On the collision of two solitons for the generalized KdV equation in the nonintegrable case

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Introduction

\[
(gKdV) \quad \partial_t u + \partial_x (\partial_x^2 u + f(u)) = 0 \quad t, x \in \mathbb{R}
\]

We call \textit{soliton} a solution \( R(t, x) = Q_c(x - ct), \ c > 0 \)

General questions about the collision of two solitons

Let \( u(t) \) be a solution such that

\[
u(t) \sim Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t) \quad \text{as} \ t \to -\infty,
\]

where \( Q_{c_1}(x - c_1 t), \ Q_{c_2}(x - c_2 t) \) are two solitons \( 0 < c_2 < c_1 \)

▶ What is the behavior of \( u(t) \) during and after the collision?
▶ Do the two solitons survive the collision at the main orders?
▶ If yes, are their speeds (size) and trajectories (shift) modified?
▶ Is the collision elastic or inelastic?
Previous results concerning the collision of solitons

- **Integrable case** ($f(u) = u^2$ or $u^3$)
  There exist explicit multi-solitons describing the interaction of solitons and explicit formulas for the shifts on the solitons.

  **The collision is elastic**
  [Fermi, Pasta and Ulam], [Zabusky and Kruskal] [Lax], [Hirota], [Miura et al.], etc.

- **Numerical results for nonintegrable models and experiments**

  **The collision seems inelastic but almost elastic**
  (small dispersive trail)
  [Eilbeck and McGuire], [Bona et al.] (BBM), [Shih] (gKdV), [Craig et al.] and references therein (Euler, KdV, experiments) etc.

  No rigorous description of collision for nonintegrable models.
We report on a recent work describing collision for nonintegrable gKdV, in the case of two nonlinear objects of different scale:

\[ 0 < c_2 \ll c_1 \]

Available results say that \( Q_{c_1} \) is globally stable and will survive the collision, up to a perturbation of order \( \| Q_{c_2} \|_{L^2} \).

However, in the nonintegrable case, without special algebraic structure, it is not clear whether \( Q_{c_2} \) survives the collision.
Assumption on \( f \): for \( p = 2, 3, 4 \)

\[
f(u) = u^p + f_1(u), \quad \lim_{u \to 0} \left| \frac{f_1(u)}{u^p} \right| = 0 \quad \text{subcriticality at 0}
\]

Then, there exists \( c_* > 0 \) s.t. \( \forall c \in (0, c_*), \exists Q_c \in H^1 \) solution of

\[
Q'' + f(Q_c) = cQ_c \quad \text{stable in } H^1
\]

For \( c \) small:

\[
Q_c(x) \sim Kc^{\frac{1}{p-1}} e^{-\sqrt{c}|x|}, \quad \| Q_c \|_{H^1} \sim \| Q_c \|_{L^2} \sim Kc^{\frac{1}{p-1} - \frac{1}{4}}
\]
Asymptotic results in the energy space

Orbital stability by conservation laws
[Cazenave and Lions, 1982], [Weinstein, 1986]

\[ \| u(0) - Q_c \|_{H^1} = \alpha_0 \text{ small} \Rightarrow \sup_t \| u(t) - Q_c(\cdot - \rho(t)) \|_{H^1} \leq K \alpha_0 \]

Asymptotic stability [Martel and Merle, 2001-2007]

Under the same assumptions, there exists \( c^+ \sim c \) such that

\[ u(t) - Q_{c^+}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x > \frac{c}{10} t) \]

\[ \rho'(t) \to c^+ \quad \text{as } t \to +\infty \]
Existence of asymptotic multi-solitons [Martel, 2005]

There exists a unique solution in $H^1$ such that

$$\lim_{t \to -\infty} \| U(t) - (Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t)) \|_{H^1} \to 0$$

The behavior of $U(t)$ as $t \to +\infty$ is not known, except that $Q_{c_1}$ is stable up to a perturbation of order $\| Q_{c_2} \|_{H^1} \sim KC_2^{p-\frac{1}{4}}$.

