_2^{\pm}}{c_2} - 1\right| \leq K c_2^{\frac{1}{4}}.$$

Theorem 1.3 is the analogue of Theorem 1.3 in [21]. Note that since $\|Q_{c_2}\|_{H^1} \sim K c_2^{\frac{1}{p-1}-\frac{1}{4}}$, (1.24) means that the two solitons (even the smaller one) are preserved through the collision. The loss of a power $\frac{1}{2}$ in c between (1.23) and (1.24) is due to the difference of sizes of Q_{c_1} and Q_{c_2}.

Our approach is the same as in [21], the main tool being the construction of an approximate solution in the collision region. The large time behavior is controlled by asymptotic arguments, from [17], [22], [18] later refined in [16], [19] and [20].

The paper is organized as follows. In Section 2, we construct an approximate solution of (1.1) in a large time region including the collision. This section contains the main new arguments. In Section 3, we recall preliminary results for the asymptotics of the 2-soliton structure in large time. In Section 4, we prove Theorems 1.1, 1.2 and 1.3.

2 Construction of an approximate 2-soliton solution

For the sake of simplicity, we can first assume by scaling that $c_*(f) > 1$ and

$$c_1 = 1 \quad \text{and} \quad c_2 = c < c_0,$$

where $c_0 > 0$ is to be chosen small enough. We denote $Q_1 = Q > 0$ and we suppose that (1.5) holds for Q. Moreover, in what follows, we assume $Q_{c_2} > 0$, the case $Q_{c_2} < 0$ (and thus $p = 3$) is treated similarly. We construct an approximate solution of equation (1.1) close to the sum of two soliton solutions related to Q and Q_c on a large time interval containing the collision time. (The general case will follow by a scaling argument, see Corollary 2.1 in section 2.5.)

Let

$$T_c = c^{-\frac{1}{2}-\frac{1}{100}}. \tag{2.1}$$

(The power $\frac{1}{100}$ in the definition of T_c above can be replaced by any small number, giving a justification of Remark 2 following Theorem 1.2.)

Proposition 2.1 (Construction of an approximate solution of the gKdV eq.) *There exist $c_0(f) > 0$ and $K_0(f) > 0$ such that for any $0 < c < c_0(f)$, there exists a function $v = v_{1,c}$ such that the following hold.*

1. *Approximate solution on $[-T_c, T_c]$: for $j = 0, 1, 2$,*

$$S(t,x) = \partial_t v + \partial_x(\partial_x^2 v - v + f(v)) \quad \textit{satisfies} \tag{2.2}$$

$$\forall t \in [-T_c, T_c], \quad \|\partial_x^j S(t)\|_{L^2(\mathbb{R})} \leq K_0 c^{\frac{2}{p-1}+\frac{3}{4}}. \tag{2.3}$$

2. *Closeness to the sum of two solitons for $t = \pm T_c$: there exist Δ, Δ_c such that*

$$\begin{aligned} &\|v(T_c) - Q(. - \tfrac{1}{2}\Delta) - Q_c(. + (1-c)T_c - \tfrac{1}{2}\Delta_c)\|_{H^1} \leq K_0 c^{\frac{2}{p-1}+\frac{1}{4}}, \\ &\|v(-T_c) - Q(. + \tfrac{1}{2}\Delta) - Q_c(. - (1-c)T_c + \tfrac{1}{2}\Delta_c)\|_{H^1} \leq K_0 c^{\frac{2}{p-1}+\frac{1}{4}}, \end{aligned} \tag{2.4}$$

where

$$\left|\Delta - c^{\frac{1}{p-1}-\frac{1}{2}}\delta\right| \leq K_0 c^{\frac{2}{p-1}-\frac{1}{2}}, \quad |\Delta_c - \delta_c| \leq K_0 c^{\frac{1}{12}}, \tag{2.5}$$

$$\delta = 2\frac{\int Q \, \frac{d}{d\widetilde{c}} \int Q_{\widetilde{c}}{}_{|\widetilde{c}=1}}{\frac{d}{d\widetilde{c}}\left(\int Q_{\widetilde{c}}^2\right)_{|\widetilde{c}=1}}. \tag{2.6}$$

3. *Closeness to the sum of two solitons: for all $t \in [-T_c, T_c]$, there exists $y(t)$ such that*

$$\|v(t) - Q(. - y(t)) - Q_c(. + (1-c)t)\|_{H^1} \leq K_0 c^{\frac{1}{p-1}}. \tag{2.7}$$

To prove Proposition 2.1, we follow the same strategy as in [21], Sections 2 and 3. Here, we recall the main steps and only mention the parts which have to be adapted. We refer to [21] for more details.

Remark. It follows from the proof of Proposition 2.1 that the constants $c_0(f)$, $K_0(f)$ depend continuouly on $f \in C^{p+4}$.

Notation. For k, k', ℓ, $\ell' \in \mathbb{N}$, we denote

$$(k',\ell') \prec (k,\ell) \quad \text{if } k' < k \text{ and } \ell' \leq \ell \text{ or if } k' \leq k \text{ and } \ell' < \ell.$$

We denote by $\mathcal{Y}$ the set of functions $g \in C^\infty(\mathbb{R})$ such that

$$\forall j \in \mathbb{N},\ \exists K_j,\ r_j > 0,\ \forall x \in \mathbb{R}, \quad |g^{(j)}(x)| \leq K_j(1+|x|)^{r_j} e^{-|x|}.$$

Note that $\mathcal{Y}$ is stable by sum, multiplication and differentiation.

2.1 Choice of a decomposition for v

We look for $v(t,x)$ with a specific structure as in [19]. Let $k_0 \geq 1$, $\ell_0 \geq 0$, and

$$\Sigma_0 = \{(k,\ell),\ 1 \leq k \leq k_0,\ 0 \leq \ell \leq \ell_0\}.$$

We set

$$y_c = x + (1-c)t \quad \text{and} \quad R_c(t,x) = Q_c(y_c),$$
$$y = x - \alpha(y_c) \quad \text{and} \quad R(t,x) = Q(y),$$

where for $(a_{k,\ell})_{(k,\ell)\in\Sigma_0}$,

$$\alpha(s) = \int_0^s \beta(s')ds', \quad \beta(s) = \sum_{(k,\ell)\in\Sigma_0} a_{k,\ell}\, c^\ell Q_c^k(s). \tag{2.8}$$

The form of $v(t,x)$ is

$$v(t,x) = Q(y) + Q_c(y_c) + W(t,x), \tag{2.9}$$

$$W(t,x) = \sum_{(k,\ell)\in\Sigma_0} c^\ell \left(Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y) \right), \tag{2.10}$$

where $a_{k,\ell}$, $A_{k,\ell}$, $B_{k,\ell}$ are to be determined.

The motivation in [21] for choosing W of the form (2.10) is the stability of the family of functions

$$\left\{ c^\ell Q_c^k,\ c^\ell (Q_c^k)',\ k \geq 1,\ \ell \geq 0 \right\} \tag{2.11}$$

by multiplication and differentiation due to the power nonlinearity in the equation (see Lemma 2.1 in [21]). In the case of equation (1.1), for a general nonlinearity this structure is preserved up to a lower order term (see Lemma 2.1). Let

$$S(t,x) = \partial_t v + \partial_x(\partial_x^2 v - v + v^p). \tag{2.12}$$

Proposition 2.2 (Decomposition of $S(t,x)$) *Assume that f is of class C^{k_0+3}. Let*

$$\mathcal{L}w = -\partial_x^2 w + w - f'(Q)w. \tag{2.13}$$

Then,

$$\begin{aligned} S(t,x) = &\sum_{(k,\ell)\in\Sigma_0} c^\ell Q_c^k(y_c) \Big[a_{k,\ell}(-3Q + 2f(Q))'(y) - (\mathcal{L}A_{k,\ell})'(y) \Big] \\ &+ \sum_{(k,\ell)\in\Sigma_0} c^\ell (Q_c^k)'(y_c) \Big[a_{k,\ell}(-3Q'')(y) + \left(3A''_{k,\ell} + f'(Q)A_{k,\ell}\right)(y) - (\mathcal{L}B_{k,\ell})'(y) \Big] \\ &+ \sum_{(k,\ell)\in\Sigma_0} c^\ell \left(Q_c^k(y_c) F_{k,\ell}(y) + (Q_c^k)'(y_c) G_{k,\ell}(y) \right) + \mathcal{E}(t,x) \end{aligned}$$

where $F_{k,\ell}$, $G_{k,\ell}$ and $\mathcal{E}$ satisfy, for any $(k,\ell) \in \Sigma_0$,

(i) *Dependence property of $F_{k,\ell}$ and $G_{k,\ell}$: The expressions of $F_{k,\ell}$ and $G_{k,\ell}$ depend only on $(a_{k',\ell'})$, $(A_{k',\ell'})$, $(B_{k',\ell'})$ for $(k',\ell') \prec (k,\ell)$.*

(ii) *Parity property of $F_{k,\ell}$ and $G_{k,\ell}$: Assume that for any (k',ℓ') such that $(k',\ell') \prec (k,\ell)$ $A_{k',\ell'}$ is even and $B_{k',\ell'}$ is odd, then $F_{k,\ell}$ is odd and $G_{k,\ell}$ is even.*

Moreover, $F_{1,0} = (f'(Q))'$ and $G_{1,0} = f'(Q)$.

(iii) *Estimate on $\mathcal{E}$: there exists $\kappa(y) > 0$ (depending on $(a_{k,\ell})$ and $(A_{k,\ell})$, $(B_{k,\ell})$) such that*

$$\forall j = 0,1,2,\ \forall (t,x) \in [-T_c, T_c] \times \mathbb{R}, \quad |\partial_x^j \mathcal{E}(t,x)| \le \kappa(y)(Q_c^{k_0}(y_c) + c^{\ell_0}) Q_c(y_c). \quad (2.14)$$

Remark. Estimate (2.14) is only a first rough estimate on the rest term, which can not be used without further information on $\kappa(y)$. In Proposition 2.5, for the functions $(A_{k,\ell})$, $(B_{k,\ell})$ to be chosen in this paper, we estimate precisely the size of $\partial_x^j \mathcal{E}$ in L^2.

Before proving the above proposition, we recall the following properties of Q_c, proved in Appendix A.

Lemma 2.1 (Properties of Q_c) *For $0 < c \le 1$, $\forall k, \widetilde{k} \in \{1, \dots, k_0\}$,*

$$\frac{1}{K} c^{\frac{1}{p-1}} e^{-\sqrt{c}|x|} \le Q_c(x) \le K c^{\frac{1}{p-1}} e^{-\sqrt{c}|x|}, \quad |Q_c'(x)| \le K c^{\frac{1}{p-1}+\frac{1}{2}} e^{-\sqrt{c}|x|}, \quad (2.15)$$

$$(Q_c^k)'(Q_c^{\widetilde{k}})' = ck\widetilde{k} Q_c^{k+\widetilde{k}} + \sum_{p+1 \le k_1 \le k_0 - k - \widetilde{k} + 2} k\widetilde{k}\, \sigma_{k_1} Q_c^{k+\widetilde{k}+k_1-2} + O(Q_c^{k_0+1}), \quad (2.16)$$

$$(Q_c^k)'' = ck^2 Q_c^k + \sum_{k+p-1 \le k_1 \le k_0} \sigma_{k_1}^{k*} Q_c^{k_1} + O(Q_c^{k_0+1}), \quad (2.17)$$

$$(Q_c^k)^{(3)} = ck^2 (Q_c^k)' + \sum_{k+p-1 \le k_1 \le k_0} \sigma_{k_1}^{k*} (Q_c^{k_1})' + O(Q_c^{k_0+1}), \quad (2.18)$$

$$(Q_c^k)^{(4)} = c^2 k^4 Q_c^k + c \sum_{k+p-1 \le k_1 \le k_0} \sigma_{k_1}^{k**} Q_c^{k_1} + \sum_{k+2p-2 \le k_1 \le k_0} \sigma_{k_1}^{k***} Q_c^{k_1} + O(Q_c^{k_0+1}), \quad (2.19)$$

where σ_{k_1}, $\sigma_{k_1}^{k*}$, $\sigma_{k_1}^{k**}$ and $\sigma_{k_1}^{k***}$ are independent of c, and where $O(Q_c^k)$ is a function $\mathcal{E}$ satisfying for $j = 0,1,2$, $|\partial_x^j \mathcal{E}(t,x)| \le K Q_c^k(y_c)$, where K is independent of c.

Proof of Proposition 2.2. Inserting $v = R + R_c + W$ in the expression of $S(t,x)$ in (2.12), and using the equations of R and R_c, we obtain the following decomposition (see also [21], Proof of Proposition 2.2)

$$S(t,x) = \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}, \quad (2.20)$$

where

$$\mathbf{I} = \partial_t R + \partial_x(\partial_x^2 R - R + f(R)), \quad \mathbf{II} = \partial_x(f(R+R_c) - f(R) - f(R_c)),$$

$$\mathbf{III} = \partial_t W - \partial_x(\overline{\mathcal{L}} W), \quad \text{where } \overline{\mathcal{L}} W = -\partial_x^2 w + w - f'(R) w,$$

$$\mathbf{IV} = \partial_x(f(R+R_c+W) - f(R+R_c) - f'(R)W).$$

Now, we follow exactly the same steps as in Section 2 of [21], replacing Lemma 2.1 in [21] by Lemma 2.1 and using Taylor expansions. For example, by (1.2) for $k_0 \le p$, we have the following Taylor expansion of f and F:

$$\begin{aligned} f(s) &= s^p + f_1(s) = s^p + \sum_{p+1\le k_1\le k_0} \frac{1}{k_1!} s^{k_1} f_1^{(k_1)}(0) + s^{k_0+1}O(1), \\ F(s) &= \frac{1}{p+1}s^{p+1} + \sum_{p+2\le k_1\le k_0} \frac{1}{k_1!} s^{k_1} f_1^{(k_1-1)}(0) + s^{k_0+1}O(1). \end{aligned} \tag{2.21}$$

Decomposition of $\mathbf{I}$. As in the proof of Lemma A.1 in [21], we claim

$$\begin{aligned} \mathbf{I} &= \beta(y_c)(-3Q+2f(Q))'(y) + \beta'(y_c)(-3Q'')(y) + c\beta(y_c)Q'(y) + \beta''(y_c)(-Q')(y) \\ &\quad + \beta^2(y_c)(3Q^{(3)})(y) + \beta'(y_c)\beta(y_c)(3Q'')(y) + \beta^3(y_c)(-Q^{(3)})(y) \\ &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5 + \mathbf{I}_6 + \mathbf{I}_7. \end{aligned}$$

Using Claim A.1 (Appendix), we deduce that $\mathbf{I}$ has the following decomposition:

$$\mathbf{I} = \sum_{(k,\ell)\in\Sigma_0} c^\ell \Big(Q_c^k(y_c) a_{k,\ell}(-3Q+2f(Q))'(y) + (Q_c^k)'(y_c) a_{k,\ell}(-3Q'')(y) \Big) \tag{2.22}$$

$$+ \sum_{(k,\ell)\in\Sigma_0} c^\ell \Big(Q_c^k(y_c) F^{\mathbf{I}}_{k,\ell}(y) + (Q_c^k)'(y_c) G^{\mathbf{I}}_{k,\ell}(y) \Big) + O(Q_c^{k_0+1}), \tag{2.23}$$

where the main terms, i.e. (2.22) are coming from $\mathbf{I}_1$ and $\mathbf{I}_2$ and $F^{\mathbf{I}}_{k,\ell}$, $G^{\mathbf{I}}_{k,\ell}$ satisfy (i)-(ii) of Proposition 2.1.

Decomposition of $\mathbf{II}$. For this term, we use the Taylor decomposition of f both at 0 and at R, i.e.

$$\begin{aligned} f(R+R_c) - f(R) - f(R_c) &= \sum_{1\le k_1\le p-1} \frac{1}{k_1!} Q_c^{k_1}(y_c) f^{(k_1)}(Q(y)) \\ &\quad + \sum_{p\le k_1\le k_0} \frac{1}{k_1!} Q_c^{k_1}(y_c)(f^{(k_1)}(Q(y)) - f^{(k_1)}(0)) + O(Q_c^{k_0+1}). \end{aligned}$$

Then, by

$$\partial_x(g(y)) = (1-\beta(y_c))g'(y), \tag{2.24}$$

applied to $g(y,y_c) = f(Q(y)+Q_c(y_c)) - f(Q(y)) - f(Q_c(y_c))$, we obtain:

$$\mathbf{II} = \sum_{(k,\ell)\in\Sigma_0} c^\ell \Big(Q_c^k(y_c) F^{\mathbf{II}}_{k,\ell}(y) + (Q_c^k)'(y_c) G^{\mathbf{II}}_{k,\ell}(y) \Big) + O(Q_c^{k_0+1}), \tag{2.25}$$

where $F^{\mathbf{II}}_{k,\ell}$, $G^{\mathbf{II}}_{k,\ell}$ satisfy (i)-(ii). Note that $F^{\mathbf{II}}_{1,0} = (f'(Q))'$ and $G^{\mathbf{II}}_{1,0} = f'(Q)$.

