Classification of characteristic points for a semilinear wave equation in one space dimension
Part 1

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The equation

$$\begin{cases} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p-1} u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a \in \mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

THE CAUCHY PROBLEM IN $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

It is a consequence of:

- ▷ the Cauchy problem in $H^1 \times L^2(\mathbb{R})$,
- ▷ the finite speed of propagation.

Maximal solution in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

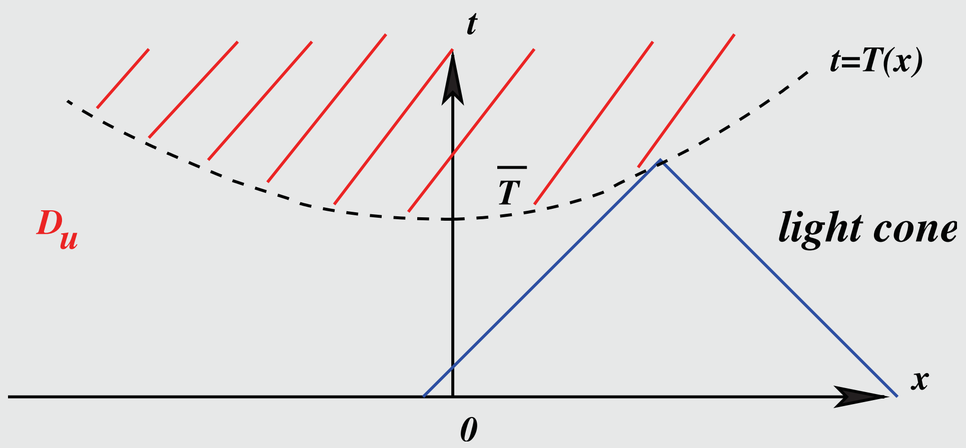
- either it exists for all $t \in [0, \infty)$ (**global solution**),
- or it exists for all $t \in [0, \bar{T})$ (**singular solution**).

Existence of singular solutions

It's a consequence of ODE techniques and the finite speed of propagation; see also the energy argument by Levine 1974:

if $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ *and* $\int_{\mathbb{R}} \left(\frac{1}{2}(u_1)^2 + \frac{1}{2}(\partial_x u_0)^2 - \frac{1}{p+1}|u_0|^{p+1} \right) dx < 0$*,*
then u *is not global*.

Singular solutions: the maximal influence domain



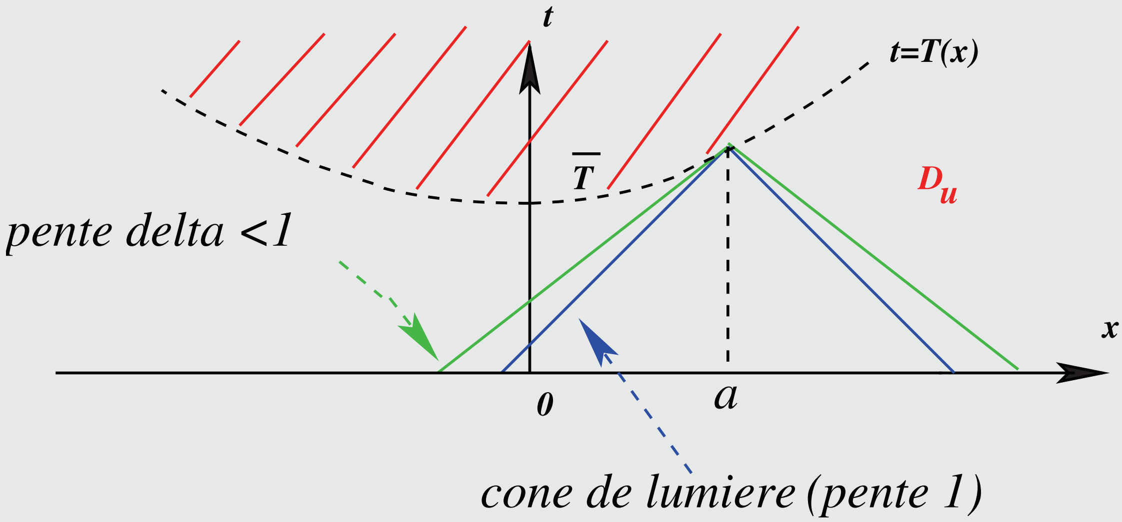
The blow-up set $t \to T(x)$ is 1-Lipschitz (**finite speed of propagation**).