Stability of multi-solitons in $H^1$ [Martel, Merle and Tsai, 2002]

For $\alpha_0$ small, $T$ large:

$$\| u(T) - (Q_{c_1}(. - c_1 T) + Q_{c_2}(. - c_2 T)) \|_{H^1} \leq \alpha_0 \quad \Rightarrow$$

$$\sup_{t \geq T} \| u(t) - (Q_{c_1}(. - \rho_1(t)) + Q_{c_2}(. - \rho_2(t))) \|_{H^1} \leq K(\alpha_0 + e^{-\gamma T})$$
PLAN

- Stability of two soliton collision for the gKdV equation
- Detailed description for the quartic gKdV equation $f(u) = u^4$
- Existence of symmetric 2-soliton-like solutions
- Classification of the nonlinearities
- Some elements of proof
THM 1. [Martel and Merle, 2007]

Assume $0 < c_2 < c_0(c_1) \ll c_1$. Let $U(t)$ be s.t.

$$\lim_{t \to -\infty} \|U(t) - (Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t))\|_{H^1(\mathbb{R})} \to 0$$

There exist $c_1^+ \sim c_1$, $c_2^+ \sim c_2$, s.t.

$$c_1^+ \geq c_1, \quad c_2^+ \leq c_2$$

$$w^+(t, x) = U(t, x) - (Q_{c_1^+}(x - \rho_1(t)) + Q_{c_2^+}(x - \rho_2(t)))$$

$$\lim_{t \to +\infty} \|w^+(t)\|_{H^1(x \geq \frac{c_2}{10} t)} = 0, \quad \sup_t \|w^+(t)\|_{H^1} \leq Kc_2^{\frac{1}{p-1}}$$
Comments on THM 1

- The two solitons are preserved through the collision
  \[ \sup_t \|w^+(t)\|_{H^1} \leq Kc_2^{\frac{1}{p-1}} \quad \text{and} \quad \|Q_{c_2}\|_{H^1} \sim Kc_2^{\frac{1}{p-1} - \frac{1}{4}} \]

- Speed change is related to a dispersive residue
  \[ \|w^+(t)\|_{H^1} \not\to 0 \text{ as } t \to +\infty \text{ iff } c_1^+ > c_1 \text{ and } c_2^+ < c_2 \]

- Center of mass: \[ \lim_{+\infty} |\rho_j'(t) - c_j^+| = 0 \]

- Stability in $H^1$ of the behavior of $U(t)$ for all time
Detailed description for the quartic KdV equation

\[
\partial_t u + \partial_x (\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}
\]

Assume \(0 < c \ll 1\). Let \(U(t)\) be s.t. \((Q = Q_1)\)

\[
\lim_{t \to -\infty} \left\| U(t) - Q(\cdot - t) - Q_c(\cdot - ct) \right\|_{H^1(\mathbb{R})} \to 0
\]

**THM 2.** [Martel and Merle, 2007]

\[
c_1^+ - 1 \geq K \frac{c^{17}}{6}, \quad 1 - \frac{c_2^+}{c} \geq K \frac{c^8}{3}
\]

\[
0 < K \frac{c^{17}}{12} \leq \left\| w_x^+(t) \right\|_{L^2} + c^\frac{1}{2} \left\| w^+(t) \right\|_{L^2} \leq K' c^{\frac{11}{12}}, \quad t \text{ large}
\]
Comments on THM 2

- **Nonexistence of a pure 2-soliton solution in this regime**

- THM 2 is the first rigorous result describing an inelastic collision between two nonlinear objects

- The collision is almost elastic

\[ \|w^+(t)\|_{L^2} \leq K \|Q_c\|_{L^2}^7 \]
THM 3. [Martel and Merle, 2007]

Assume $0 < c \ll 1$. There exists a solution $\varphi(t)$ s.t.

$$\varphi(-t, -x) = \varphi(t, x)$$

$$w^-(t, x) = \varphi(t, x) - Q(x - t + \frac{1}{2} \Delta) - Q_c(x - ct + \frac{1}{2} \Delta_c)$$

$$w^+(t, x) = \varphi(t, x) - Q(x - t - \frac{1}{2} \Delta) - Q_c(x - ct - \frac{1}{2} \Delta_c)$$

$$\lim_{-\infty} \| w^-(t) \|_{H^1(x \leq \frac{1}{10} ct)} = 0, \quad \lim_{+\infty} \| w^+(t) \|_{H^1(x \geq \frac{1}{10} ct)} = 0$$

$$K c \frac{17}{12} \leq \| w_x^\pm(t) \|_{L^2} + c^\frac{1}{2} \| w^\pm(t) \|_{L^2} \leq K' c \frac{17}{12}, \quad \pm t \text{ large}$$

$$\Delta \sim_0 \frac{K_1}{c^{1/6}} < 0, \quad \Delta c \sim_0 -K_2 < 0$$
Comments on THM 3

- The solution $\varphi(t)$ is a generalization of multi-soliton in the nonintegrable situation. We can obtain $\varphi(t)$ at any order of $c$.