Decomposition of $\mathbf{III}$. Since $W(t,x) = \sum_{(k,\ell)\in\Sigma_0} c^\ell \left(Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y) \right)$, we are reduced to compute $\partial_t w - \partial_x(\overline{\mathcal{L}} w)$ for terms of the type $w(t,x) = Q_c^k(y_c)A(y)$ and $w(t,x) =$

$(Q_c^k)'(y_c)B(y)$. We recall (see Claim A.3 in [21]), for $A(x) \in C^3$,

$$
\begin{aligned}
&\partial_t(Q_c^k(y_c)A(y)) - \partial_x(\overline{\mathcal{L}}(Q_c^k(y_c)A(y)))\\
&\quad = Q_c^k(y_c)(-\mathcal{L}A)'(y) + (Q_c^k)'(y_c)(3A'' + f'(Q)A - cA)(y)\\
&\quad + Q_c^k(y_c)\beta(y_c)(-3A'' - f'(Q_c)A + cA)'(y) + Q_c^k(y_c)\beta'(y_c)(-3A'')(y)\\
&\quad + Q_c^k(y_c)\beta''(y_c)(-A')(y) + Q_c^k(y_c)\beta^2(y_c)(3A^{(3)})(y)\\
&\quad + Q_c^k(y_c)\beta'(y_c)\beta(y_c)(3A'')(y) + Q_c^k(y_c)\beta^3(y_c)(-A^{(3)})(y)\\
&\quad + (Q_c^k)'(y_c)\beta(y_c)(-6A'')(y) + (Q_c^k)'(y_c)\beta'(y_c)(-3A')(y) + (Q_c^k)'(y_c)\beta^2(y_c)(3A'')(y)\\
&\quad + (Q_c^k)''(y_c)(3A')(y) + (Q_c^k)''(y_c)\beta(y_c)(-3A')(y) + (Q_c^k)^{(3)}(y_c)A(y).
\end{aligned}
$$

Note that a similar formula holds for $w(t,x) = (Q_c^k)'(y_c)B(y)$ (see Claim A.4 in [21]).

Then, from Lemma 2.1 and the decompositions of $\beta(y_c)$, $\beta''(y_c)$, $\beta^2(y_c)$, $\beta'(y_c)\beta(y_c)$ and $\beta^3(y_c)$ (see Claim A.1), we obtain the following decomposition for **III**:

$$
\mathbf{III} = \sum_{(k,\ell)\in\Sigma_0} c^\ell \Big(Q_c^k(y_c)(-\mathcal{L}A_{k,\ell})'(y) + (Q_c^k)'(y_c)\big(3A''_{k,\ell} + f'(Q)A_{k,\ell} - (\mathcal{L}B_{k,\ell})'\big)(y)\Big) \quad (2.26)
$$

$$
+ \sum_{(k,\ell)\in\Sigma_0} c^\ell \Big(Q_c^k(y_c)F^{\mathbf{III}}_{k,\ell}(y) + (Q_c^k)'(y_c)G^{\mathbf{III}}_{k,\ell}(y)\Big) + \mathcal{E}_{\mathbf{III}}(t,x) \quad (2.27)
$$

where $F^{\mathbf{III}}_{k,\ell}$, $G^{\mathbf{III}}_{k,\ell}$ satisfy (i)-(ii) and $\mathcal{E}_{\mathbf{III}}(t,x)$ satisfies (iii).

Decomposition of **IV**. Let $\mathbf{N} = f(R+R_c+W) - f(R+R_c) - f'(R)W$. Using Taylor formula and (2.24), we obtain

$$
\mathbf{N} = \sum_{k=2}^{k_0} \frac{1}{k!}((R_c+W)^k - R_c^k)f^{(k)}(R) + \mathcal{E}_{\mathbf{N}}(t,x), \quad (2.28)
$$

$$
\mathbf{IV} = \sum_{\substack{2\le k\le k_0\\ 0\le \ell\le \ell_0}} c^\ell \Big(Q_c^k(y_c)F^{\mathbf{IV}}_{k,\ell}(y) + (Q_c^k)'(y_c)G^{\mathbf{IV}}_{k,\ell}(y)\Big) + \mathcal{E}_{\mathbf{IV}}(t,x),
$$

where $F^{\mathbf{IV}}_{k,\ell}$ and $G^{\mathbf{IV}}_{k,\ell}$ satisfy (i)-(ii) and $\mathcal{E}_{\mathbf{IV}}(t,x)$ satisfies (iii).

2.2 Resolution of the systems $(\Omega_{k,\ell})$

Proposition 2.2 leads to the following decomposition of $S(t,x)$:

$$
\begin{aligned}
S(t,x) = &- \sum_{(k,\ell)\in\Sigma_0} c^\ell Q_c^k(y_c)\Big((\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2f(Q))' - F_{k,\ell}\Big)(y)\\
&- \sum_{(k,\ell)\in\Sigma_0} c^\ell (Q_c^k)'(y_c)\Big((\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - \big(3A''_{k,\ell} + f'(Q)A_{k,\ell}\big) - G_{k,\ell}\Big)(y) + \mathcal{E}(t,x).
\end{aligned}
$$

Therefore, we want to solve by induction on (k,ℓ) the following systems:

$$
(\Omega_{k,\ell}) \quad \begin{cases} (\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2f(Q))' = F_{k,\ell} \\ (\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - f'(Q)A_{k,\ell} = G_{k,\ell}. \end{cases}
$$

The first step is to establish a general existence result for the model system:

$$(\Omega)\quad \begin{cases} (\mathcal{L}A)' + a(3Q - 2f(Q))' = F \\ (\mathcal{L}B)' + a(3Q'') - 3A'' - f'(Q)A = G. \end{cases}$$

We introduce some notation and we recall well-known results concerning the operator $\mathcal{L}$.

Claim 2.1 *The function $\varphi(x) = -\frac{Q'(x)}{Q(x)}$ is odd and satisfies:*

(i) $\lim_{x\to-\infty}\varphi(x) = -1$*;* $\lim_{x\to+\infty}\varphi(x) = 1$*;*

(ii) $\forall x \in \mathbb{R}$, $|\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \le Ce^{-|x|}$.

(iii) $\varphi' \in \mathcal{Y}$, $(1 - \varphi^2) \in \mathcal{Y}$.

Proof of Claim 2.1. By (A.1), we have $\varphi^2 = \frac{Q'^2}{Q^2} = 1 - \frac{2F(Q)}{Q^2}$, thus (i) is a consequence of (1.2). Next, $\varphi' = \frac{1}{Q^2}((Q')^2 - Q''Q) = \frac{1}{Q^2}(Qf(Q) - 2F(Q))$, and (ii), (iii) follow from (1.2) and the decay of Q.

Lemma 2.2 (Properties of $\mathcal{L}$) *The operator $\mathcal{L}$ defined in $L^2(\mathbb{R})$ by* (2.13) *is self-adjoint and satisfies the following properties:*

(i) *There exist a unique $\lambda_0 > 0$, $\chi_0 \in H^1(\mathbb{R})$, $\chi_0 > 0$ such that $\mathcal{L}\chi_0 = -\lambda_0\chi_0$.*

(ii) *The kernel of $\mathcal{L}$ is $\{\lambda Q', \lambda \in \mathbb{R}\}$. Let $\Lambda Q = \frac{d}{dc}Q_{c|c=1}$, then $\mathcal{L}(\Lambda Q) = -Q$.*

(iii) *For all $h \in L^2(\mathbb{R})$ such that $\int hQ' = 0$, there exists a unique $\widetilde{h} \in H^2(\mathbb{R})$ such that $\int \widetilde{h}Q' = 0$ and $\mathcal{L}\widetilde{h} = h$; moreover, if h is even (respectively, odd), then $\widetilde{h}$ is even (respectively, odd).*

(iv) *For $h \in H^2(\mathbb{R})$, $\mathcal{L}h \in \mathcal{Y}$ implies $h \in \mathcal{Y}$.*

(v) *If $\frac{d}{d\widetilde{c}}\int Q^2_{\widetilde{c}|\widetilde{c}=c} > 0$ then there exists $\lambda_c > 0$ such that*

$$\int wQ_c = \int wQ_c' = 0 \quad\Rightarrow\quad \int (w_x^2 + cw^2 - f'(Q_c)w^2) \ge \lambda_c \int w^2.$$

Proof of Lemma 2.2. See Weinstein [28] and proof of Lemma 2.2 in [21].

We claim the following general existence result for (Ω) (similar to Proposition 2.3 in [21]):

Proposition 2.3 (Existence for the model problem (Ω)) *Let $F, G : \mathbb{R} \to \mathbb{R}$ such that*

$$F(x) = \overline{F}(x) + \widetilde{F}(x) + \varphi(x)\widehat{F}(x), \quad G(x) = \overline{G}(x) + \widetilde{G}(x) + \varphi(x)\widehat{G}(x),$$

- $\overline{F}$, $\overline{G} \in \mathcal{Y}$*;* $\overline{F}$ *is odd and* $\overline{G}$ *is even;*
- $\widetilde{F}$ *and* $\widehat{G}$ *are odd polynomial functions;* $\widehat{F}$ *and* $\widetilde{G}$ *are even polynomial functions.*

Then, there exist $a \in \mathbb{R}$ and two functions $A(x)$, $B(x)$ satisfying (Ω) and such that

$$A(x) = \overline{A}(x) + \widetilde{A}(x) + \varphi(x)\widehat{A}(x), \quad B(x) = \overline{B}(x) + \widetilde{B}(x) + \varphi(x)\widehat{B}(x),$$

- $\overline{A}$, $\overline{B} \in \mathcal{Y}$; $\overline{A}$ *is even and* $\overline{B}$ *is odd;*
- $\widetilde{A}$ *and* $\widehat{B}$ *are even polynomial functions;* $\widehat{A}$ *and* $\widetilde{B}$ *are odd polynomial functions.*

Moreover,

$$\text{if } \widetilde{F} = 0 \text{ (respectively, } \widehat{F} = 0\text{) then } \widetilde{A} = 0 \text{ (respectively, } \widehat{A} = 0\text{);} \tag{2.29}$$

$$\text{if } \widetilde{A}'' = 0 \text{ and } \widetilde{G} = 0 \text{ then } \widetilde{B} = 0; \quad \text{if } \widehat{A}'' = 0 \text{ and } \widehat{G} = 0 \text{ then } \deg \widehat{B} = 0. \tag{2.30}$$

Remark. In Proposition 2.3, we find one solution of system (Ω). This solution is not unique but this does not play a role in this paper. See Corollary 3.1 in [21] for the uniqueness question.

Note that as a consequence of (2.30), it could be that $\widehat{B} = b \in \mathbb{R}$ while $\widehat{A}'' = \widehat{G} = 0$. This has the consequence to possibly develop polynomial growths in the functions $A_{k,\ell}$, $B_{k,\ell}$. In the rest of this paper, it will be sufficient to consider indices (k, ℓ) for which $\widehat{B}_{k,\ell}$ is a constant and the other polynomials $\widetilde{A}, \widehat{A} = 0, \widetilde{B} = 0$ are zero, see Proposition 2.4. However, if one wants to solve the systems $(\Omega_{k,\ell})$ for large k, ℓ, polynomial growths appear in general, see [21].

Sketch of the proof of Proposition 2.3. As in the proof of Proposition 2.3 in [21], we first reduce the proof to the case where the second members do not contain polynomials and thus are in $\mathcal{Y}$.

Step 1. Following step 1 of the proof of Proposition 2.3 in [21], considering

$$-\widetilde{A}''(x) + \widetilde{A}(x) = \int_0^x \widetilde{F}(z)dz, \quad -\widehat{A}''(x) + \widehat{A}(x) = \int_0^x \widehat{F}(z)dz,$$

$$-\widetilde{B}''(x)+\widetilde{B}(x) = \int_0^x \left(\widetilde{G}(z)+3\widetilde{A}''(z)\right) dz, \quad -(\widehat{B}^*)''(x)+\widehat{B}^*(x) = \int_0^x \left(\widehat{G}(z)+3\widehat{A}''(z)\right) dz,$$

where $\widehat{B} = \widehat{B}^* + b$, and using the exponential decay of $f'(Q)$, we reduce ourselves to solving the following system in $(a, b, \overline{A}, \overline{B})$:

$$\begin{cases} (\mathcal{L}\overline{A})' + a(3Q - 2f(Q))' = \mathcal{F} \\ (\mathcal{L}\overline{B})' + a(3Q'') - 3\overline{A}'' - f'(Q)\overline{A} = \mathcal{G} + b(\mathcal{L}\varphi)', \end{cases}$$

where $\mathcal{F} \in \mathcal{Y}$ is odd, $\mathcal{G} \in \mathcal{Y}$ is even and $\mathcal{F}$, $\mathcal{G}$ do not depend on the parameters a and b. See [21] for more details.

Step 2. Existence of a solution to the reduced system. Set $\mathcal{H}(x) = \int_{-\infty}^x \mathcal{F}(z)dz$. Since $\mathcal{F}$ is odd, $\int_{\mathbb{R}} \mathcal{F} = 0$ and so $\mathcal{H} \in \mathcal{Y}$ is even. To find a solution $(a, b, \overline{A}, \overline{B})$ of $(\overline{\Omega})$, it is sufficient to solve

$$(\overline{\Omega}) \qquad \begin{cases} \mathcal{L}\overline{A} + a(3Q - 2f(Q)) = \mathcal{H} \\ (\mathcal{L}\overline{B})' + a(3Q'') - 3\overline{A}'' - f'(Q)\overline{A} = \mathcal{G} + b(\mathcal{L}\varphi)'. \end{cases}$$

Since $\int \mathcal{H}Q' = 0$ (by parity) and $\mathcal{H} \in \mathcal{Y}$, it follows from Lemma 2.2 (iii)-(iv) that there exists

$$\overline{H} \in \mathcal{Y}, \text{ even, such that } \mathcal{L}\overline{H} = \mathcal{H}. \tag{2.31}$$

By Lemma 2.2, there also exists

$$V_0 \in \mathcal{Y}, \text{ even, such that } \mathcal{L}V_0 = 3Q - 2f(Q). \tag{2.32}$$

It follows that, for all a,

$$\overline{A} = \overline{H} - aV_0 \tag{2.33}$$

is solution of $\mathcal{L}\overline{A} + a(3Q - 2f(Q)) = \mathcal{H}$, moreover, $\overline{A}$ is even and $\overline{A} \in \mathcal{Y}$. Note that at this point (a, b) are still free, they will be used to solve the second equation. Indeed, replacing $\overline{A}$ by $\overline{H} - aV_0$ in this equation, solving $(\overline{\Omega})$ is equivalent to finding $(a, b, \overline{B})$ such that

$$(\mathcal{L}\overline{B})' = -aZ_0 + D + b(\mathcal{L}\varphi)', \tag{2.34}$$

where

$$D = 3\overline{H}'' + f'(Q)\overline{H} + \mathcal{G}, \quad Z_0 = 3Q'' + 3V_0'' + f'(Q)V_0.$$

It follows from the properties of Q, V_0, $\mathcal{G}$ and $\overline{H}$ that D and Z_0 are even and satisfy Z_0, $D \in \mathcal{Y}$. To solve (2.34), it suffices to find $\overline{B} \in \mathcal{Y}$ such that

$$\mathcal{L}\overline{B} = E \quad \text{where} \quad E = \int_0^x (D - aZ_0)(z)dz + b\mathcal{L}\varphi. \tag{2.35}$$

We now choose (a, b) such that the function E is orthogonal to Q' and has decay at ∞. First, we claim a nondegeneracy condition on Z_0, related to the strict stability of the soliton Q (i.e. assumption (1.5)). This is a nontrivial extension of Claim 2.3 in [21], which means that the solvability of (Ω) is related to the noncriticality of Q.