Remark : $\bar{T} = \inf T(x)$ is the **blow-up time**. For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

The aim of this talk : To describe precisely the blow-up set, and the solution near the blow-up set, *for an arbitrary blow-up solution*.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

Known results, for an arbitrary solution

- The blow-up set $\Gamma = \{(x, T(x))\} \subset \mathbb{R}^2$.
- By definition, Γ is 1-Lipschitz.
- $\mathcal{R} \neq \emptyset$ (Indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x \in \mathbb{R}} T(x)$ is non characteristic).
- Caffarelli and Friedman (1985 and 1986) had two criteria to have $\mathcal{R} = \mathbb{R}$ and $x \mapsto T(x)$ of class C^1 (using the **positivity of the fundamental solution**):
  - ▷ either when $p \geq 3$, with $u_0 \geq 0$, $u_1 \geq 1$ and $(u_0, u_1) \in C^4 \times C^3(\mathbb{R})$,
  - ▷ or under conditions on initial data that ensure that

$$u \geq 0 \text{ and } \partial_t u \geq (1 + \delta_0)|\partial_x u|$$

for some $\delta_0 > 0$.

Questions and new results

- ▷ **Existence**
  - Are there characteristic points? *yes,* $\mathcal{S} \neq \emptyset$.
- ▷ **Regularity**
  - Is $\mathcal{R}$ open? *yes*
  - Is Γ (or $\Gamma_{\mathcal{R}}$) of class C^1 ? *yes*
- ▷ **Asymptotic behavior (profile)**
  - How does the solution behave near a non characteristic point? *we have the profile*
  - and near a characteristic point? *we have a precise decomposition into solitons*

Rk. Regularity and asymptotic behavior are linked.

The plan

- The first talk:

- ▷ Part 1: Existence of characteristic points.
- ▷ Part 2: A Liouville theorem and regularity of the blow-up set.
- ▷ Part 3: A Lyapunov functional and the blow-up rate.

- The second talk

- ▷ Part 4: Asymptotic behavior near *non characteristic* points (the blow-up profile).
- ▷ Part 5: Asymptotic behavior near *characteristic* points (decomposition into solitons).

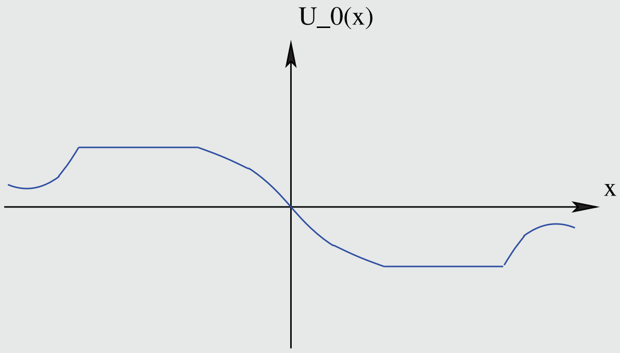
Rk. The order of this presentation goes from the easiest (to state) to the most complicated. The chronological order is actually 3, 4, 1, 2, 5.

Part 1 : Existence of characteristic points

We recall: Any *solution to the Cauchy problem has (at least) a non characteristic point (the minimum of the blow-up set).*

Th. There exist *initial data which give solutions with a characteristic point.*

Example : We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0),\ u(0,t) = 0$.



Th. If we perturb initial data, then then new solution blows up and has a characteristic point.