- Speeds at $t = \pm\infty$ are identical. The shift $\Delta$ on $Q$ becomes negative infinite as $c \to 0$. The shift $\Delta_c$ on $Q_c$ is negative and of size 1

- $\varphi(t)$ is not unique but lower bound on the defect is universal.

- From critical Cauchy theory [Tao, 2006], we conjecture that $w^+$ is pure dispersion.

- THM 3 extends to general nonlinearity $f(u)$, except the lower bound on $w^+(t)$ ($\Delta \sim K_1 \int Q_c$).
Classification of the nonlinearities

We go back to the general framework: for $p = 2, 3, m \geq p + 1$,

$$f(u) = u^p + f_1(u), \quad \lim_{u \to 0} \left| \frac{f_1(u)}{u^p} \right| = 0, \quad f_1^{(m)}(0) \neq 0.$$ 

Considering small solutions, after scaling, we reduce to

$$f(u) = u^p + \varepsilon u^m + f_2(u), \quad \lim_{u \to 0} \left| \frac{f_2(u)}{u^m} \right| = 0.$$ 

**THM 4.** [Muñoz, 2009]

Assume $0 < \varepsilon \ll 1, 0 < c \ll 1$.

$$0 < K \varepsilon c^{\frac{2}{p-1} + \frac{3}{4}} \leq \| w_x^+(t) \|_{L^2} + c^2 \| w^+(t) \|_{L^2}, \quad t \text{ large}$$

Non existence of a pure 2-soliton solution in this regime.
Sketch of the method (quartic case)

Proofs are based on both algebraic computations (during the interaction) and asymptotic analysis. Define $T_c = c^{-\frac{1}{2}} - \frac{1}{100}$.

- **Asymptotic arguments for $|t| > T_c$**
  
  For $|t| > T_c$, we expect the solitons to be decoupled. We apply refinements of previous stability and asymptotic stability arguments ([Martel and Merle, 2001]). Monotonicity of localized $L^2$ quantities, Viriel type identities.

- **Construction of an explicit approximate solution**
  
  Algebraic computations relevant in the collision region $|t| < T_c$.

- **Justification of the algebra on $[-T_c, T_c]$**
  
  Stability arguments using a modified Hamiltonian structure (refinement of [Weinstein, 1986]).
Approximate solution at all order for $|t| < T_c$

Let

$$y_c = x + (1 - c)t, \quad y = x - \alpha(y_c)$$

$$v(t, x) = Q(y) + Q_c(y_c) + W(t, x)$$

$$\alpha'(s) = \sum_{1 \leq k \leq k_0} \sum_{0 \leq \ell \leq \ell_0} a_{k, \ell} c^\ell Q^k_c(s)$$

$$W(t, x) = \sum_{1 \leq k \leq k_0} \sum_{0 \leq \ell \leq \ell_0} c^\ell \left( Q_c^k(y_c)A_{k, \ell}(y) + (Q_c^k)'(y_c)B_{k, \ell}(y) \right)$$

where $(a_{k, \ell}, A_{k, \ell}, B_{k, \ell})$ are to be determined so that

$$\| \partial_t v + \partial_x(\partial_x^2 v - v - v^4) \|_{L^2} \leq Kc^{N(k_0, \ell_0)}$$

$N(k_0, \ell_0) \to +\infty$ as $k_0, \ell_0 \to +\infty$

(Introduction of parameters $(a_{k, \ell})$ is related to the shift of $Q$)
For each \((k, \ell)\), we obtain the system

\[
(\Omega_{k,\ell}) \quad \left\{ \begin{array}{l}
(\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2Q^4)' = F_{k,\ell} \\
(\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - 4Q^3 A_{k,\ell} = G_{k,\ell}
\end{array} \right.
\]

where \(F_{k,\ell}\) and \(G_{k,\ell}\) are given in terms of \((a_{k',\ell'}, A_{k',\ell'}, B_{k',\ell'})\), for \(k' \leq k\), \(\ell' \leq \ell\), with either \(k' < k\) or \(\ell' < \ell\) and where \(\mathcal{L}A = -A'' + A - 4Q^3 A\).

System \((\Omega_{k,\ell})\) can be solved when \(F_{k,\ell}\) and \(G_{k,\ell}\) have certain parity properties (no uniqueness: two free parameters).