Claim 2.2 (Nondegeneracy condition)

$$\int Z_0 Q = -\frac{1}{2}\frac{d}{dc}\int Q_c^2\Big|_{c=1} = -\int \Lambda Q\, Q \neq 0. \tag{2.36}$$

Assuming Claim 2.2, we finish the proof of Proposition 2.3. Let

$$a = \frac{\int DQ}{\int Z_0 Q} \quad \text{and} \quad b = -\int_0^{+\infty} (D - aZ_0)(z)dz. \tag{2.37}$$

Then, E defined by (2.35) satisfies

$$E \in \mathcal{Y}, \quad E \text{ is odd}, \quad \int EQ' = 0. \tag{2.38}$$

Indeed, by integration by parts, and decay properties of Q, we have

$$\int EQ' = -\int (D - aZ_0)Q + b\int (\mathcal{L}\varphi)Q' = -\int DQ + a\int Z_0 Q + b\int \varphi(\mathcal{L}Q') = 0,$$

by (2.37) and $\mathcal{L}Q' = 0$. By Claim 2.1 and (2.37), we have

$$\lim_{+\infty} E = \int_0^{+\infty} (D - aZ_0)\, dz + b\lim_{+\infty}(\mathcal{L}\varphi) = 0 \quad \text{and so } E \in \mathcal{Y}.$$

For (a, b) fixed as in (2.37), from (2.38) and Lemma 2.2, it follows that there exists $\overline{B} \in \mathcal{Y}$ such that $\mathcal{L}\overline{B} = E$. Setting

$$A = \overline{A} + \widetilde{A} + \widehat{A}, \quad B = \overline{B} + \widetilde{B} + \widehat{B},$$

we have constructed a solution of system (Ω). $\square$

Proof of Claim 2.2. Let ΛQ be defined in Lemma 2.2; recall that $\mathcal{L}(\Lambda Q) = -Q$. Note also that $\mathcal{L}(xQ') = -2Q''$ (since $\mathcal{L}Q' = 0$). Thus, V_0 defined by (2.32) is $V_0 = -\Lambda Q - xQ'$. Therefore,

$$\begin{aligned}\int Z_0 Q &= 3\int Q''Q + \int (3v_0'' + f'(Q)V_0)Q == 3\int Q''Q + \int V_0(3Q'' + Qf'(Q)) \\ &= -3\int (Q')^2 - \int (\Lambda Q + xQ')(3Q'' + Qf'(Q)).\end{aligned}$$

First,

$$\begin{aligned}-\int xQ'(3Q'' + Qf'(Q)) &= -\int xQ'(4Q'' - Q + f(Q) + Qf'(Q)) \\ &= 2\int (Q')^2 - \frac{1}{2}\int Q^2 + \int Qf(Q).\end{aligned}$$

Since $\mathcal{L}Q = -Q'' + Q - Qf'(Q)$, we also have $\mathcal{L}(Q + \Lambda Q + xQ') = -3Q'' - Qf'(Q)$ and thus

$$\begin{aligned}-\int \Lambda Q(3Q'' + Qf'(Q)) &= \int \Lambda Q \mathcal{L}(Q + \Lambda Q + xQ') = -\int Q(Q + \Lambda Q + xQ') \\ &= -\frac{1}{2}\int Q^2 - \int \Lambda Q\, Q.\end{aligned}$$

Thus, we obtain by $\int (Q')^2 + \int Q^2 = \int Qf(Q)$,

$$\int Z_0 Q = -\int (Q')^2 - \int Q^2 + \int Qf(Q) - \int \Lambda Q\, Q = -\int \Lambda Q\, Q.$$

Proposition 2.3 allows us to solve the systems $(\Omega_{k,\ell})$ for all $(k,\ell) \in \Sigma_0$, for any $k_0 \geq 1$, $\ell_0 \geq 0$ (as in [21]). In the present paper, for the sake of simplicity, we work for the minimal set of indices so that we are able to prove Theorems 1 and 2. Indeed, let us define

$$\Sigma_p = \{(k,\ell) \mid \ell = 0,\ 1 \leq k \leq p,\ \text{or}\ \ell = 1,\ k = 1\}. \tag{2.39}$$

Using Propositions 2.2 and 2.3, we solve the systems $(\Omega_{k,\ell})$ by induction on $(k,\ell) \in \Sigma_p$, following [21].

Proposition 2.4 (Resolution of $(\Omega_{k,\ell})$ for $(k,\ell) \in \Sigma_p$) *For all $(k,\ell) \in \Sigma_p$, there exists $(a_{k,\ell}, A_{k,\ell}, B_{k,\ell})$ of the form*

$$\begin{aligned}&A_{k,\ell}(x) = \overline{A}_{k,\ell}(x) \in \mathcal{Y}, \quad B_{k,\ell}(x) = \overline{B}_{k,\ell}(x) + \varphi(x) b_{k,\ell}(x), \quad b_{k,0} \in \mathbb{R},\ \overline{B}_{k,\ell} \in \mathcal{Y}, \\ &A_{k,\ell} \text{ is even and } B_{k,\ell} \text{ is odd,}\end{aligned} \tag{2.40}$$

satisfying

$$(\Omega_{k,\ell}) \quad \begin{cases} (\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2f(Q))' = F_{k,\ell} \\ (\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - f'(Q)A_{k,\ell} = G_{k,\ell}, \end{cases}$$

where $F_{k,\ell}$, $G_{k,\ell}$ are defined in Proposition 2.2.

As a consequence of Proposition 2.4, we see that by restricting the sum defining $v(t,x)$ to the set of indices Σ_p, all the functions $A_{k,\ell}$ belong to $\mathcal{Y}$ and the functions $B_{k,\ell}$ are bounded with derivatives in $\mathcal{Y}$. This will simplify the proof of the estimates in Proposition 2.5 with respect to the general estimates proved in [21].

Proof of Proposition 2.4. 1. Case $k=1$, $\ell=0$. Recall that from Proposition 2.2, the functions $F_{1,0}, G_{1,0} \in \mathcal{Y}$ are explicit. Thus, from Proposition 2.3 (2.29)-(2.30), the system $(\Omega_{1,0})$ has a solution $(a_{1,0}, A_{1,0}, B_{1,0})$ such that

$$\widetilde{A}_{1,0} = \widehat{A}_{1,0} = \widetilde{B}_{1,0} = 0 \quad \text{and} \quad \widehat{B}_{1,0} = b_{1,0},\ b_{1,0} \in \mathbb{R}.$$

2. Case $2 \le k \le p$, $\ell = 0$. In this case, by induction on $1 \le k \le p$, we solve $(\Omega_{k,0})$, and we prove

$$\widetilde{A}_{k,0} = \widehat{A}_{k,0} = \widetilde{B}_{k,0} = 0 \quad \text{and} \quad \widehat{B}_{k,0} = b_{k,0},\ b_{k,0} \in \mathbb{R}. \tag{2.41}$$

The argument consists in proving that if property (2.41) is satisfied for all $1 \le k' < k$, then $F_{k,0}, G_{k,0} \in \mathcal{Y}$, and thus by Proposition 2.3, (2.41) holds for k as well. This has been checked in detail in [21], see Claim 2.4 and Lemma B1 (except for the case $k=p$). First, it is quite clear that **I** and **II** (see Proposition 2.2) contribute of terms $F^{\mathbf{I,II}}_{k,0}, G^{\mathbf{I,II}}_{k,0} \in \mathcal{Y}$, see also proof of Lemma B.1 in [21]. For the term **III** in the decomposition of $S(t,x)$, which is linear in W, the proof is exactly the same as in Claim 2.4 of [21].

Now, we give some details concerning the term **IV**. Recall first that $\mathbf{IV} = \partial_x \mathbf{N}$, where $\mathbf{N} = f(R+R_c+W) - f(R+R_c) - f'(R)W$. In the Taylor expansion (2.28), for $2 \le k_1 \le p-1$, the term $f^{(k-1)}(R(x))$ decays as $e^{-|x|}$, by (1.2), thus the contribution of these terms to $F_{k',\ell'}$, $G_{k',\ell'}$ are in $\mathcal{Y}$. For $k=p$, the term $f^{(k-1)}(R(x))$ is bounded and the term of lower order in $((R_c+W)^p - R_c^p)$ which is not in $\mathcal{Y}$ comes from $B_{1,0} = \overline{B}_{1,0} + b_{1,0}\varphi$. Thus, the lowest order term not localized in the y variable is

$$pb_{1,0}(Q_c^{p-1}Q_c'\varphi)_x = pb_{1,0}(Q_c^{p-1}Q_c'' + (p-1)Q_c^{p-2}(Q_c')^2) + pb_{1,0}Q_c^{p-1}Q_c'\varphi'.$$

Using Lemma 2.1, this term does not give contribution for $\ell = 0$, $k = p$.

It follows that $F_{k,0}, G_{k,0} \in \mathcal{Y}$, and thus by Proposition 2.3, we obtain a solution satisfying (2.41).

3. Case $k=1$, $\ell=1$. This case is handled in the same way, we notice that $F_{1,1}, G_{1,1} \in \mathcal{Y}$, and conclude that

$$\widetilde{A}_{1,1} = \widehat{A}_{1,1} = \widetilde{B}_{1,1} = 0 \quad \text{and} \quad \widehat{B}_{1,1} = b_{1,1},\ b_{1,1} \in \mathbb{R}. \tag{2.42}$$

2.3 Definition of $v(t)$ and estimates on $S(t,x)$

We define the function $v(t,x)$ as follows. For $(k,\ell) \in \Sigma_p$, we consider $(a_{k,\ell}, A_{k,\ell}, B_{k,\ell})$ defined in Proposition 2.4, and $v(t,x)$ defined by

$$v(t,x) = Q(y) + Q_c(y_c) + \sum_{(k,\ell)\in\Sigma_p} c^\ell \Big(Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y) \Big) \tag{2.43}$$

where $y_c = x + (1-c)t$, $y = x - \alpha(y_c)$ and

$$\alpha(s) = \int_0^s \beta(s')ds', \quad \beta(s) = \sum_{(k,\ell)\in\Sigma_p} a_{k,\ell}\, c^\ell Q_c^k(s). \tag{2.44}$$

For this choice of function $v(t,x)$ and for $S(t,x)$ defined by (2.2), we claim the following estimates.

Proposition 2.5 (Estimates on V **and** S**)** *For any* $0 < c < 1$, *for any* $t \in [-T_c, T_c]$, $W(t)$, $S(t)$ *belong to* $H^s(\mathbb{R})$ *for all* $s \geq 1$ *and satisfy*

$$\|W(t)\|_{H^1} = \|v(t) - R(t) - R_c(t)\|_{H^1} \leq K c^{\frac{1}{p-1}}, \tag{2.45}$$

$$\inf_{y_1 \in \mathbb{R}} \|v(t) - Q(. - y_1) - Q_c(. + (1-c)t)\|_{H^1} \leq K c^{\frac{1}{p-1}}. \tag{2.46}$$

$$j = 0,1,2, \quad \|\partial_x^{(j)} S(t)\|_{L^2} \leq K_j c^{\frac{2}{p-1} + \frac{3}{4}}, \tag{2.47}$$

Proof of Proposition 2.5. The proof of Proposition 2.5 is based on explicit estimates on $|\alpha'|$ and on all terms of $v(t,x)$ and $S(t,x)$. Recall from Proposition 2.4 that since $v(t,x)$ is defined only with $(k,\ell) \in \Sigma_p$, we have $A_{k,\ell} \in \mathcal{Y}$ and $B_{k,\ell} \in L^\infty$, with derivatives in $\mathcal{Y}$.

First, we claim

$$\forall s \in \mathbb{R}, \quad |\alpha(s)| \leq K c^{\frac{1}{p-1} - \frac{1}{2}}, \quad |\beta(s)| = |\alpha'(s)| \leq K c^{\frac{1}{p-1}}. \tag{2.48}$$

Indeed, for c small,

$$|\alpha(s)| \leq \sum_{(k,\ell)\in\Sigma_p} \left| a_{k,\ell}\, c^\ell \int_0^s Q_c^k(s')ds' \right| \leq \max_{(k,\ell)\in\Sigma_p} |a_{k,\ell}| \ \times \sum_{(k,\ell)\in\Sigma_p} \int Q_c^k \leq K \int Q_c.$$

Since $Q_c(s') \leq K c^{\frac{1}{p-1}} \exp(-\sqrt{c}|s'|)$, $\|\alpha\|_{L^\infty} \leq K \int Q_c \leq K c^{\frac{1}{p-1} - \frac{1}{2}}$. Similarly, $\|\alpha'\|_{L^\infty} \leq K c^{\frac{1}{p-1}}$.

Proof of (2.45). For all $(k,\ell) \in \Sigma_p$, since $A_{k,\ell} \in \mathcal{Y}$ and $B_{k,\ell} \in L^\infty$, we have

$$\|c^\ell Q_c^k(y_c) A_{k,\ell}(y)\|_{L^2} \leq K c^\ell \|Q_c^k\|_{L^\infty} \leq K c^{\frac{1}{p-1}},$$
$$\|c^\ell (Q_c^k)'(y_c) B_{k,\ell}(y)\|_{L^2} \leq K c^\ell \|(Q_c^k)'\|_{L^2} \leq K c^{\frac{1}{p-1} + \frac{1}{4}}.$$

The same is true for $\partial_x W(t,x)$ using (2.48).

Proof of (2.46). Since $R_c(t) = Q_c(x + (1-c)t)$, we only have to prove that, for all $t \in [-T_c, T_c]$,

$$\inf_{y \in \mathbb{R}} \|R(t) - Q(. - y)\|_{H^1} \leq K c^{\frac{1}{p-1}}. \tag{2.49}$$

By (2.48), taking c small enough so that $|\alpha'(t)| < \frac{1}{2}$, for all $t \in [-T_c, T_c]$, there exists a unique $y(t)$ such that $y(t) - \alpha(y(t) + (1-c)t) = 0$. Then,

$$\begin{aligned}\|R(t) - Q(. - y(t))\|_{H^1} &= \|Q(. - (\alpha(x + y(t) + (1-c)t) - y(t))) - Q\|_{H^1} \\ &= \|Q(. - (\alpha(x + y(t) + (1-c)t) - \alpha(y(t) + (1-c)t))) - Q\|_{H^1}\end{aligned}$$

By (2.48), we have $|\alpha(x + y(t) + (1-c)t) - \alpha(y(t) + (1-c)t)| \leq K c^{\frac{1}{p-1}} |x|$. Thus, we obtain (2.49).

Proof of (2.47). By the decomposition of $S(t,x)$ in the proof of Proposition 2.2, and the choice of $A_{k,\ell}$, $B_{k,\ell}$ in Proposition 2.4, we obtain $S(t,x) = \mathcal{E}(t,x)$ as defined in Proposition 2.2.

Thus, we only have to estimate $\mathcal{E}(t)$. Since for any $(k,\ell) \in \Sigma_p$, $A_{k,\ell}$, $B_{k,\ell} \in L^\infty$ (with derivatives in $\mathcal{Y}$), it follows from the decomposition of $S(t,x)$ (see proof of Proposition 2.2) that all functions of the y variable in the expression of $S(t,x)$ are bounded. Thus, we have

$$|S(t,x)| \leq K(|Q_c^{p+1}(y_c)| + c|Q_c^2(y_c)|),$$

where $K > 0$ is independent of y and c. Since $\|Q_c^{p+1}(y_c)\|_{L^2} + c\|Q_c^2(y_c)\|_{L^2} \leq Kc^{\frac{2}{p-1}+\frac{3}{4}}$, we obtain

$$\|\mathcal{E}(t)\|_{L^2} \leq Kc^{\frac{2}{p-1}+\frac{3}{4}}.$$

The estimates on the derivatives of S are obtained in the same way.

2.4 Proof of Proposition 2.1

In what follows, we will see that the first order of the shift Δ on Q is $a_{1,0}\int Q_c$. We first derive an explicit formula for $a_{1,0}$ in order to prove Proposition 2.1.

Lemma 2.3 (Computation of the first order of the shift on Q)

$$a_{1,0} = 2\frac{\frac{d}{d\widetilde{c}}\int Q_{\widetilde{c}|\widetilde{c}=1}}{\frac{d}{d\widetilde{c}}\left(\int Q_{\widetilde{c}}^2\right)_{|\widetilde{c}=1}}.$$

Proof of Lemma 2.3. From Proposition 2.2 and Proposition 2.4, the system $(\Omega_{1,0})$ writes, for $p = 2$, 3 and 4:

$$(\Omega_{1,0}) \quad \begin{cases} \mathcal{L}A_{1,0} + a_{1,0}(3Q - 2f(Q)) = f'(Q) \\ (\mathcal{L}B_{1,0})' + a_{1,0}(3Q'') - 3A''_{1,0} - f'(Q)A_{1,0} = f'(Q). \end{cases}$$

Recall from Claim 2.2 that $V_0 = -\Lambda Q - xQ'$ solves $\mathcal{L}V_0 = 3Q - 2f(Q)$. Let V_1 be the even H^1 solution of $\mathcal{L}V_1 = f'(Q)$. Then, the function $A_{1,0} = V_1 - a_{1,0}V_0$ solves the first line of $(\Omega_{1,0})$, independently of the value of $a_{1,0}$. By replacing $A_{1,0}$ in the second line of the system $(\Omega_{1,0})$, we obtain

$$(\mathcal{L}B_{1,0})' + a_{1,0}Z_0 = Z_1,$$

where

$$Z_0 = 3Q'' + 3V_0'' + f'(Q)V_0, \quad Z_1 = 3V_1'' + pQ^{p-1}V_1 + f'(Q). \tag{2.50}$$

Since $\mathcal{L}Q' = 0$, we have $\int(\mathcal{L}B_{1,0})'Q = 0$ and so

$$a_{1,0}\int Z_0Q = \int Z_1Q.$$

In Claim 2.2, we have obtained

$$\int Z_0Q = -\int \Lambda Q\, Q = -\frac{1}{2}\frac{d}{d\widetilde{c}}\int Q^2_{\widetilde{c}|\widetilde{c}=1}.$$

Now, we compute $\int Z_1Q$ similarly as in Claim 2.2,

$$\begin{aligned}\int Z_1Q &= \int Q(3V_1'' + f'(Q)V_1 + f'(Q)) = \int V_1(3Q'' + Qf'(Q)) + \int Qf'(Q) \\ &= -\int \mathcal{L}V_1(Q + \Lambda Q + xQ') + \int Qf'(Q) = -\int f'(Q)\Lambda Q + \int f(Q).\end{aligned}$$

Now, since $\mathcal{L}(\Lambda Q)=-Q$, we have $\int \Lambda Q = -\int Q + \int \Lambda Q\, f'(Q)$ and since $-Q''+Q=f(Q)$, we have $\int Q = \int f(Q)$. Thus, $\int Z_1 Q = -\int \Lambda Q = -\frac{d}{dc}\int Q_{c|c=1}$, which completes the proof.