Part 2 : Regularity of the blow-up set

- Near a non characteristic point:
  Th. *The set of non characteristic points* $\mathcal{R}$ **is open** *and* $T(x)$ **is of class** C^1 *on this set (*$C^{1,\alpha}$ *by N. Nouaili CPDE 2008).*

- Near a characteristic point:
  Th. *The set of characteristic points* $\mathcal{S}$ **has an empty interior.**
  If $a \in \mathcal{S}$*, then* $T_l'(a) = 1$ *and* $T_r'(a) = -1$.

  Cor. *There is no solution with* $a \in \mathcal{S}$ *and* $T'(a) = 1$.

Part 2 : The corner property near a *characteristic* point

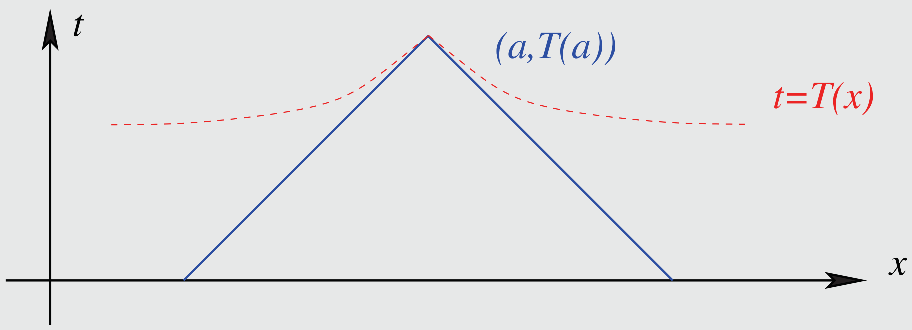
Th. (the corner property) *If $a \in \mathcal{S}$, then for all x near a,*

$$\frac{1}{C}|x-a||\log|x-a||^{-\gamma(a)} \le T(x) - T(a) + |x-a| \le C|x-a||\log|x-a||^{-\gamma(a)} \quad (1)$$

where

$$\gamma(a) = \frac{(k(a)-1)(p-1)}{2} \text{ with } k(a) \in \mathbb{N}, k(a) \ge 2.$$

Rk. Estimate (1) remains valid after differentiation.



Comments

Rk. We recall the result of Caffarelli and Friedman:

If for all $x \in \mathbb{R}$ and $t < T(x)$, we have $u(x,t) \ge 0$ and $\partial_t u \ge (1+\delta_0)|\partial_x u|$ for some $\delta_0 > 0$, then $\mathcal{R} = \mathbb{R}$.

Here, We improve their criterion:
If for all $x \in [a,b]$ and $t < T(x)$, we have $u(x,t) \ge 0$, then $(a,b) \subset \mathcal{R}$.

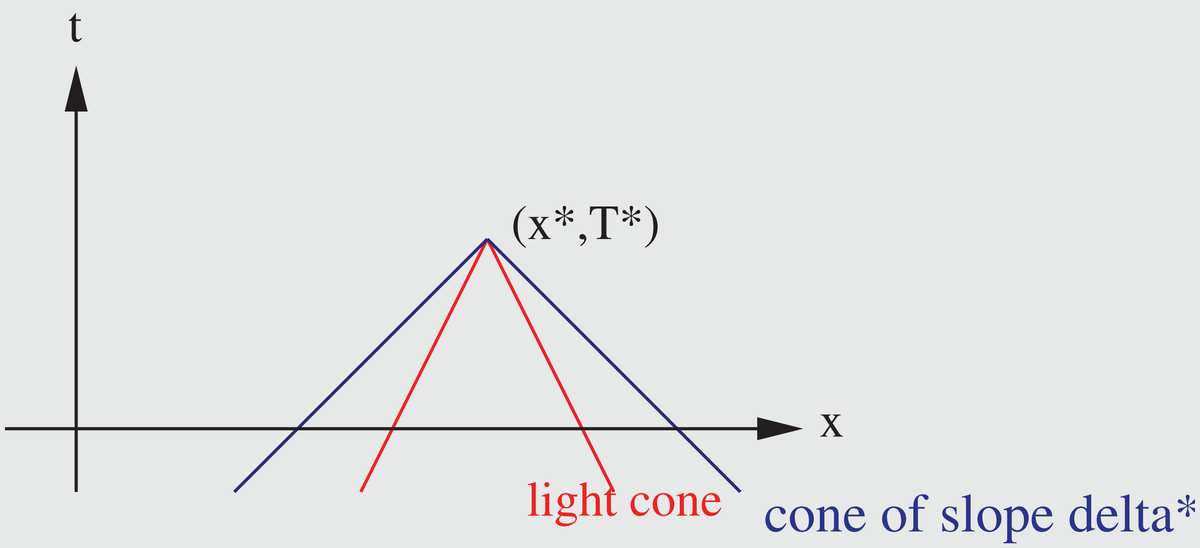
Idea of the proof of the regularity in the *non characteristic* case:
The techniques are based on

- - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable (see Part 3 of this talk).
- - a **Liouville Theorem** (see next slide).

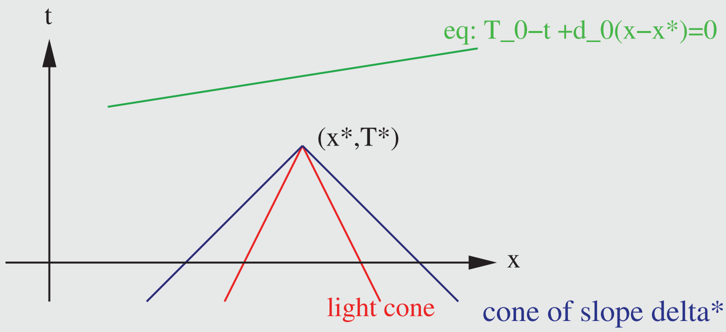
Idea of the proof of the regularity in the *characteristic* case: At the end of the talk.

A Liouville Theorem

Th. Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:
- u is defined in the *infinite* blue cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the green line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x - x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the green line.

Comments

- The limiting case $\delta^* = 1$ is still open.

The proof:

- The proof has a completely different structure from the proof for the heat equation.
- The proof is based on various energy arguments and on a dynamical result.

Part 3: A Lyapunov functional and the blow-up rate

Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in B(0,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in B(0,1) \times [-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_B \left(q_1^2 + \left(\partial_y q_1\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

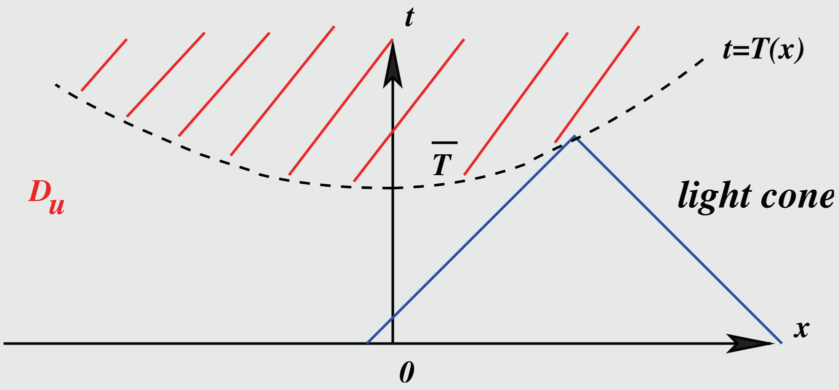
Lemma 1 (Monotonicity (Antonini-Merle)) *For all s_1 and s_2:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1} \int_{s_1}^{s_2} \int_B (\partial_s w)^2 (1-|y|^2)^{\frac{2}{p-1}-1} dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

The blow-up rate

We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint : Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-|y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1}|w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on p and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Getting rid of the weights

Reducing $(-1,1)$ to $(-\frac{1}{2}, \frac{1}{2})$, we get:

Cor. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left((\partial_s w)^2 + (\partial_y w)^2 + w^2 + |w|^{p+1} \right) dy \leq K.$$

Upper bound in the original $u(x,t)$ variables

Th. sup. *For all* $x_0 \in I\!R$ *and* $t \in [\frac{3}{4}T(x_0), T(x_0))$:

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}} + \frac{\|\partial_x u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}\right) \le K.$$

Rk. We have a lower bound of the same size when x_0 is non characteristic (see Part 4 on profiles near a non characteristic point).