We obtain \(A_{k,\ell}, B_{k,\ell}\) which are localized functions plus polynomial.
Recomposition at $t = \pm T_c$ and identification of a defect

We find $A_{1,0}, A_{2,0} \in L^2$ and

$$B_{1,0}(x) = -b_{1,0} \frac{Q'(x)}{Q(x)} + \tilde{B}_{1,0}(x), \quad \tilde{B}_{1,0} \in L^2, \quad b_{1,0} \neq 0$$

$$B_{2,0}(x) = -b_{2,0} \frac{Q'(x)}{Q(x)} + \tilde{B}_{2,0}(x), \quad \tilde{B}_{2,0} \in L^2, \quad b_{2,0} \neq 0$$

For $t = + T_c$, we have $y_c << y$, the two solitons are decoupled and

$$v(T_c, x) \sim Q(y) + Q_c(y_c) - b_{1,0} Q_c'(y_c) - b_{2,0} (Q_c^2)'(y_c) + \ldots$$

$$\sim Q(y) + Q_c(y_c - b_{1,0}) - b_{2,0} (Q_c^2)'(y_c - b_{1,0}) + \ldots$$

But the term $-b_{2,0} (Q_c^2)'$ is a defect of size $\|(Q_c^2)\|_{L^2} = K_c^{11/12}$.

We cannot recompose $v(T_c, x)$ as the sum of two solitons at this order ($Q_c''$ is related to $Q_c^4$ and $Q_c^{(3)}$ is related to $(Q_c^4)'$)
Nonexistence of a pure 2-soliton (quartic case)

- By contradiction, assume that there exists a pure 2-soliton $U(t)$:
  \[ \| U(t) - Q(. - t - x_1, \pm) - Q_c(. - ct - x_2, \pm) \|_{H^1} \to 0 \quad \text{as} \quad t \to \pm \infty \]
  By stability, after time and space translations, $\exists T_+, \delta_+$
  \[ \| U(-T_c, . - T_c) - Q(. + \frac{1}{2} \Delta) - Q_c(. -(1 - c) T_c + \frac{1}{2} \Delta_c) \|_{H^1} \leq Kc \]
  \[ \| U(T_+, . - \delta_+) - Q(. - \frac{1}{2} \Delta) - Q_c(. +(1 - c) T_c - \frac{1}{2} \Delta_c) \|_{H^1} \leq Kc \]
  A priori no relation between $T_+$ and $T_c$.
- From the algebra, there exists a nonsymmetric approx. solution
  \[ \| \tilde{v}(-T_c) - Q(. + \frac{1}{2} \Delta) - Q_c(. -(1 - c) T_c + \frac{1}{2} \Delta_c) \|_{H^1} \leq Kc \]
  \[ \| \tilde{v}(T_c) - Q(. - \frac{1}{2} \Delta) - Q_c(. +(1 - c) T_c - \frac{1}{2} \Delta_c) + 2b_{2,0}(Q_c^2)' \|_{H^1} \leq Kc \]
By stability analysis on \([-T_c, T_c]\) applied on \(\tilde{v}\), \(\exists \delta\) s.t.
\[
\| U(T_c) - \tilde{v}(T_c, \cdot - \delta) \|_{H^1} \leq Kc
\]

- \(T_+ \sim T_c\) by stability of 2 soliton structure
  \[
  \| U(T_c) - Q(\cdot - \rho_1) - Q_c(\cdot - \rho_2) \|_{H^1} \leq Kc^{\frac{11}{12}}, \quad \rho_1 - \rho_2 \sim T_c,
  \]

  \[
  \Rightarrow \quad \forall t > T_c, \quad \| U(t) - Q(\cdot - \rho_1(t)) - Q_c(\cdot - \rho_2(t)) \|_{H^1} \leq Kc^{\frac{5}{12}}
  \]

  with \(\rho_1(t) - \rho_2(t) \sim (1 - c)t\)

- Contradiction follows from

\[
\| U(T_+) - Q(\cdot - \tilde{\rho}_1) - Q_c(\cdot - \tilde{\rho}_2) \|_{H^1} \leq Kc
\]

\[
\| U(T_+) - Q(\cdot - \rho_1) - Q_c(\cdot - \rho_2) + 2b_{2,0}(Q^2_c)'(\cdot - \rho_2) \|_{H^1} \leq Kc
\]
Case of the BBM equation [Martel, Merle and Mizumachi]

For the BBM equation, we study the collision of a soliton of speed $c_0 > 1$ with a small soliton of speed $c > 1$ close to 1.

After renormalization, it is equivalent to study the collision of $Q$ by a small soliton $R_\sigma \sim Q_\sigma$, $\sigma = c - 1 > 0$ small, for the equation

$$(1 - \lambda \partial_x^2) \partial_t u + \partial_x (\partial_x^2 u - u + u^2) = 0, \quad \lambda = \frac{c_0 - 1}{c_0} \in (0, 1)$$

Similar analysis can be done, and shows the existence of a nonzero defect.