Proof of Proposition 2.1. From what precedes (in particular Proposition 2.5), we only need to recompose the function $v(t,x)$ at time $\pm T_c$, combining the first terms of the decomposition of $v(t,x)$. By symmetry, we consider only $t=T_c$. This proof follows closely the proof of Proposition 3.1 in [21].

1. First, we claim

$$\|v(T_c)-Q(y)-Q_c(y_c)-b_{1,0}Q_c'(y_c)\|_{H^1} \le K c^{\frac{2}{p-1}+\frac14}. \tag{2.51}$$

Indeed, from the definition of $v(t,x)$, and the fact for $(k,\ell)\in\Sigma_p$, $A_{k,\ell}\in\mathcal{Y}$, $B_{k,\ell}\in L^\infty$, we have:

$$|v(T_c)-Q(y)-Q_c(y_c)-b_{1,0}Q_c'(y_c)| \le K\left[Q_c(y_c)e^{-\frac{|y|}{2}} + |(Q_c^2)'(y_c)| + c|Q_c'(y_c)|)\right].$$

By (2.15), for all $t\in[-T_c,T_c]$, $\|Q_c(y_c)e^{-\frac{|y|}{2}}\|_{H^1}\le K\exp(-\frac12\sqrt{c}t)$, and thus at $t=T_c$, for c small enough,

$$\|Q_c(y_c)e^{-\frac{|y|}{2}}\|_{H^1}\le K\exp(-\tfrac12 c^{-\frac{1}{100}})\le Kc^{10}.$$

By (2.15), $\|(Q_c^2)'(y_c)\|_{H^1}+c\|Q_c'(y_c)\|_{H^1}\le Kc^{\frac{2}{p-1}+\frac14}$, and thus the estimate is proved for the L^2 norm. We proceed similarly for the estimate on $\partial_x(v(T_c)-Q(y)-Q_c(y_c)-b_{1,0}Q_c'(y_c))$.

2. Position of the soliton Q at $t=T_c$. Let

$$\Delta = \sum_{(k,\ell)\in\Sigma_p} a_{k,\ell}c^\ell \int Q_c^k.$$

We claim

$$\text{for } x\ge -T_c/2 \text{ and } t=T_c,\quad |\alpha(y_c)-\tfrac12\Delta|\le Ke^{-\frac14 c^{-\frac{1}{100}}}, \tag{2.52}$$

$$\text{for } t=T_c,\quad \|Q(y)-Q(.-\tfrac12\Delta)\|_{H^1}\le Ke^{-\frac12 c^{-\frac{1}{100}}}. \tag{2.53}$$

Proof of (2.52). For any $k\ge1$, for any $y_c>0$, we have, by (2.15),

$$0\le \int_{y_c}^{\infty} Q_c^k(s)ds \le Kc^{\frac{1}{p-1}}\int_{y_c}^{\infty} e^{-\sqrt{c}\,s}ds = Kc^{\frac{1}{p-1}-\frac12}e^{-\sqrt{c}\,y_c},$$

we obtain

$$\left|\alpha(y_c)-\tfrac12\Delta\right|\le Kc^{\frac{1}{p-1}-\frac12}e^{-\sqrt{c}\,y_c}.$$

For $x\ge -T_c/2$ and $t=T_c$, we have $y_c = x+(1-c)T_c \ge (\frac12 - c)T_c$, thus $\sqrt{c}\,y_c\ge \frac12 c^{-\frac{1}{100}}-1$, and so

$$|\alpha(y_c)-\tfrac12\Delta|\le Kc^{-1/6}e^{-\frac12 c^{-\frac{1}{100}}}\le Ke^{-\frac14 c^{-\frac{1}{100}}}.$$

Proof of (2.53). For $x\ge -T_c/2$, by (2.52), we have $|\alpha(y_c)-\frac12\Delta|\le Kc^{\frac{1}{p-1}-\frac12}e^{-\frac12 c^{-\frac{1}{100}}}$, and so

$$\|Q(y)-Q(.-\tfrac12\Delta)\|_{H^1(x>-T_c/2)}\le Kce^{-\frac14 c^{-\frac{1}{100}}}.$$

For $x < -T_c/2$, since $y = x - \alpha(y_c)$, and $|\alpha(y_c)| \leq Kc^{\frac{1}{p-1}-\frac{1}{2}}$, we have $y < -T_c/4$. Thus,

$$\begin{aligned}&\|Q(y) - Q(. - \tfrac{1}{2}\Delta)\|_{H^1(x<-T_c/2)}\\ &\leq \|Q(y)\|_{H^1(x<-T_c/2)} + \|Q(. - \tfrac{1}{2}\Delta)\|_{H^1(x<-T_c/2)} \leq Ke^{-\frac{1}{2}c^{-\frac{1}{100}}}.\end{aligned}$$

3. Position of the soliton Q_c at $t = T_c$. We claim

$$\|Q_c(y_c) - b_{1,0}Q_c'(y_c) - Q_c(. + (1-c)T_c - b_{1,0})\|_{H^1} \leq Kc^{\frac{1}{p-1}+\frac{3}{4}}. \tag{2.54}$$

Indeed, for the L^2-norm, we have by a scaling argument

$$\begin{aligned}\|Q_c - b_{1,0}Q_c' - Q_c(. - b_{1,0})\|_{L^2} &= c^{\frac{1}{p-1}-\frac{1}{4}}\|Q - \sqrt{c}\, b_{1,0}Q' - Q(. - \sqrt{c}\, b_{1,0})\|_{L^2}\\ &\leq Kc^{\frac{1}{p-1}-\frac{1}{4}}(\sqrt{c}\, b_{1,0})^2 = Kc^{\frac{1}{p-1}+\frac{3}{4}},\end{aligned}$$

and similarly for the estimate on the x derivative.

Thus Proposition 2.1 is proved.

2.5 Extension of Proposition 2.1 by scaling

Let

$$T_{c_1,c_2} = c_1^{-\frac{3}{2}}T_c = c_1^{-\frac{3}{2}}\left(\frac{c_2}{c_1}\right)^{-\frac{1}{2}-\frac{1}{100}}.$$

By a scaling argument, we have from Proposition 2.1 the following

Theorem 2.1 *Let $0 < c_1 < c_*(f)$ be such that (1.5) holds. There exist $c_0(c_1)$ and $K_0(c_1) > 0$, continuous in c_1 such that for any $0 < c_2 < c_0(c_1)$, there exist function $v = v_{c_1,c_2}$ satisfying $v(0,x) = v(0,-x)$ and such that the following hold.*

1. *Approximate solution on $[-T_{c_1,c_2}, T_{c_1,c_2}]$: for $j = 0,1,2$,*

$$\forall t \in [-T_{c_1,c_2}, T_{c_1,c_2}], \quad \|\partial_x^j S(t)\|_{L^2(\mathbb{R})} \leq K_0 c_2^{\frac{2}{p-1}+\frac{3}{4}}. \tag{2.55}$$

2. *Closeness to the sum of two solitons for $t = \pm T_{c_1,c_2}$: there exist Δ_1, Δ_2 such that*

$$\begin{aligned}&\|v(T_{c_1,c_2}) - Q_{c_1}(. - \tfrac{1}{2}\Delta_1) - Q_{c_2}(. + (c_1 - c_2)T_{c_1,c_2} - \tfrac{1}{2}\Delta_2)\|_{H^1} \leq K_0 c_2^{\frac{2}{p-1}+\frac{1}{4}},\\ &\|v(-T_{c_1,c_2}) - Q_{c_1}(. + \tfrac{1}{2}\Delta_1) - Q_{c_2}(. - (c_1 - c_2)T_{c_1,c_2} + \tfrac{1}{2}\Delta_2)\|_{H^1} \leq K_0 c_2^{\frac{2}{p-1}+\frac{1}{4}},\end{aligned} \tag{2.56}$$

where

$$\left|\Delta_1 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-1}-\frac{1}{2}}\delta_1\right| \leq Kc_2^{\frac{2}{p-1}-\frac{1}{2}}, \quad \delta_1 = 2\,\frac{\int Q_{c_1} \frac{d}{dc}\int Q_{c}{}_{|c=c_1}}{\frac{d}{dc}\left(\int Q_c^2\right)_{|c=c_1}}. \tag{2.57}$$

3. *Closeness to the sum of two solitons: for all $t \in [-T_{c_1,c_2}, T_{c_1,c_2}]$, there exists $y_1(t)$ such that*

$$\|v(t,x) - Q_{c_1}(. - y_1(t)) - Q_{c_2}(. - (c_2 - c_1)t)\|_{H^1} \leq K_0 c^{\frac{1}{p-1}}. \tag{2.58}$$

Proof of Theorem 2.1. Fix a nonlinearity f satisfying (1.2). Fix $0 < c_1 < c_*(f)$ such that (1.5) holds. Let

$$\widetilde{f}(\widetilde{u}) = \widetilde{u}^p + \widetilde{f}_1(\widetilde{u}) \quad \text{where} \quad \widetilde{f}_1(\widetilde{u}) = c_1^{-\frac{p}{p-1}} f_1(c_1^{\frac{1}{p-1}} \widetilde{u}).$$

Then $u(t)$ is solution of (1.1) if and only if

$$\widetilde{u}(t,x) = c_1^{-\frac{1}{p-1}} u(c_1^{-\frac{3}{2}} t, c_1^{-\frac{1}{2}} x) \text{ is solution of } \partial_t \widetilde{u} + \partial_x(\partial_x^2 \widetilde{u} + \widetilde{f}(\widetilde{u})) = 0. \tag{2.59}$$

First, we observe that $\widetilde{f}$ satisfies assumption (1.2). Second, for any $0 < c < c_*(f)$, let Q_c be the positive even solution of (1.4). For $0 < \widetilde{c} = \frac{c}{c_1} < \frac{c_*(f)}{c_1}$,

$$\widetilde{Q}_{\widetilde{c}}(x) = c_1^{-\frac{1}{p-1}} Q_c(c_1^{-\frac{1}{2}} x) \quad \text{solves} \quad \widetilde{Q}_{\widetilde{c}}'' + \widetilde{f}(\widetilde{Q}_{\widetilde{c}}) = \widetilde{c}\, \widetilde{Q}_{\widetilde{c}}. \tag{2.60}$$

Thus, $c_*(\widetilde{f}) \geq \frac{c_*(f)}{c_1} > 1$ (in fact, $c_*(\widetilde{f}) = \frac{c_*(f)}{c_1}$). Moreover, for any $0 < c < c_*(f)$, we have

$$\begin{aligned}
&\int Q_c^2 = c_1^{\frac{2}{p-1}-\frac{1}{2}} \int \widetilde{Q}^2_{\frac{c}{c_1}}, \quad \int Q_c = c_1^{\frac{1}{p-1}-\frac{1}{2}} \int \widetilde{Q}_{\frac{c}{c_1}}, \\
&\frac{d}{dc}\int Q_c^2{}_{|c=c_1} = c_1^{\frac{2}{p-1}-\frac{1}{2}} \frac{d}{dc}\left(\int \widetilde{Q}^2_{\frac{c}{c_1}}\right)_{|c=c_1} = c_1^{\frac{2}{p-1}-\frac{3}{2}} \frac{d}{d\widetilde{c}}\int \widetilde{Q}^2_{\widetilde{c}}{}_{|\widetilde{c}=1}, \\
&\frac{d}{dc}\int Q_c{}_{|c=c_1} = c_1^{\frac{1}{p-1}-\frac{1}{2}} \frac{d}{dc}\left(\int \widetilde{Q}_{\frac{c}{c_1}}\right)_{|c=c_1} = c_1^{\frac{1}{p-1}-\frac{3}{2}} \frac{d}{d\widetilde{c}}\int \widetilde{Q}_{\widetilde{c}}{}_{|\widetilde{c}=1}.
\end{aligned} \tag{2.61}$$

In particular, $\frac{d}{dc}\int Q_c^2{}_{\big|c=c_1} > 0$ is equivalent to $\frac{d}{d\widetilde{c}}\int \widetilde{Q}_{\widetilde{c}}^2{}_{\big|\widetilde{c}=1} > 0$.

Let $c_0 = \frac{1}{4} c_0(\widetilde{f})$, $K_0 = K_0(\widetilde{f})$, where $c_0(\widetilde{f})$, $K_0(\widetilde{f})$ are defined in Proposition 2.1 (these constants thus depend continuously upon c_1, see Remark after Proposition 2.1). Let $0 < c_2 < c_0$, and let $c = \frac{c_2}{c_1}$. We consider $\widetilde{v} = \widetilde{v}_{1,c}$ as defined in Proposition 2.1 for the nonlinearity $\widetilde{f}$ and $\widetilde{S} = \partial_t \widetilde{v} + \partial_x(\partial_x^2 \widetilde{v} - \widetilde{v} + \widetilde{f}(\widetilde{v}))$. From Proposition 2.1, we have

$$\forall t \in [-T_c, T_c], \quad \|\partial_x^j \widetilde{S}(t)\|_{L^2(\mathbb{R})} \leq K_0 c^{\frac{2}{p-1}+\frac{3}{4}}. \tag{2.62}$$

$$\|\widetilde{v}(T_c) - \widetilde{Q}(. - \tfrac{1}{2}\widetilde{\Delta}) - \widetilde{Q}_c(. + (1-c)T_c - \tfrac{1}{2}\widetilde{\Delta}_c)\|_{H^1} \leq K_0 c^{\frac{2}{p-1}+\frac{1}{4}}, \tag{2.63}$$

$$\left|\widetilde{\Delta} - c^{\frac{1}{p-1}-\frac{1}{2}} \widetilde{\delta}\right| \leq K c^{\frac{2}{p-1}-\frac{1}{2}}, \quad \widetilde{\delta} = 2\, \frac{\int \widetilde{Q}\, \frac{d}{d\widetilde{c}} \int \widetilde{Q}_{\widetilde{c}}{}_{|\widetilde{c}=1}}{\frac{d}{d\widetilde{c}}\left(\int \widetilde{Q}_{\widetilde{c}}^2\right)_{|\widetilde{c}=1}}. \tag{2.64}$$

Then, we set

$$v(t,x) = v_{c_1,c_2}(t,x) = c_1^{\frac{1}{p-1}} \widetilde{v}(c_1^{\frac{3}{2}} t, c_1^{\frac{1}{2}} x), \tag{2.65}$$

$$S(t,x) = \partial_t v + \partial_x(\partial_x^2 v - v + f(v)). \tag{2.66}$$

Since $\partial_x^j S(t,x) = c_1^{\frac{3+j}{2}+\frac{1}{p-1}} \partial_x^j \widetilde{S}$, estimate (2.62) gives $j = 0,1,2$, $\|\partial_x^j S(t)\|_{L^2(\mathbb{R})} \leq K c_2^{\frac{2}{p-1}+\frac{3}{4}}$.

From (2.63)

$$\|v(T_{c_1,c_2}) - Q_{c_1}(. - \tfrac{1}{2} c_1^{-\frac{1}{2}} \widetilde{\Delta}) - Q_{c_2}(. + (c_1 - c_2) T_{c_1,c_2} - \tfrac{1}{2} c_1^{-\frac{1}{2}} \widetilde{\Delta}_c)\|_{H^1} \leq K c_2^{\frac{2}{p-1}+\frac{1}{4}}.$$

Setting $\Delta_1 = c_1^{-\frac{1}{2}}\widetilde{\Delta}$ and $\Delta_2 = c_1^{-\frac{1}{2}}\widetilde{\Delta}_c$, by (2.64) and (2.61), we have

$$\left|\Delta_1 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-1}-\frac{1}{2}}\delta_1\right| \leq K c_2^{\frac{2}{p-1}-\frac{1}{2}}$$

$$\delta_1 = c_1^{-\frac{1}{2}}\widetilde{\delta} = 2\, c_1^{-\frac{1}{2}} \frac{\int \widetilde{Q} \frac{d}{d\widetilde{c}} \int \widetilde{Q}_{\widetilde{c}|\widetilde{c}=1}}{\frac{d}{d\widetilde{c}}\left(\int \widetilde{Q}_{\widetilde{c}}^2\right)_{|\widetilde{c}=1}} = 2\, \frac{\int Q_{c_1} \frac{d}{dc}\int Q_{c|c=c_1}}{\frac{d}{dc}\left(\int Q_c^2\right)_{|c=c_1}}.$$

Estimate (2.58) follows from (2.7).

3 Preliminary results for stability of the 2-soliton structure

This section is similar to Section 4 in [21].