Idea of the proof of the upper bound

- Selfsimilar transformation and existence of a Lyapunov functional
- Interpolation to gain regularity
- Gagliardo-Nirenberg estimates.

Classification of characteristic points for a semilinear wave equation in one space dimension
Part 2

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The equation

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where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a \in \mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

The plan

- The first talk:

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Rk. The order of this presentation goes from the easiest (to state) to the most complicated. The chronological order is actually 3, 4, 1, 2, 5.

Part 4: Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic**. Using a covering argument for x near x_0, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1,1)}$ is bounded.

Question: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible**.

See for similar difficulty and approach, results for

- ▷ the **critical KdV** (Martel and Merle),
- ▷ **NLS** (Merle and Raphaël).

Stationary solutions.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1} \left(r'^2(1-y^2) + r^2\right) \rho dy < +\infty\}.$$

Prop. Consider a stationary solution in $\mathcal{H}_0$. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components.

Blow-up profile near a non characteristic point

Th. *There exist $C_0 > 0$ and $\mu_0 > 0$ such that if x_0 is* **non characteristic**, *then there exist $d(x_0) \in (-1,1)$, $e(x_0) = \pm 1$ and $s^*(x_0) \geq -\log T(x_0)$ such that :*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

where the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2(1-y^2) + q_2^2\right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.
Rk. We have exp. fast convergence (hence, C^{1,μ_0} regularity of $\mathcal{R}$, see Nouaili).
Rk. $\|w_{x_0}(y,s) - \kappa(d(x_0),y)\|_{L^\infty(-1,1)} \to 0$.
Rk. The parameter of the profile $d(x_0)$ has a geometrical interpretation $(T'(x_0))$.

Difficulties of the proof of convergence

- The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **modulation theory**.
- The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Part 5: Asymptotic behavior at a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s)$ for $i = 1, ..., k$ such that:
(i)

$$\|w_{x_0}(s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \kappa(d_i(s), \cdot)\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing

$$\bar{w}_{x_0}(\xi, s) = (1-y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(x_0) = -\tanh^{-1} d_i(s),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

Part 5: Asymptotic behavior at a *characteristic* point (cont.)

(iii) For all $i = 1, ..., k(x_0)$ and s large enough,

$$\left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s - C_0 \le \zeta_i(s) \le \left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s + C_0.$$

(iv) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ as $s \to \infty$.

Rk.
- As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of *equidistant* stationary solutions ("solitons"), with *alternate* signs.
- In the ξ variable, half of the solitons go to $-\infty$, and the other half to $+\infty$.
- The main difficulty in the proof is to prove that $k(x_0) \ge 2$ (the case $k(x_0) = 0$ is harder to eliminate).

The energy behavior

. Defining

$$k(x_0) = 1 \text{ when } x_0 \in \mathcal{R},$$

we get the following:

Cor.
(i) *For all $x_0 \in \mathbb{R}$ and $s \ge -\log T(x_0)$, we have*

$$E(w_{x_0}(s)) \ge k(x_0)E(\kappa_0).$$

(ii) **(An energy criterion for non characteristic points)** *If for some $x_0 \in \mathbb{R}$ and $s_0 \ge -\log T(x_0)$, we have*

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

Idea of the proof of the results in the *characteristic* case

The results are: the decomposition into solitons, the corner property and the fact that the interior of S is empty.

5 main steps are needed:

- Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons, with no information on the signs or the distance between the solitons' centers (in the ξ variable).
- Step 2: Characterization of the case $k(x_0) \geq 2$. Proof of the corner property.
- Step 3: Excluding the case $k(x_0) = 0$ if $x_0 \in \partial\mathcal{S}$ (note that $\partial\mathcal{S} \subset \mathcal{S}$ since $\mathcal{R} = \mathbb{R}\backslash\mathcal{S}$ is open).
- Step 4: Characterization of the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 1$.
- Step 5: Conclusion (we prove that the interior of S is empty, then that $k(x_0) \geq 2$ for all $x_0 \in \mathcal{S}$).