3.1 Dynamic stability in the interaction region

Proposition 3.1 (Exact solution close to the approximate solution v) *Let $0 < c_1 < c_*(f)$ be such that (1.5) holds. There exist $c_0(c_1)$ and $K_0(c_1) > 0$, continuous in c_1 such that for any $0 < c_2 < c_0(c_1)$, the following holds. Let $v = v_{c_1,c_2}$ be defined in Theorem 2.1. Suppose that for some $\theta > \frac{1}{p-1}$, for some $T_0 \in [-T_{c_1,c_2}, T_{c_1,c_2}]$,*

$$\|u(T_0) - v(T_0)\|_{H^1(\mathbb{R})} \leq c_2^{\theta}, \tag{3.1}$$

where $u(t)$ is an H^1 solution of (1.1). Then, $u(t)$ is global and there exists $\rho(t)$ such that, for all $t \in [-T_{c_1,c_2}, T_{c_1,c_2}]$,

$$\|u(t) - v(t, . - \rho(t))\|_{H^1} + |\rho'(t) - c_1| \leq K_0 \left(c_2^{\theta} + c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}\right). \tag{3.2}$$

The fact that $u(t)$ is global follows from the stability of Q_{c_1}.

Sketch of the proof of Proposition 3.1. The proof is similar to the one of Proposition 4.1 in [21]. For the sake of simplicity, we give a sketch of the proof in the special case $c_1 = 1$ and $c_2 = c$ small, i.e. we work in the context of Proposition 2.1. The general case follows by the same scaling argument as in Section 2.5. In view of (3.2), we may assume that

$$\theta \leq \frac{2}{p-1} + \frac{1}{4} - \frac{1}{100}. \tag{3.3}$$

We prove the result on $[T_0, T_c]$. By using the transformation $x \to -x$, $t \to -t$, the proof is the same on $[-T_c, T_0]$.

Let $K^* > 1$ be a constant to be fixed later. Since $\|u(T_0) - v(T_0)\|_{H^1} \leq c^{\theta}$, by continuity in time in $H^1(\mathbb{R})$, there exists $T_0 < T^* \leq T_c$ such that

$$T^* = \sup\left\{T \in [T_0, T_c] \text{ s.t. } \forall t \in [T_0, T],\ \exists r(t) \in \mathbb{R} \text{ with } \|u(t) - v(t, . - r(t))\|_{H^1} \leq K^* c^{\theta}\right\}.$$

The objective is to prove that $T^* = T_c$ for K^* large. For this, we argue by contradiction, assuming that $T^* < T_c$ and reaching a contradiction with the definition of T^* by proving independent estimates on $\|u(t) - v(t, . - r)\|_{H^1}$ on $[T_0, T^*]$.

We claim (see Lemma 4.1 in [21]).

Claim 3.1 *Assume that $0 < c < c(K^*)$ small enough. There exists a unique C^1 function $\rho(t)$ such that, for all $t \in [T_0, T^*]$,*

$$z(t,x) = u(t, x+\rho(t)) - v(t,x) \quad \textit{satisfies} \quad \int z(t,x)Q'(y)dx = 0. \tag{3.4}$$

Moreover, we have, for all $t \in [T_0, T^*]$,

$$|\rho(T_0)| + \|z(T_0)\|_{H^1} \leq Kc^\theta, \ \|z(t)\|_{H^1} \leq 2K^* c^\theta, \tag{3.5}$$

$$\partial_t z + \partial_x(\partial_x^2 z - z + f(z+v) - f(v)) = -S(t) + (\rho'(t) - c_1)\partial_x(v+z). \tag{3.6}$$

$$|\rho'(t) - 1| \leq K\|z(t)\|_{H^1} + K\|S(t)\|_{H^1}, \tag{3.7}$$

Recall that the existence, uniqueness and regularity of $\rho(t)$ is obtained by a standard use of the Implicit Function Theorem applied to $u(t)$ at each fixed time t. Estimate (3.7) is obtained by equation (3.6).

Step 1. Energy estimates on $z(t)$. We extend to the case of the general power nonlineartity the definition given in [21] of the energy functional for $z(t)$:

$$\mathcal{F}(t) = \frac{1}{2}\int \left((\partial_x z)^2 + (1 + \alpha'(y_c))z^2\right) - \int (F(v+z) - F(v) - f(v)z).$$

Lemma 3.1 (Properties of $\mathcal{F}$) *Assume that $0 < c < c(K^*)$ small enough. There exists $K > 0$ (independent of K^* and c) such that*

(i) *Coercivity of $\mathcal{F}$ under orthogonality conditions:*

$$\forall t \in [T_0, T^*], \quad \|z(t)\|_{H^1}^2 \leq K\mathcal{F}(t) + K\left|\int z(t)Q(y)\right|^2. \tag{3.8}$$

(ii) *Control of the direction Q:*

$$\forall t \in [T_0, T^*], \quad \left|\int z(t)Q(y)\right| \leq Kc^\theta + Kc^{\frac{1}{p-1}-\frac{1}{4}}\|z(t)\|_{L^2} + K\|z(t)\|_{L^2}^2. \tag{3.9}$$

(iii) *Control of the variation of the energy fonctional:*

$$\mathcal{F}(T^*) - \mathcal{F}(T_0) \leq Kc^{2\theta}\left((K^*)^2(1+K^*)c^{\frac{1}{2(p-1)}-\frac{1}{8}} + K^*\right). \tag{3.10}$$

Proof of Lemma 3.1. (i) For this property, see proof of Claim 4.2 in Appendix D of [21]. Recall that the proof of such property is related to assumption (1.5) (nonlinear stability of Q) and to the choice of $\rho(t)$ in Claim 3.1.

(ii) This estimate follows from the conservation of $\int u^2(t)$ and a similar approximate conservation for $v(t)$. Indeed, we have $|\frac{1}{2}\frac{d}{dt}\int v^2| = |\int S(t,x)v(t,x)dx| \leq K\|S(t)\|_{L^2}$ from the equation of $v(t)$. Thus,

$$\forall t \in [T_0, T^*], \quad \left|\int v^2(t) - \int v^2(T_0)\right| \leq KT_c \sup_{t\in[-T_c,T_c]} \|S(t)\|_{H^1} \leq Kc^\theta. \tag{3.11}$$

Since $u(t)$ is a solution of the (gKdV) equation, we have

$$\int u^2(t) = \int (v(t)+z(t))^2 = \int u^2(T_0) = \int (v(T_0)+z(T_0))^2 . \tag{3.12}$$

By expanding (3.12) and using (3.11) and (3.5), we obtain:

$$2\left|\int v(t)z(t)\right| \le Kc^\theta + 2\left|\int v(T_0)z(T_0)\right| + \|z(T_0)\|_{L^2}^2 + \|z(t)\|_{L^2}^2 \le Kc^\theta + \|z(t)\|_{L^2}^2 .$$

Using this and $\|v(t)-Q(y)\|_{L^2} \le Kc^{\frac{1}{p-1}-\frac14}$, we obtain:

$$\left|\int z(t)Q(y)\right| \le \left|\int z(t)v\right| + \left|\int z(t)(v-Q(y))\right| \le Kc^\theta + Kc^{\frac{1}{p-1}-\frac14}\|z(t)\|_{L^2} + \|z(t)\|_{L^2}^2 .$$

(iii) The computations of the proof of Lemma 4.3 in [21] are extended as follows:

$$\mathcal{F}'(t) = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3,$$

where

$$\mathbf{F}_1 = \int \partial_t z(-\partial_x^2 z + z - (f(v+z)-f(v))), \quad \mathbf{F}_2 = \int \partial_t z\, \alpha'(y_c) z,$$

$$\mathbf{F}_3 = \int \left\{\frac12 (1-c)\alpha''(y_c)z^2 - \partial_t v\left(f(v+z)-f(v)-zf'(v)\right).\right\}$$

Let $m_0 = \min\left(\frac{2}{p-1}, \frac{1}{p-1}+\frac12\right)$. We claim the following estimates.

Claim 3.2

$$\left|\mathbf{F}_1 + (\rho'(t)-1)\int \alpha'(y_c)Q'(y)z\right| \le Kc^{\frac{1}{p-1}+\frac14}\|z(t)\|_{L^2}^2 + K\|z(t)\|_{L^2}(\|\partial_x^2 S(t)\|_{L^2} + \|S(t)\|_{L^2}), \tag{3.13}$$

$$\begin{aligned}&\left|\mathbf{F}_2 - (\rho'(t)-1)\int \alpha'(y_c)Q'(y)z + \frac12\int \alpha'(y_c)Q'(y)f''(Q(y))z^2\right| \\ &\le K\|z(t)\|_{H^1}^2\left(c^{m_0} + c^{\frac{1}{p-1}}\|z(t)\|_{H^1}\right) + K\|z(t)\|_{H^1}(\|\partial_x^2 S(t)\|_{L^2} + \|S(t)\|_{L^2}),\end{aligned} \tag{3.14}$$

$$\left|\mathbf{F}_3 - \frac12\int \alpha'(y_c)Q'(y)f''(Q(y))z^2\right| \le Kc^{m_0}\|z(t)\|_{H^1}^2 + Kc^{\frac{1}{p-1}}\|z(t)\|_{H^1}^3 . \tag{3.15}$$

Estimates (3.13)–(3.15) are obtained exactly as in [21]. For the reader's convenience, we reproduce the computations in Appendix B. Now, we conclude the proof of Lemma 3.1.

From the cancellations of the main terms of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, and then from (3.5) and Theorem 2.1, (2.55), we get

$$\begin{aligned}|\mathcal{F}'(t)| &\le K\|z(t)\|_{H^1}^2\left(c^{\frac{1}{p-1}+\frac14} + c^{\frac{1}{p-1}}\|z(t)\|_{H^1}\right) + K\|z(t)\|_{H^1}\left(\|\partial_x^2 S(t)\|_{L^2} + \|S(t)\|_{L^2}\right) \\ &\le Kc^{2\theta}\left[(K^*)^2(c^{\frac{1}{p-1}+\frac14} + K^* c^{\frac{1}{p-1}+\theta}) + K^* c^{\frac{2}{p-1}+\frac34-\theta}\right].\end{aligned}$$

Integrating on the time interval $[T_0, T^*]$, since $T^* - T_0 \leq 2T_c = 2c^{\frac{1}{2}+\frac{1}{100}}$, and $\theta > \frac{1}{p-1} > \frac{1}{4}$, we obtain

$$|\mathcal{F}(T^*) - \mathcal{F}(T_0)| \leq Kc^{2\theta}\left((K^*)^2(1+K^*)c^{\frac{1}{p-1}-\frac{1}{4}-\frac{1}{100}} + K^* c^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}-\theta}\right].$$

Note that by (3.3), we have $\frac{2}{p-1} + \frac{1}{4} - \frac{1}{100} - \theta \geq 0$ and $\frac{1}{p-1} - \frac{1}{4} - \frac{1}{100} \geq \frac{1}{2(p-1)} - \frac{1}{8} > 0$, since $\frac{1}{2(p-1)} \geq \frac{1}{6} \geq \frac{1}{8} + \frac{1}{100}$. Thus, Lemma 3.1 is proved.

Step 2. Conclusion of the proof. By (3.9), we have

$$\left|\int z(T^*)Q(y)\right| \leq Kc^{\theta} + Kc^{\frac{1}{p-1}-\frac{1}{4}}\|z(T^*)\|_{L^2} + \|z(T^*)\|_{L^2}^2,$$

and thus by (3.8),

$$\|z(T^*)\|_{H^1}^2 \leq K\mathcal{F}(T^*) + K(c^{\theta} + c^{\frac{1}{p-1}-\frac{1}{4}}\|z(T^*)\|_{L^2} + \|z(T^*)\|_{L^2}^2)^2.$$

Since $\frac{1}{p-1} - \frac{1}{4} > 0$, it follows that for c small enough,

$$\|z(T^*)\|_{H^1}^2 \leq (K+1)\mathcal{F}(T^*) + Kc^{2\theta}.$$

Next, by (3.10) and $|\mathcal{F}(T_0)| \leq Kc^{2\theta}$, we obtain

$$\|z(T^*)\|_{H^1}^2 \leq (K+1)(\mathcal{F}(T^*) - \mathcal{F}(T_0)) + Kc^{2\theta} \leq K_1 c^{2\theta}\left((K^*)^2(1+K^*)c^{\frac{1}{2(p-1)}-\frac{1}{8}} + K^* + 1\right),$$

where K_1 is independent of c and K^*. Choose $c_* = c_*(K^*)$ such that

$$(K^*)^2(1+K^*)c_*^{\frac{1}{2(p-1)}-\frac{1}{8}} < 1.$$

Then, for $0 < c < c_*$,

$$\|z(T^*)\|_{H^1}^2 \leq K_1 c^{2\theta}(2+K^*).$$

Next, fix K^* such that $K_1(2+K^*) < \frac{1}{2}(K^*)^2$. Then

$$\|z(T^*)\|_{H^1}^2 \leq \frac{1}{2}(K^*)^2 c^{2\theta}.$$

This contradict the definition of T^*, thus proving that $T^* = T_c$. Thus estimate (3.2) is proved on $[T_0, T_c]$.

3.2 Stability and asymptotic stability for large time

In this section, we consider the stability of the 2-soliton structure after the collision. This question has been considered in [19], [20]. See also [17], [22], [16]. We recall the following.

Proposition 3.2 (Stability and asymptotic stability [19], [20]) *Let $0 < c_1 < c_*(f)$ be such that* (1.5) *holds. There exist $c_0(c_1)$ and $K_0(c_1) > 0$, continuous in c_1 such that for any $0 < c_2 < c_0(c_1)$ and for any $\omega > 0$, the following hold. Let $u(t)$ be an H^1 solution of* (1.1) *such that for some $t_1 \in \mathbb{R}$ and $\frac{1}{2}T_{c_1,c_2} \leq X_0 \leq \frac{3}{2}T_{c_1,c_2}$,*

$$\|u(t_1) - Q_{c_1} - Q_{c_2}(.+X_0)\|_{H^1} \leq c_2^{\omega+\frac{1}{p-1}+\frac{1}{4}}. \tag{3.16}$$

Then, there exist C^1 functions $\rho_1(t)$, $\rho_2(t)$ defined on $[t_1, +\infty)$ such that

1. Stability.

$$\sup_{t\geq t_1}\|u(t)-Q_{c_1}(.-\rho_1(t))-Q_{c_2}(.-\rho_2(t))\|_{H^1}\leq Kc^{\omega+\frac{1}{p-1}-\frac{1}{4}}, \tag{3.17}$$

$$\begin{aligned}&\forall t\geq t_1,\ \tfrac{1}{2}c_1\leq(\rho_1-\rho_2)'(t)\leq\tfrac{3}{2}c_1,\\ &|\rho_1(t_1)|\leq Kc_2^{\omega+\frac{1}{p-1}+\frac{1}{4}},\quad |\rho_2(t_1)-X_0|\leq Kc_2^{\omega}.\end{aligned} \tag{3.18}$$

2. Convergence of $u(t)$. There exist $c_1^+,c_2^+>0$ such that

$$\lim_{t\to+\infty}\|u(t)-Q_{c_1^+}(x-\rho_1(t))-Q_{c_2^+}(x-\rho_2(t))\|_{H^1(x>\frac{c_2t}{10})}=0. \tag{3.19}$$

$$\left|\frac{c_1^+}{c_1}-1\right|\leq Kc^{\omega+\frac{1}{p-1}+\frac{1}{4}},\quad \left|\frac{c_2^+}{c_2}-1\right|\leq Kc^{\omega}. \tag{3.20}$$

The proof of Proposition 3.2 is based on energy arguments, monotonicity results on local energy quantities, and a Virial argument on the linearized problem around solitons.

The loss of $\frac{1}{2}$ in the exponent between (3.16) and (3.17) is due to the fact that the natural norm to study the stability of Q_{c_2} is not $\|.\|_{H^1}$ but $\|\partial_x(.)\|_{L^2}+c^{\frac{1}{2}}\|.\|_{L^2}$.

3.3 Monotonicity results

Recall a more precise decomposition of $u(t)$ used in the proof of Proposition 3.2 in [19], [20].

Claim 3.3 (Decomposition of the solution) *Under the assumptions of Proposition 3.2, there exist C^1 functions $\rho_1(t)$, $\rho_2(t)$, $c_1(t)$, $c_2(t)$, defined on $[t_1,+\infty)$, such that the function $\eta(t)$ defined by*

$$\eta(t,x)=u(t,x)-R_1(t,x)-R_2(t,x),$$

where for $j=1,2$, $R_j(t,x)=Q_{c_j(t)}(x-\rho_j(t))$, satisfies for all $t\geq t_1$,

$$\int R_j(t)\eta(t)=\int(x-\rho_j(t))R_j(t)\eta(t)=0,\quad j=1,2, \tag{3.21}$$

$$\|\eta(t)\|_{H^1}+\left|\frac{c_1(t)}{c_1}-1\right|+c_2^{\frac{1}{p-1}-\frac{1}{4}}\left|\frac{c_2(t)}{c_2}-1\right|\leq Kc_2^{\omega+\frac{1}{p-1}-\frac{1}{4}}, \tag{3.22}$$

Now, we recall some monotonicity results for two localized quantities defined in $\eta(t)$. Define

$$\psi(x)=\tfrac{2}{\pi}\arctan(\exp(-\tfrac{1}{4}x)), \tag{3.23}$$

$$g_j(t)=\int(\eta_x^2+c_j\eta^2)(t,x)e^{-\frac{1}{4}\sqrt{c_j}|x-\rho_j(t)|}dx,\quad j=1,2. \tag{3.24}$$

For $0\leq t_0\leq t$, $x_0\geq 0$, $j=1,2$, let

$$\mathcal{M}_j(t)=\int\eta^2\psi_j,$$

$$\mathcal{E}_j(t)=\int\left[\frac{1}{2}\eta_x^2-(F(R_1+R_2+\eta)-(f(R_1)+f(R_2))\eta-F(R_1+R_2))\right]\psi_j,$$

$$\text{where}\quad \begin{aligned}&\psi_1(x)=\psi(\sqrt{c_1}\widetilde{x}_1),\quad \widetilde{x}_1=x-\rho_1(t)+x_0+\tfrac{c_1}{2}(t-t_0),\\ &\psi_2(x)=\psi(\sqrt{c_2}\widetilde{x}_2),\quad \widetilde{x}_2=x-\rho_2(t)+x_0+\tfrac{c_2}{2}(t-t_0).\end{aligned}$$

Claim 3.4 (Monotonicity results in $\eta(t)$) *Let $x_0 > 0$, $t_0 > 0$. For all $t \geq t_0$,*

$$\frac{d}{dt}\left(\int Q^2_{c_1(t)} + \mathcal{M}_1(t)\right) \leq Ke^{-\frac{\sqrt{c_1}}{16}(c_1(t-t_0)+x_0)}g_1(t) + Ke^{-\frac{1}{32}c_1\sqrt{c_2}(t+T_{c_1,c_2})},$$

$$\frac{d}{dt}\left(2E(Q_{c_1(t)}) + 2\mathcal{E}_1(t) + \frac{c_1}{100}\left(\int Q^2_{c_1(t)} + \mathcal{M}_1(t)\right)\right)$$

$$\leq Ke^{-\frac{1}{16}\sqrt{c_1}(c_1(t-t_0)+x_0)}g_1(t) + Ke^{-\frac{1}{32}c_1\sqrt{c_2}(t+T_{c_1,c_2})}.$$

$$\frac{d}{dt}\left(\int Q^2_{c_1(t)} + \int Q^2_{c_2(t)} + \mathcal{M}_2(t)\right) \leq Ke^{-\frac{c_2\sqrt{c_2}}{16}(t-t_0)}e^{-\frac{\sqrt{c_2}}{16}x_0}\sqrt{c_2}\,g_2(t) + Ke^{-\frac{1}{32}c_1\sqrt{c_2}(t+T_{c_1,c_2})},$$

$$\frac{d}{dt}\left(2E(Q_{c_1(t)})+2E(Q_{c_2(t)}) + 2\mathcal{E}_2(t) + \frac{c_2}{100}\left(\int Q^2_{c_1(t)}+\int Q^2_{c_2(t)}\right) + \mathcal{M}_2(t)\right)$$

$$\leq Ke^{-\frac{c_2\sqrt{c_2}}{16}(t-t_0)}e^{-\frac{c_2}{16}x_0}c_2^{\frac{3}{2}}g_2(t) + Ke^{-\frac{1}{32}c_1\sqrt{c_2}(t+T_{c_1,c_2})}.$$

Claim 3.4 is proved in [20] for the power case. The proof is exactly the same for a nonlinearity $f(u)$ satisfying (1.2).

4 Proof of the main Theorems

4.1 Proof of Theorem 1.1

Let $0 < c_1 < c_*(f)$ such that (1.5) holds and $c_2 > 0$ small enough. Let $u(t)$ be the unique solution of (1.1) such that (see Theorem 1 and Remark 2 in [15])

$$\lim_{t\to-\infty}\|u(t) - Q_{c_1}(.-c_1t) - Q_{c_2}(.-c_2t)\|_{H^1} = 0.$$

1. Behavior at $-T_{c_1,c_2}$. We claim that

$$\forall t < -\frac{1}{32}T_{c_1,c_2}, \quad \|u(t) - Q_{c_1}(.-c_1t) - Q_{c_2}(.-c_2t)\|_{H^1} \leq Ke^{\frac{1}{4}\sqrt{c_2}(c_1-c_2)t}. \tag{4.1}$$

This is a consequence of the proof of existence of $u(t)$ in [15]. See Proposition 5.1 in [21] for a proof in the power case.

Now, let Δ_1, Δ_2 be defined in Theorem 2.1 and

$$T^-_{c_1,c_2} = T_{c_1,c_2} + \frac{1}{2}\frac{\Delta_1 - \Delta_2}{c_1 - c_2}, \quad a = \frac{1}{2}\Delta_1 - T^-_{c_1,c_2}.$$

Since $|\Delta_1| \leq Kc^{-\frac{1}{6}}$ and Δ_2 is independent of c, we have $-T^-_{c_1,c_2} \leq -\frac{1}{32}T_{c_1,c_2}$, and thus, for c_2 small enough:

$$\|u(-T^-_{c_1,c_2},.+a)-Q_{c_1}(.+\tfrac{\Delta_1}{2})-Q_{c_2}(.-(c_1-c_2)T_{c_1,c_2}+\tfrac{\Delta_2}{2})\|_{H^1} \leq Ke^{-\frac{1}{4}\sqrt{c_2}(c_1-c_2)T^-_{c_1,c_2}} \leq Kc_2^{10}.$$

Let $\widetilde{u}(t,x) = u(t+T_{c_1,c_2} - T^-_{c_1,c_2}, x-a)$. Then $\widetilde{u}(t,x)$ is also solution of (1.1) and satisfies

$$\|\widetilde{u}(-T_{c_1,c_2}) - Q_{c_1}(.+\tfrac{\Delta_1}{2}) - Q_{c_2}(.-(c_1-c_2)T_{c_1,c_2} + \tfrac{\Delta_2}{2})\|_{H^1} \leq Kc_2^{10}. \tag{4.2}$$

In what follows, we work with $\widetilde{u}(t)$ satisfying (4.2) and we denote $\widetilde{u}$ by u.

2. Behavior at $+T_{c_1,c_2}$. Now, consider $v=v_{c_1,c_2}$ constructed in Theorem 2.1 (possibly taking a smaller c_2). By (2.56) and (4.2), we have

$$\|u(-T_{c_1,c_2})-v(-T_{c_1,c_2})\|_{H^1}\leq Kc_2^{\frac{2}{p-1}+\frac{1}{4}}.$$

Applying Proposition 3.1 with

$$T_0=-T_{c_1,c_2},\quad \theta=\frac{2}{p-1}+\frac{1}{4},$$

it follows that there exists a function $\rho(t)$ such that

$$\forall t\in[-T_{c_1,c_2},T_{c_1,c_2}],\quad \|u(t)-v(t,.-\rho(t))\|_{H^1}\leq Kc^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}.$$

In particular, by (2.56), for some a_-, b_- such that $\frac{1}{2}T_{c_1,c_2}<a_--b_-<2T_{c_1,c_2}$,

$$\|u(T_{c_1,c_2})-Q_{c_1}(.-a_-)-Q_{c_2}(.-b_-)\|_{H^1}\leq Kc^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}. \tag{4.3}$$

3. Behavior as $t\to+\infty$. From (4.3), it follows that we can apply Proposition 3.2 to $u(t)$ for $t\geq T_{c_1,c_2}$, with

$$\omega=\frac{1}{p-1}-\frac{1}{100}.$$

It follows that there exist $\rho_1(t)$, $\rho_2(t)$, c_1^+, c_2^+ so that

$$w^+(t,x)=u(t,x)-Q_{c_1^+}(x-\rho_1(t))-Q_{c_2^+}(x-\rho_2(t))\quad\text{satisfies} \tag{4.4}$$

$$\sup_{t\geq T_{c_1,c_2}}\|w^+(t)\|_{H^1}\leq Kc_2^{\frac{2}{p-1}-\frac{1}{4}-\frac{1}{100}},\quad \lim_{t\to+\infty}\|w^+(t)\|_{H^1(x>\frac{c_2}{10}t)}=0, \tag{4.5}$$

$$|c_1^+-c_1|\leq Kc_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}},\quad |c_2^+-c_2|\leq Kc_2^{1+\frac{1}{p-1}-\frac{1}{100}}. \tag{4.6}$$

4. Estimates on $c_1^+-c_1$ and $c_2^+-c_2$. By (4.1) and conservation of the L^2 norm, we have

$$M_0=\int u^2(t)=\int Q_{c_1}^2+\int Q_{c_2}^2.$$

By the definition of $w^+(t)$, we have

$$\forall t,\quad M_0=\int Q_{c_1^+}^2+\int Q_{c_2^+}^2+\int (w^+)^2(t)+2\int w^+(t)(Q_{c_1^+}+Q_{c_2^+})+2\int Q_{c_1^+}Q_{c_2^+}.$$

Thus, by (4.5), passing to the limit as $t\to+\infty$, we obtain $M^+=\lim_{t\to+\infty}\int(w^+)^2(t)$ exists and

$$M^+=\int Q_{c_1}^2+\int Q_{c_2}^2-\int Q_{c_1^+}^2-\int Q_{c_2^+}^2, \tag{4.7}$$

Similarly, using the conservation of energy, $E^+=\lim_{t\to+\infty}E(w^+(t))$ exists and

$$E^+=E(Q_{c_1})+E(Q_{c_2})-E(Q_{c_1^+})-E(Q_{c_2^+}). \tag{4.8}$$

By (4.5), we have $\|w^+(t)\|_{L^\infty}^{p-1} \le K\|w^+(t)\|_{H^1}^{p-1} \le Kc_2^{\frac{9}{8}}$, for t large enough. Thus,

$$E(w^+(t)) = \frac{1}{2}\int (w_x^+)^2(t) - \int F(w^+(t)) \ge \frac{1}{2}\int (w_x^+)^2(t) - K\|w^+(t)\|_{L^\infty}^{p-1}\int (w^+)^2(t)$$
$$\ge \frac{1}{2}\int (w_x^+)^2(t) - K\|w^+(t)\|_{L^\infty}^{p-1}\int (w^+)^2(t) \ge \frac{1}{2}\int (w_x^+)^2(t) - Kc_2^{\frac{9}{8}}\int (w^+)^2(t).$$

Passing to the limit $t \to +\infty$, we obtain (1.11).

If $\limsup_{t\to+\infty} \|w_x^+(t)\|_{L^2} + \|w^+(t)\|_{L^2} = 0$, then $w^+(t) \to 0$ in H^1 as $t \to +\infty$, and $u(t)$ is a pure two soliton solution at $+\infty$, $c_1^+ = c_1$ and $c_2^+ = c_2$ so that (1.12)–(1.13) hold.

Assume now that $\limsup_{t\to+\infty} \|w_x^+(t)\|_{L^2} + \|w^+(t)\|_{L^2} > 0$, so that $E^+ + \frac{1}{2}c_2M^+ > 0$. Recall that ([28]) by assumption (1.5),

$$\frac{d}{dc}E(Q_c) = -\frac{1}{2}c\frac{d}{dc}\int Q_c^2 < 0, \quad \text{for } c = c_1 \text{ and } c = c_2. \tag{4.9}$$

Let $\bar{c}_2$ be such that $\bar{c}_2\left(\int Q_{c_2}^2 - \int Q_{c_2^+}^2\right) = 2(E(Q_{c_2}) - E(Q_{c_2}))$. Then, by (4.6) and (4.9) on c_2 we have $|\frac{\bar{c}_2}{c_2} - 1| \le \frac{1}{4}$. Multiplying (4.7) by $\bar{c}_2$ and summing (4.8), we find:

$$E^+ + \tfrac{\bar{c}_2}{2}M^+ = E(Q_{c_1}) - E(Q_{c_1^+}) + \tfrac{\bar{c}_2}{2}\Big(\int Q_{c_1}^2 - \int Q_{c_1^+}^2\Big).$$

Using (4.6) and (4.9) on c_1, we find

$$\frac{1}{K}(2E^+ + c_2M^+) \le \frac{c_1^+}{c_1} - 1 \le K(2E^+ + c_2M^+), \tag{4.10}$$

Let $\bar{c}_1$ be such that $\bar{c}_1\left(\int Q_{c_1}^2 - \int Q_{c_1^+}^2\right) = 2(E(Q_{c_1}) - E(Q_{c_1}))$. Arguing similarly, we have $|\bar{c}_1 - c_1| \le \frac{1}{4}c_1$ and

$$E^+ + \tfrac{\bar{c}_1}{2}M^+ = E(Q_{c_2}) - E(Q_{c_2^+}) + \tfrac{\bar{c}_1}{2}\Big(\int Q_{c_2}^2 - \int Q_{c_2^+}^2\Big).$$

By (1.2), since c_2 is small, we have $\frac{d}{dc}\int Q_{c|c=c_2}^2 \sim (\frac{2}{p-1} - \frac{1}{2})c_2^{\frac{2}{p-1}-\frac{3}{2}}$, and thus

$$\frac{1}{K}c_2^{\frac{2}{p-1}-\frac{1}{2}}(2E^+ + c_1M^+) \le 1 - \frac{c_2^+}{c_2} \le Kc_2^{\frac{2}{p-1}-\frac{1}{2}}(2E^+ + c_1M^+). \tag{4.11}$$

This concludes the proof of Theorem 1.1.

4.2 Proof of existence. Theorem 1.2

For $0 < c_1 < c_*(f)$ such that (1.5) holds and $c_2 > 0$ small enough, we denote by $u_{c_1,c_2}(t)$ the global solution of

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \quad u(0,x) = v_{c_1,c_2}(0,x), \tag{4.12}$$

where $v_{c_1,c_2}(t)$ is the approximate solution constructed in Theorem 2.1 (note that $u_{c_1,c_2}(t)$ is global by stability of Q_{c_1}). By the parity property of $x \mapsto v_{c_1,c_2}(0,x)$ and since equation (1.1) is invariant under the transformation $x \to -x$, $t \to -t$, the solution $u_{c_1,c_2}(t)$ has the following symmetry:

$$u_{c_1,c_2}(t,x) = u_{c_1,c_2}(-t,-x). \tag{4.13}$$

Thus, we shall only study $u_{c_1,c_2}(t)$ for $t \ge 0$. We claim the following concerning $u_{c_1,c_2}(t)$.

Proposition 4.1 *Let $0 < c_1 < c_*(f)$ be such that (1.5) holds. There exist $c_0(c_1) > 0$ and $K_0(c_1) > 0$, continuous in c_1 such that for any $0 < c_2 < c_0(c_1)$, there exist $0 < c_2^+(c_1, c_2) < c_1^+(c_1, c_2) < c_*(f)$, and $\rho_1(t; c_1, c_2)$, $\rho_2^+(t; c_1, c_2) \in \mathbb{R}$, such that the following hold for*

$$w^+_{c_1,c_2}(t,x) = u_{c_1,c_2}(t,x) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t)).$$

1. *Asymptotic behavior:*

$$\lim_{t\to+\infty} \|w^+_{c_1,c_2}(t)\|_{H^1(x>c_2t/10)} = 0. \tag{4.14}$$

$$\text{for } t \text{ large,} \quad \|w^+_{c_1,c_2}(t)\|_{H^1} \le K_0 c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}, \tag{4.15}$$

$$\left|\frac{c_1^+}{c_1} - 1\right| \le K_0 c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}, \quad \left|\frac{c_2^+}{c_2} - 1\right| \le K_0 c_2^{\frac{1}{p-1}-\frac{1}{100}}, \tag{4.16}$$

$$|\rho_1(T_{c_1,c_2}) - (c_1 T_{c_1,c_2} + \tfrac{1}{2}\Delta_1)| \le K c_2^{\frac{2}{p-1}-\frac{1}{2}}, \quad |\rho_2(T) - (c_2 T_{c_1,c_2} + \tfrac{1}{2}\Delta_2)| \le K c_2^{\frac{1}{p-1}-\frac{1}{100}}, \tag{4.17}$$

where Δ_1 and Δ_2 are defined in Theorem 2.1.

2. *The map $(c_1, c_2) \mapsto (c_1^+(c_1, c_2), c_2^+(c_1, c_2))$ is continuous.*

Proof of Theorem 1.2 assuming Proposition 4.1. Fix $0 < \bar{c}_1 < c_*(f)$ and $0 < \epsilon_0 < \frac{c_*(f)}{\bar{c}_1} - 1$ small enough so that Q_{c_1} satisfies (1.5) for all $c_1 \in [\bar{c}_1(1-\epsilon_0), \bar{c}_1(1+\epsilon_0)]$. Let

$$\bar{c}_0 = \min_{c_1 \in [\bar{c}_1(1-\epsilon_0), \bar{c}_1(1+\epsilon_0)]} c_0(c_1), \quad \bar{K}_0 = 2 \max_{c_1 \in [\bar{c}_1(1-\epsilon_0), \bar{c}_1(1+\epsilon_0)]} K_0(c_1),$$

where $c_0(c_1)$ and $K_0(c_1)$ are defined in Proposition 4.1.