Comments

Rk. 1: A good understading of the *non-characteristic* case is *crucial*.

Rk. 2: Excluding the case $k(x_0) = 0$ is more difficult than excluding the case $k(x_0) = 1$.
In particular, we can't exclude directly the case $k(x_0) = 0$ for all $x_0 \in \mathcal{S}$. We do it first when $x_0 \in \partial\mathcal{S}$, then prove that the interior of $\mathcal{S}$ is empty, hence $\partial\mathcal{S} = \mathcal{S}$.

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons

.
The upper bound on the blow-up rate and the Lyapunov functional in the $w(y,s)$ are crucial in this step.
We get the decomposition,

$$\|w_{x_0}(s) - \sum_{i=1}^{k(x_0)} e_i(x_0)\kappa(d_i(s),\cdot)\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

with $k(x_0) \geq 0$, such that

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty \text{ with } d_i(s) = -\tanh \zeta_i(s).$$

At this level, we don't know that $k(x_0) = 0$ and $k(x_0) = 1$ don't occur.
We have no information on the signs $e_i(x_0)$.
We have no equivalent for $\zeta_i(s)$ as $s \to \infty$.

Step 2: Case $k(x_0) \geq 2$; A differential equation on the solitons' centers

Here, we already know that $k(x_0) \geq 2$.
Linearizing the equation in the $w(y,s)$ setting around the sum of the solitons, we get the following ODE system on the solitons' centers in the ξ variable: for all $i = 1, ..., k$ and s large enough, we have

$$\frac{1}{c_1}\zeta_i' = -e_{i-1}e_i e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} + e_i e_{i+1} e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)} + R_i$$

where

$$|R_i| \leq CJ^{1+\delta_0}, \quad J(s) = \sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}(s) - \zeta_j(s))},$$

$e_0 = e_{k+1} = 0$, for some $c_1 > 0$ and $\delta_0 > 0$.

Step 2: Case $k(x_0) \geq 2$ (cont.)

Since for all $i = 1, ..., k(x_0) - 1$, we have

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty,$$

using ODE techniques, we find that

$$e_i e_{i+1} = -1 \text{ and } \zeta_i(s) \sim \left(i - \frac{k(x_0)+1}{2}\right) \frac{(p-1)}{2} \log s.$$

The upper bound on the blow-up rate gives the corner property.

Step 3: Excluding the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 0$

By contradiction, if $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 0$, then

$$\|w_{x_0}(s)\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to 0 \text{ as } s \to \infty.$$

Fixing s_0 large enough such that $E(w_{x_0}(s_0)) \leq \frac{1}{4}E(\kappa_0)$, we find x_1 near x_0 such that

$$x_1 \in \mathcal{R} \text{ and } E(w_{x_1}(s_0)) \leq \frac{1}{2}E(\kappa_0).$$

Since $E(w_{x_1}(s)) \to E(\kappa_0)$ as $s \to \infty$ and $E(w_{x_1}(s))$ is decreasing, it follows that

$$E(w_{x_1}(s_0)) \geq E(\kappa_0).$$

Contradiction.

Step 4: Characterization of the case where $x_0 \in \mathcal{S}$ and $k(x_0) = 1$

In this case,

$$\|w_{x_0}(s) - e_1\kappa(d_1(s), y)\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty \text{ and } E(w_{x_0}(s)) \geq E(\kappa_0).$$

Our "trapping" result implies that for some $d(x_0) \in (-1, 1)$,

$$w_{x_0}(s) \to \kappa(d(x_0)) \text{ as } s \to \infty.$$

Some elementary geometry and the precise knowledge of the case of non characteristic points gives that x_0 is either left-non-characteristic or right-non-characteristic.

Open questions

- The higher-dimensional case $N \geq 2$: everything in our proof is valid for $N \geq 2$, except the classification of stationary solutions in the w variable (an elliptic problem).
- At least, the radial case for $N \geq 2$.