Fix an arbitrary $0 < \bar{c}_2 < \min(\bar{c}_0, \epsilon_0^{12})$. We define $\Omega = [1 - \bar{c}_2^{\frac{1}{12}}, 1 + \bar{c}_2^{\frac{1}{12}}]^2$, and the continuous map

$$\Phi \ : \ (\lambda_1, \lambda_2) \in \Omega \mapsto \left(\frac{c_1^+(\lambda_1 \bar{c}_1, \lambda_2 \bar{c}_2)}{\bar{c}_1}, \frac{c_2^+(\lambda_1 \bar{c}_1, \lambda_2 \bar{c}_2)}{\bar{c}_2}\right).$$

By (4.16), we have

$$\text{for } j = 1, 2, \quad \left|\frac{c_j^+(\lambda_1 \bar{c}_1, \lambda_2 \bar{c}_2)}{\bar{c}_j} - \lambda_j\right| \le \bar{K}_0 \bar{c}_2^{\frac{1}{3}}.$$

This means that

$$\|\Phi - \mathrm{Id}\| \le \bar{K}_0 \bar{c}_2^{\frac{1}{3}}. \tag{4.18}$$

Moreover, by possibly taking a smaller ϵ_0,

$$\mathrm{dist}((1,1), \Phi(\partial\Omega)) \ge \bar{c}_2^{\frac{1}{12}} - \bar{K}_0 \bar{c}_2^{\frac{1}{3}} \ge \frac{1}{2} \bar{c}_2^{\frac{1}{12}} > \|\Phi - \mathrm{Id}\|. \tag{4.19}$$

From (4.18) and (4.19), we have $\deg(\Phi, \Omega, (1,1)) = \deg(\mathrm{Id}, \Omega, (1,1)) = 1$. Therefore, from degree theory there exist $(\bar{\lambda}_1, \bar{\lambda}_2) \in \Omega$ such that $\Phi(\bar{\lambda}_1, \bar{\lambda}_2) = (1,1)$ (see for example Theorems 2.3 and 2.1, p30 of [7].)

Now, for $j = 1, 2$, we set $c_j = \bar{\lambda}_j \bar{c}_j$, and we check that the function $u_{c_1,c_2}(t)$ has the property announced in Theorem 1.2. Indeed, since $\Phi(\bar{\lambda}_1, \bar{\lambda}_2) = (1,1)$, we have $c_j^+(c_1, c_2) = \bar{c}_j$ for $j = 1, 2$. Moreover, (4.14) and (4.15) imply (1.17) and (1.19). Finally, (1.21) and (1.22) follow from (4.17) and (2.57).

Proof of Proposition 4.1. Let c_1, c_2 be as in the statement of Proposition 4.1 for $0 < c_2 < c_0(c_1)$ small enough. Let $u(t,x) = u_{c_1,c_2}(t,x)$ be the solution of (4.12). Denote for simplicity $T = T_{c_1,c_2}$ (defined in (2.65)).

Step 1. Control of the modulation parameters of $u(t)$ for $t \geq T$. From Proposition 3.1 applied with $T_0 = 0$ and $\theta = \frac{2}{p-1} + \frac{1}{4}$, since $u(0) - v_{c_1,c_2}(0) = 0$, we obtain, for some $\rho(t)$,

$$\forall t \in [0,T], \quad \|u(t) - v(t,. - \rho(t))\|_{H^1} \leq K c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}, \tag{4.20}$$

where $|\rho'(t) - c_1| \leq K c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}$, $\rho(0) = 0$ and so

$$|\rho(T) - c_1 T| \leq K c_2^{\frac{2}{p-1}-\frac{1}{4}-\frac{1}{50}}. \tag{4.21}$$

By (2.56) and (4.20), we have

$$\|u(T) - Q_{c_1}(. - a) - Q_{c_2}(. - b)\|_{H^1} \leq K c_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}, \tag{4.22}$$

for $a = \frac{1}{2}\Delta_1 + \rho(T)$, $b = (c_1 - c_2)T + \frac{1}{2}\Delta_2 + \rho(T)$, so that

$$\frac{1}{2} c_1 T \leq a - b \leq 2 c_1 T.$$

Therefore, we can apply Proposition 3.2 (1) to $u(t)$ with $\omega = \frac{1}{p-1} - \frac{1}{100}$. Then, by Claim 3.3 we have the decomposition of $u(t)$ in terms of $\eta(t)$, $c_j(t)$, $\rho_j(t)$ ($j = 1,2$) defined for all $t \geq T$:

$$\eta(t,x) = u(t,x) - Q_{c_1(t)}(x - \rho_1(t)) - Q_{c_2(t)}(x - \rho_2(t)), \tag{4.23}$$

with for all $t \geq T$,

$$\forall t \geq T, \quad \|\eta(t)\|_{H^1} \leq K c^{\frac{2}{p-1}-\frac{1}{4}-\frac{1}{100}}. \tag{4.24}$$

Now, we claim

$$|\rho_1(T) - c_1 T - \tfrac{1}{2}\Delta_1| \leq K c_2^{\frac{2}{p-1}-\frac{1}{2}}, \quad |\rho_2(T) - c_2 T - \tfrac{1}{2}\Delta_2| \leq K c_2^{\frac{1}{p-1}-\frac{1}{100}}. \tag{4.25}$$

Proof of (4.25). From (4.20), (4.21) and $\|v(T)\|_{H^2} \leq K$, we have

$$\|u(T) - v(T, . - c_1 T)\|_{H^1} \leq K c_2^{\frac{2}{p-1}-\frac{1}{4}-\frac{1}{50}}. \tag{4.26}$$

Remark that for a small,

$$\tfrac{1}{K}|a| \leq \|Q_{c_1} - Q_{c_1}(. - a)\|_{L^2} \leq K|a|, \quad \tfrac{1}{K}|a| \leq c_2^{-\frac{1}{p-1}+\frac{1}{4}} \|Q_{c_2} - Q_{c_2}(. - a)\|_{L^2} \leq K|a|. \tag{4.27}$$

By (2.56) we have

$$\|v(T) - Q_{c_1}(. - \tfrac{1}{2}\Delta_1) - Q_{c_2}(. + (c_1 - c_2)T - \tfrac{1}{2}\Delta_2))\|_{H^1} \leq K c^{\frac{2}{p-1}+\frac{1}{4}}.$$

Thus by (4.23), (4.26) and (4.27), we deduce (4.25).

Step 2. Asymptotic stability. From (4.24), we can apply Proposition 3.2 (2) to $u(. + T)$ with $\omega = \frac{1}{p-1} - \frac{1}{100}$. We deduce that there exist c_1^+, $c_2^+ > 0$, such that

$$c_j(t) \to c_j^+, \quad \rho_j'(t) \to c_j^+, \quad \text{as } t \to +\infty,\ j = 1,2, \tag{4.28}$$

$$\lim_{t\to+\infty}\|w^+(t)\|_{H^1(x>c_2t/10)}=0, \tag{4.29}$$

where

$$w^+(t,x)=u(t,x)-Q_{c_1^+}(x-\rho_1(t))-Q_{c_2^+}(x-\rho_2(t)),$$

$$\left|\frac{c_1^+}{c_1}-1\right|\leq Kc^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}},\quad \left|\frac{c_2^+}{c}-1\right|\leq Kc^{\frac{1}{p-1}-\frac{1}{100}}. \tag{4.30}$$

From (4.28), $\|\eta(t)-w^+(t)\|_{H^1}\to 0$ as $t\to+\infty$ and thus, from (4.24), we obtain $\|w^+(t)\|_{H^1}\leq Kc^{\frac{2}{p-1}-\frac{1}{4}-\frac{1}{100}}$ for t large. This concludes the proof of the first part of Proposition 4.1.

Step 3. Continuity of $c_1^+(c_1,c_2)$ and $c_2^+(c_1,c_2)$. The proof is the same as in [21]. Let us give a sketch.

Let $\bar{c}_1<c_*(f)$ such that (1.5) holds for $\bar{c}_1$ and $0<\bar{c}_2<c_0$ small enough. First, we prove that the map $(c_1,c_2)\mapsto c_1^+(c_1,c_2)$ defined in a neighboorhood of $(\bar{c}_1,\bar{c}_2)$ is continuous.

Denote by $\eta_{c_1,c_2}(t)$, $c_{c_1,c_2,j}(t)$, $c_j^+(c_1,c_2)$, the parameters in the decomposition of $u_{c_1,c_2}(t)$. We claim an estimate on $|c_1^+(c_1,c_2)-c_{c_1,c_2,1}(t)|$ which is related to the quantities $\mathcal{M}_1(t)$, $\mathcal{E}_1(t)$ defined in section 3.3.

Claim 4.1 *For all $t\geq T_c$,*

$$|c_1^+(c_1,c_2)-c_{c_1,c_2,1}(t)|\leq K_0\int((\eta_{c_1,c_2})_x^2+\eta_{c_1,c_2}^2)(t,x)\psi(x-\rho_1(t)+c_1\tfrac{t}{4})dx+K_0e^{-\frac{1}{64}c_1\sqrt{c_2}t}. \tag{4.31}$$

Assuming this claim, let us complete the proof of continuity of $c_1^+(c_1,c_2)$.

Since $\|\eta_{\bar{c}_1,\bar{c}_2}(t)\|_{H^1(x>\frac{\bar{c}_2t}{10})}\to 0$ as $t\to+\infty$, for $\varepsilon>0$, there exits $T_\varepsilon>0$ such that

$$K_0\int((\eta_{\bar{c}_1,\bar{c}_2})_x^2+\eta_{\bar{c}_1,\bar{c}_2}^2)(T_\varepsilon,x)\psi(x-\rho_1(T_\varepsilon)+c_1\tfrac{T_\varepsilon}{4})dx+K_0e^{-\frac{1}{64}c_1\sqrt{c_2}T_\varepsilon}\leq\varepsilon.$$

We fix $T_\varepsilon>0$ to such value. Then, by continuous dependence in H^1 of $u_{c_1,c_2}(t)$ solution of (1.1) upon the initial data on $[0,T_\varepsilon]$ (see [12]) and of its decomposition in Claim 3.3, and the fact that $u_{c_1,c_2}(0)=v_{c_1,c_2}(0)$ is continuous upon the parameters (c_1,c_2) (see proofs of Proposition 2.1 and Theorem 2.1), there exists $\delta(\varepsilon)>0$ such that if $|(c_1,c_2)-(\bar{c}_1,\bar{c}_2)|\leq\delta$, then

$$K_0\int((\eta_{c_1,c_2})_x^2+\eta_{c_1,c_2}^2)(T_\varepsilon,x)\psi(x-\rho_1(T_\varepsilon)+c_1\tfrac{T_\varepsilon}{4})dx+K_0e^{-\frac{1}{64}c_1\sqrt{c_2}T_\varepsilon}\leq 2\varepsilon,$$

$$|c_{\bar{c}_1,\bar{c}_2,1}(T_\varepsilon)-c_{c_1,c_2,1}(T_\varepsilon)|\leq\varepsilon.$$

From Claim 4.1, applied to η_{c_1,c_2}, $\eta_{\bar{c}_1,\bar{c}_2}$, we have $|c_1^+(c_1,c_2)-c_{c_1,c_2,1}(T_\varepsilon)|\leq 2\varepsilon$ and $|c_1^+(\bar{c}_1,\bar{c}_2)-c_{\bar{c}_1,\bar{c}_2,1}(T_\varepsilon)|\leq\varepsilon$. Therefore, $|c_1^+(\bar{c}_1,\bar{c}_2)-c_1^+(c_1,c_2)|\leq 4\varepsilon$. Thus, $(c_1,c_2)\mapsto c_1^+(c_1,c_2)$ is continuous.

We argue similarly for $(c_1,c_2)\mapsto c_2^+(c_1,c_2)$. This concludes the proofs of Proposition 4.1 and of Theorem 1.2.

Proof of Claim 4.1. For $T \le t_0 \le t$, let $\mathcal{M}_1(t)$ and $\mathcal{E}_1(t)$ be defined in section 3.3, with $x_0 = c_1 \frac{t_0}{4}$. From Claim 3.4 integrated on $[t_0, t]$, we obtain

$$\int Q_{c_1(t)}^2 - \int Q_{c_1(t_0)}^2 \le (\mathcal{M}_1(t_0) - \mathcal{M}_1(t)) + Ke^{-\frac{1}{64}c_1\sqrt{c_2}t_0},$$

$$\Big(-E(Q_{c_1(t)}) + E(Q_{c_1(t_0)}) - \frac{c_1^+}{100}\Big(\int Q_{c_1(t)}^2 - \int Q_{c_1(t_0)}^2\Big)\Big)$$
$$\ge 2\mathcal{E}_1(t) - 2\mathcal{E}_1(t_0) + \frac{1}{100}(\mathcal{M}_1(t) - \mathcal{M}_1(t_0)) - Ke^{-\frac{1}{64}c_1\sqrt{c_2}t_0}.$$

Note in particular that $\int_{t_0}^t e^{-\frac{1}{16}\sqrt{c_1}(c_1(t-t_0)+x_0)} g_1(t)dt \le Ke^{-\frac{1}{16}\sqrt{c_1}x_0} \le Ke^{-\frac{1}{64}c_1^{\frac{3}{2}}t_0}$. Letting $t \to +\infty$, by the asymptotic stability, this gives

$$\int Q_{c_1^+}^2 - \int Q_{c_1(t_0)}^2 \le \mathcal{M}_1(t_0) + Ke^{-\frac{1}{64}\sqrt{c_2}t_0},$$

$$E(Q_{c_1^+}) - E(Q_{c_1(t_0)}) + \frac{c_1^+}{100}\Big(\int Q_{c_1^+}^2 - \int Q_{c_1(t_0)}^2\Big) \le 2\mathcal{E}_1(t_0) + \frac{c_1^+}{100}\mathcal{M}_1(t_0) + Ke^{-\frac{1}{64}c_1\sqrt{c_2}t_0}.$$

By (4.9), we obtain:

$$|c_1^+ - c_1(t_0)| \le K \int (\eta_x^2 + \eta^2)(t_0, x)\psi(x - \rho_1(t_0) + \tfrac{t_0}{4})dx + Ke^{-\frac{1}{64}c_1\sqrt{c_2}t_0},$$

which concludes the proof of Claim 4.1.

4.3 Proof of stability. Theorem 1.3

Theorem 1.3 follows directly from Proposition 3.1, Proposition 3.2 and the proof of Theorem 1.2. Let $0 < \bar{c}_1 < c_*(f)$ such that (1.5) holds for $\bar{c}_1$. Let $0 < \bar{c}_2 < c_0(\bar{c}_1)$ small enough. We assume

$$\|u(0) - \varphi(0)\|_{H^1} \le K\bar{c}_2^{\frac{1}{p-1}+\frac{1}{2}}, \tag{4.32}$$

where $\varphi = \varphi_{\bar{c}_1,\bar{c}_2}$ is the solution constructed in Theorem 1.2.

From the proof of Theorem 1.2, there exist (c_1, c_2) close to $(\bar{c}_1, \bar{c}_2)$ in the following sense (see (4.16)):

$$\left|\frac{\bar{c}_1}{c_1} - 1\right| \le Kc_2^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}}, \quad \left|\frac{\bar{c}_2}{c_2} - 1\right| \le Kc_2^{\frac{1}{p-1}-\frac{1}{100}}, \tag{4.33}$$

so that $\varphi(0) = v_{c_1,c_2}$. The assumption (4.32) on $u(0)$ is thus equivalent to

$$\|u(0) - v_{c_1,c_2}(0)\|_{H^1} \le Kc_2^{\frac{1}{p-1}+\frac{1}{2}}. \tag{4.34}$$

By invariance of (1.1) by the transformation $x \to -x$, $t \to -t$, it is enough to prove the result for $t \ge 0$.

(i) Estimates on $[0, T_{c_1,c_2}]$.

By (4.34) and Proposition 3.1 (applied with $T_0 = 0$ and $\theta = \frac{1}{p-1} + \frac{1}{2}$) we obtain, for all $t \in [0, T_{c_1,c_2}]$, for some $\rho(t)$,

$$\|u(t) - v(t, x - \rho(t))\|_{H^1} \le Kc_2^{\frac{1}{p-1}+\frac{1}{2}} + Kc^{\frac{2}{p-1}+\frac{1}{4}-\frac{1}{100}} \le Kc_2^{\frac{1}{p-1}+\frac{1}{2}},$$

for c_2 small. From (2.58), we obtain (1.24) on $[0, T_{c_1,c_2}]$.

From Theorem 2.1, we deduce, for some a, b, with $a - b \geq \frac{1}{2} T_{c_1,c_2}$,

$$\|u(T_{c_1,c_2}) - Q_{c_1}(. - a) - Q_{c_2}(. - b)\|_{H^1} \leq K c_2^{\frac{1}{p-1}+\frac{1}{2}}. \tag{4.35}$$

(ii) Estimates on $[T_{c_1,c_2}, +\infty)$.

By (4.35) and Proposition 3.2 (applied with $\omega = \frac{1}{4}$) for all $t \in [T_{c_1,c_2}, +\infty)$, there exist $\rho_1(t)$, $\rho_2(t)$ and c_1^+, c_2^+, such that

$$\begin{gathered} \|u(t) - Q_{c_1^+}(. - \rho_1(t)) - Q_{c_2^+}(. - \rho_2(t))\|_{H^1} \leq K c_2^{\frac{1}{p-1}}, \\ \left|\frac{c_1^+}{c_1} - 1\right| \leq K c_2^{\frac{1}{p-1}+\frac{1}{2}}, \quad \left|\frac{c_2^+}{c_2} - 1\right| \leq K c^{\frac{1}{4}}. \end{gathered} \tag{4.36}$$

(iii) Combining (4.33) and (4.36), we obtain

$$\left|\frac{c_1^+}{\bar{c}_1} - 1\right| \leq K c_2^{\frac{1}{p-1}+\frac{1}{2}}, \quad \left|\frac{c_2^+}{\bar{c}_2} - 1\right| \leq K c^{\frac{1}{4}}.$$

4.4 Open problem and monotonicity of speeds

The main question following Theorem 1 concerns the case where c_2 is not small with respect to c_1. In this case, we expect the following.

Open problem. *Assume that f satisfies* (1.2) *with $p = 4$ and assume that for all $c \in (0, c_*(f))$ the positive solution Q_c of* (1.4) *satisfies* (1.5)*. Let $0 < c_2 < c_1 < c_*(f)$ and let $u(t)$ be the solution of* (1.1) *such that*

$$\lim_{t\to-\infty} \|u(t) - [Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t)]\|_{H^1} = 0. \tag{4.37}$$

There exist $0 < c_2^+ < c_1^+ < c_*(f)$, $v_0 \in H^1$, such that

$$\lim_{t\to+\infty} \left\|u(t) - \left[Q_{c_1^+}(. - c_1^+ t) + Q_{c_2^+}(. - c_2^+ t) + W(t)v_0\right]\right\|_{H^1} = 0, \tag{4.38}$$

where $W(t)v_0$ is the solution of the linear Airy equation $v_t + v_{xxx} = 0$ with $v(0) = v_0$.

Note that for $p = 4$, a scattering estimate such as (4.38) is suggested by results of Tao [27]. For $p = 2$ or 3, we cannot expect scattering to be true, and we can replace (4.38) by a weaker result on $\eta(t) = u(t) - \left[Q_{c_1^+}(. - c_1^+ t) + Q_{c_2^+}(. - c_2^+ t)\right]$:

$$\lim_{t\to+\infty} \|\eta(t)\|_{H^1(x>0)} = 0, \quad \lim_{t\to+\infty} E(\eta(t)) = E^+ \geq 0. \tag{4.39}$$

This means that the nonlinear term in $E(\eta)$ is controlled by $\frac{1}{2}\int \eta_x^2$ for large positive time.

In the framework of this open problem , we claim the following monotonicity principle on the velocities c_j, c_j^+.

Claim 4.2 (Monotonicity principle) *Let $u(t)$ be a solution of* (1.1) *satisfying* (4.37)–(4.38)*. Then*

$$c_2^+ \leq c_2 \leq c_1 \leq c_1^+. \tag{4.40}$$

Proof. We prove (4.40) by a convexity argument. By L^2 norm and energy conservation it follows that $\lim_{t\to+\infty}\int \eta^2(t) = M^+$ and $\lim_{t\to+\infty} E(\eta(t)) = E^+ \geq 0$ exist and satisfy

$$\int Q_{c_1}^2 + \int Q_{c_2}^2 = \int Q_{c_1^+}^2 + \int Q_{c_2^+}^2 + M^+ \geq \int Q_{c_1^+}^2 + \int Q_{c_2^+}^2, \tag{4.41}$$

$$E(Q_{c_1}) + E(Q_{c_2}) = E(Q_{c_1^+}) + E(Q_{c_2^+}) + E^+ \geq E(Q_{c_1^+}) + E(Q_{c_2^+}). \tag{4.42}$$

Recall that we assume

$$-\frac{d}{dc}E(Q_c) = \frac{1}{2}c\frac{d}{dc}\int Q_c^2 > 0 \quad \text{for } c \in (0, c_*). \tag{4.43}$$

We consider the following two cases.

• $c_1^+ \geq c_1$. Then it follows from (4.41) and (4.43) that $c_2^+ \leq c_2$, and the result holds in this case.

• $c_1^+ < c_1$. Similarly, using (4.42) and (4.43), this implies $c_2^+ > c_2$ and thus $c_2 < c_2^+ < c_1^+ < c_1$. We claim that this implies a contradiction. Indeed, set

$$\alpha_1 = -\frac{E(Q_{c_1}) - E(Q_{c_1^+})}{\int Q_{c_1}^2 - \int Q_{c_1^+}^2}, \quad \alpha_2 = -\frac{E(Q_{c_2^+}) - E(Q_{c_2})}{\int Q_{c_2^+}^2 - \int Q_{c_2}^2}.$$

On the one hand, by (4.43), we have

$$-(E(Q_{c_1}) - E(Q_{c_1^+})) = \frac{1}{2}\int_{c_1^+}^{c_1} c(\tfrac{d}{dc}\int Q_c^2)dc \geq \frac{1}{2}c_1^+\int_{c_1^+}^{c_1}(\tfrac{d}{dc}\int Q_c^2)dc \geq \frac{1}{2}c_1^+(\int Q_{c_1}^2 - \int Q_{c_1^+}^2),$$

which means that $\alpha_1 \geq \frac{1}{2}c_1^+$. Similarly, we have $\alpha_2 \leq \frac{1}{2}c_2^+$.

On the other hand, by (4.41), (4.42), we have

$$\alpha_1 = -\frac{E(Q_{c_1}) - E(Q_{c_1^+})}{\int Q_{c_1}^2 - \int Q_{c_1^+}^2} \leq -\frac{E(Q_{c_2^+}) - E(Q_{c_2})}{\int Q_{c_2^+}^2 - \int Q_{c_2}^2} = \alpha_2. \tag{4.44}$$

Thus, we obtain $\frac{1}{2}c_1^+ \leq \alpha_1 \leq \alpha_2 \leq \frac{1}{2}c_2^+$, which is a contradiction.

A Proof of Lemma 2.1.

Proof of (2.15): it follows from the equation of Q_c, (1.2) and standard arguments.

Note that for any $0 < c < c_*$; from (1.4) multiplying by Q_c' and integrating, we get

$$(Q_c')^2 + 2F(Q_c) = c\,Q_c^2. \tag{A.1}$$

Using the Taylor decomposition of $F(Q_c)$ (see (2.21)), we obtain

$$(Q_c')^2 = cQ_c^2 + \sum_{p+1\leq k_1\leq k_0} \sigma_{k_1}Q_c^{k_1} + O(Q_c^{k_0+1}),$$

and (2.16) follows from $(Q_c^k)'(Q_c^{\widetilde{k}})' = k\widetilde{k}(Q_c')^2 Q_c^{k+\widetilde{k}-2}$.

Proof of (2.17)–(2.19). We prove (2.17) and (2.19), (2.18) is obtained in a similar way. Note that from (1.4) and (2.21), we get (2.17) for $k = 1$. For $k \geq 1$, we have from direct calculations:

$$\begin{aligned}(Q_c^k)'' &= k(k-1)(Q_c')^2 Q_c^{k-2} + kQ_c'' Q_c^{k-1} \\ &= k(k-1)cQ_c^k - 2k(k-1)Q_c^{k-2}F(Q_c) + ckQ_c^k - kf(Q_c)Q_c^{k-1} \\ &= k^2 cQ_c^k - 2k(k-1)Q_c^{k-2}F(Q_c) - kf(Q_c)Q_c^{k-1},\end{aligned} \tag{A.2}$$

and we get (2.17) by using (2.21) for f and F. Now, we prove (2.19), from (A.2),

$$(Q_c^k)^{(4)} = ((Q_c^k)'')'' = ck^2(Q_c^k)'' - 2k(k-1)(Q_c^{k-2}F(Q_c))'' - k(f(Q_c)Q_c^{k-1})''.$$

For the first term, we use (2.17). Now, we consider the term $(f(Q_c)Q_c^{k-1})''$, the term $(Q_c^{k-2}F(Q_c))''$ is similar. We have

$$\begin{aligned}(f(Q_c)Q_c^{k-1})'' &= (Q_c^{k-1})'' f(Q_c) + (Q_c')^2 Q_c^{k-2}(2(k-1)f'(Q_c) + Q_c f''(Q_c)) + Q_c'' Q_c^{k-1} f'(Q_c) \\ &= c\Big[(k-1)^2 Q_c^{k-1} f(Q_c) + Q_c^k(2(k-1)f'(Q_c) + Q_c f''(Q_c)) + Q_c^k f'(Q_c)\Big] \\ &\quad - 2F(Q_c)Q_c^{k-2}(2(k-1)f'(Q_c) + Q_c f''(Q_c)) - f(Q_c)Q_c^{k-1}f'(Q_c).\end{aligned}$$

Now, using Taylor expansions for f (i.e. (2.21)) and for f' and f'', we get (2.19). Thus Lemma 2.1 is proved.

Claim A.1 *(i) For any integer $r > 0$,*

$$Q_c^r(y_c)\beta(y_c) = \sum_{\substack{1+r\leq k\leq k_0+r \\ 0\leq \ell\leq \ell_0}} c^\ell Q_c^k(y_c) a_{k-r,\ell}.$$

(ii) Decomposition of β'', β^2, $\beta'\beta$ and β^3:

$$\beta''(y_c) = \sum_{\substack{1\leq k\leq k_0+p-1 \\ 0\leq \ell\leq \ell_0+1}} c^\ell Q_c^k(y_c) a_{k,\ell}^{1*} + O(Q_c^{k_0+1}), \quad \beta^2(y_c) = \sum_{\substack{2\leq k\leq 2k_0 \\ 0\leq \ell\leq 2\ell_0}} c^\ell Q_c^k(y_c) a_{k,\ell}^{2*},$$

$$\beta'(y_c)\beta(y_c) = \sum_{\substack{2\leq k\leq 2k_0 \\ 0\leq \ell\leq 2\ell_0}} c^\ell (Q_c^k)'(y_c) a_{k,\ell}^{3*}, \quad \beta^3(y_c) = \sum_{\substack{3\leq k\leq 3k_0 \\ 0\leq \ell\leq 3\ell_0}} c^\ell Q_c^k(y_c) a_{k,\ell}^{4*},$$

where for any $k \geq 1$, $\ell \geq 0$, the coefficients $a_{k,\ell}^{1*}$, $a_{k,\ell}^{2*}$, $a_{k,\ell}^{3*}$ and $a_{k,\ell}^{4*}$ depend on some $(a_{k',\ell'})$ for $(k', \ell') \prec (k, \ell)$.

See proof of Claim A.1 in [21].

B Proof of Claim 3.2

First, we claim the following estimates

Claim B.1

$$\|\partial_t v(t)\|_{L^\infty} + \|v^{p-2} - Q^{p-2}(y)\|_{L^\infty} + \|\partial_x v - Q'(y)\|_{L^2} \le Kc^{\frac{1}{p-1}}, \tag{B.1}$$

$$\|\partial_t v(t) + \alpha'(y_c)Q'(y)\|_{L^2} \le Kc^{\frac{1}{p-1}+\frac{1}{4}}, \quad \|\partial_t v(t) + \alpha'(y_c)Q'(y)\|_{L^\infty} \le Kc^{m_0}, \tag{B.2}$$

$$\|\alpha''(y_c)\|_{L^\infty} + \frac{1}{c}\|\alpha^{(4)}(y_c)\|_{L^\infty} \le Kc^{\frac{1}{2}+\frac{1}{p-1}}, \tag{B.3}$$

where $m_0 = \min\left(\frac{2}{p-1}, \frac{1}{p-1} + \frac{1}{2}\right)$.

The proof of Claim B.1 is omitted, it is a direct consequence of the definition of v and the expression of Q_c. See also proof of Claim 4.1 in [21].

Proof of (3.13). We replace $\partial_t z$ by its expression:

$$\mathbf{F}_1 = -\int S(t)\left(-\partial_x^2 z + z - (f(v+z) - f(v))\right)$$
$$+ (\rho'(t) - 1)\int \partial_x(v+z)\left(-\partial_x^2 z + z - (f(v+z) - f(v))\right) = \mathbf{g}_1 + \mathbf{g}_2.$$

By integration by parts, the Cauchy-Schwarz' inequality, we have

$$|\mathbf{g}_1| \le K\|z(t)\|_{L^2}\left(\|\partial_x^2 S(t)\|_{L^2} + \|S(t)\|_{L^2}\right).$$

Since $\int \partial_x(v+z)f(v+z) = 0$, and by the definition of $S(t)$

$$\mathbf{g}_2 = (\rho'(t)-1)\int \partial_x(v+z)(-\partial_x^2 z + z + f(v)) = (\rho'(t)-1)\int (\partial_x v(-\partial_x^2 z + z) + \partial_x z\, f(v))$$
$$= (\rho'(t)-1)\int z\partial_x(-\partial_x^2 v + v - f(v)) = (\rho'(t)-1)\int z(\partial_t v - S(t)).$$

By (3.5), Claim B.1 and the definition of v, we obtain:

$$\left|\mathbf{g}_2 + (\rho'(t)-1)\int \alpha'(y_c)Q'(y)z\right| \le K|\rho'(t)-1|\|z(t)\|_{L^2}\left(\|\partial_t v + \alpha'(y_c)Q'(y)\|_{L^2} + \|S(t)\|_{L^2}\right)$$
$$\le K\|z(t)\|_{L^2}(\|z(t)\|_{H^1} + \|S(t)\|_{L^2})(c^{\frac{1}{p-1}+\frac{1}{4}} + \|S(t)\|_{L^2}).$$

Proof of (3.14). The term $\mathbf{F}_2$ was introduced in the expression of $\mathcal{F}$ to cancel the main terms in $\mathbf{F}_1$ and $\mathbf{F}_3$.

$$\mathbf{F}_2 = \int \alpha'(y_c)z\partial_x(-\partial_x^2 z + z - (f(z+v) - f(v)))$$
$$-\int \alpha'(y_c)zS(t) + (\rho'(t)-1)\int \alpha'(y_c)\partial_x(v+z)z = \mathbf{g}_3 + \mathbf{g}_4.$$

First,

$$\mathbf{g}_4 = -\int \alpha'(y_c)zS(t) + (\rho'(t)-1)\int \alpha'(y_c)\partial_x v\, z - \frac{1}{2}(\rho'(t)-1)\int z^2\alpha''(y_c).$$

By (3.5) and the definition of v, we have

$$\left|\mathbf{g}_4 - (\rho'(t)-1)\int \alpha'(y_c)Q'(y)z\right| \leq Kc^{m_0}\|z(t)\|_{H^1}(\|z(t)\|_{H^1} + \|S(t)\|_{H^1}).$$

Second, for the term $\mathbf{g}_3$, we integrate by parts, to obtain:

$$\mathbf{g}_3 = -\int \alpha''(y_c)(\tfrac{3}{2}(\partial_x z)^2 + \tfrac{1}{2}z^2) + \int \alpha^{(4)}(\tfrac{1}{2}z^2) - \int \alpha'(y_c)z\partial_x(f(z+v)-f(v)). \qquad \text{(B.4)}$$

Estimating the terms $\alpha''(y_c)$ and $\alpha^{(4)}(y_c)$ from Claim B.1 we obtain

$$\left|-\int \alpha''(y_c)(\tfrac{3}{2}(\partial_x z)^2 + \tfrac{1}{2}z^2) + \int \alpha^{(4)}(\tfrac{1}{2}z^2)\right| \leq Kc^{\frac{1}{p-1}+\frac{1}{2}}\|z(t)\|_{H^1}^2.$$

In the last term of (B.4), cubic and higher order terms are controlled by $Kc^{\frac{1}{p-1}}\|z(t)\|_{H^1}^3$. The quadratic term is

$$\int \alpha'(y_c)z\partial_x(-f'(v)z) = \frac{1}{2}\int \alpha''(y_c)z^2 f'(v) - \frac{1}{2}\int \alpha'(y_c)z^2\partial_x(f'(v)) = \mathbf{g}_5 + \mathbf{g}_6.$$

As before, $|\mathbf{g}_5| \leq Kc^{\frac{1}{2}+\frac{1}{p-1}}\|z(t)\|_{H^1}^2$. Finally, by Claim B.1

$$\left|\mathbf{g}_6 + \frac{1}{2}\int \alpha'(y_c)z^2 Q'(y)f''(Q(y))\right| \leq Kc^{\frac{2}{p-1}}\|z(t)\|_{H^1}^2.$$

Proof of (3.15). First note $\left|\frac{1}{2}(1-c)\int \alpha''(y_c)z^2\right| \leq Kc^{\frac{1}{2}+\frac{1}{p-1}}\|z(t)\|_{L^2}^2$. We now estimate

$$-\int \partial_t v\left(f(v+z) - f(v) - f'(v)z - \tfrac{1}{2}f''(v)z^2\right) - \frac{1}{2}\int \partial_t v\, f''(v)z^2 = \mathbf{g}_7 + \mathbf{g}_8$$

We have $|\mathbf{g}_7| \leq Kc^{\frac{1}{p-1}}\|z(t)\|_{H^1}^3$ and by $|\alpha'(y_c)| \leq Kc^{\frac{1}{p-1}}$,

$$\left|\mathbf{g}_8 - \frac{1}{2}\int \alpha'(y_c)Q'(y)f''(Q)z^2\right| \leq Kc^{m_0}\|z\|_{H^1}^2,$$

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