On the collision of two solitons for the generalized KdV equation in the nonintegrable case

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Introduction

$$(gKdV) \quad \partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0 \quad t, x \in \mathbb{R}$$

We call **soliton** a solution $R(t,x) = Q_c(x - ct)$, $c > 0$

General questions about the collision of two solitons

Let $u(t)$ be a solution such that

$$u(t) \sim Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t) \quad \text{as } t \to -\infty,$$

where $Q_{c_1}(x - c_1 t)$, $Q_{c_2}(x - c_2 t)$ are two solitons $(0 < c_2 < c_1)$

- What is the behavior of $u(t)$ during and after the collision ?
- Do the two solitons survive the collision at the main orders ?
- If yes, are their speeds (size) and trajectories (shift) modified ?
- Is the collision elastic or inelastic ?

Previous results concerning the collision of solitons

- Integrable case ($f(u) = u^2$ or u^3)

  There exist explicit multi-solitons describing the interaction of solitons and explicit formulas for the shifts on the solitons.

  The collision is elastic

  [Fermi, Pasta and Ulam], [Zabusky and Kruskal] [Lax], [Hirota], [Miura et al.], etc.

- Numerical results for nonintegrable models and experiments

  The collision seems inelastic but almost elastic

  (small dispersive trail)

  [Eilbeck and McGuire], [Bona et al.] (BBM), [Shih] (gKdV), [Craig et al.] and references therein (Euler, KdV, experiments) etc.

  No rigorous description of collision for nonintegrable models.

We report on a recent work describing collision for nonintegrable gKdV, in the case of two nonlinear objects of different scale:

$$0 < c_2 \ll c_1$$

Available results say that Q_{c_1} is globally stable and will survive the collision, up to a perturbation of order $\|Q_{c_2}\|_{L^2}$.

However, in the nonintegrable case, without special algebraic structure, it is not clear whether Q_{c_2} survives the collision.

General setting

Assumption on f: for $p = 2, 3, 4$

$$f(u) = u^p + f_1(u), \quad \lim_{u \to 0} \left| \frac{f_1(u)}{u^p} \right| = 0 \quad \textbf{subcriticality at } 0$$

Then, there exists $c_* > 0$ s.t. $\forall c \in (0, c_*)$, $\exists Q_c \in H^1$ solution of

$$Q_c'' + f(Q_c) = cQ_c \quad \textbf{stable in } H^1$$

For c small:

$$Q_c(x) \sim K c^{\frac{1}{p-1}} e^{-\sqrt{c}|x|}, \quad \|Q_c\|_{H^1} \sim \|Q_c\|_{L^2} \sim K c^{\frac{1}{p-1} - \frac{1}{4}}$$

Asymptotic results in the energy space

Orbital stability by conservation laws
[Cazenave and Lions, 1982], [Weinstein, 1986]

$$\|u(0) - Q_c\|_{H^1} = \alpha_0 \text{ small} \Rightarrow \sup_t \|u(t) - Q_c(. - \rho(t))\|_{H^1} \leq K\alpha_0$$

Asymptotic stability [Martel and Merle, 2001-2007]

Under the same assumptions, there exists $c^+ \sim c$ such that

$$u(t) - Q_{c^+}(. - \rho(t)) \to 0 \quad \text{in } H^1(x > \tfrac{c}{10}t)$$

$$\rho'(t) \to c^+ \quad \text{as } t \to +\infty$$

Existence of asymptotic multi-solitons [Martel, 2005]

There exists a unique solution in H^1 such that

$$\lim_{t\to-\infty} \left\| U(t) - (Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t)) \right\|_{H^1} \to 0$$

The behavior of $U(t)$ as $t \to +\infty$ is not known, except that Q_{c_1} is stable up to a perturbation of order $\|Q_{c_2}\|_{H^1} \sim K c_2^{\frac{1}{p-1}-\frac{1}{4}}$.

Stability of multi-solitons in H^1 [Martel, Merle and Tsai, 2002]

For α_0 small, T large:

$$\left\| u(T) - (Q_{c_1}(. - c_1 T) + Q_{c_2}(. - c_2 T)) \right\|_{H^1} \leq \alpha_0 \quad \Rightarrow$$

$$\sup_{t\geq T} \left\| u(t) - (Q_{c_1}(. - \rho_1(t)) + Q_{c_2}(. - \rho_2(t))) \right\|_{H^1} \leq K(\alpha_0 + e^{-\gamma T})$$

PLAN

- Stability of two soliton collision for the gKdV equation
- Detailed description for the quartic gKdV equation $f(u) = u^4$
- Existence of symmetric 2-soliton-like solutions
- Classification of the nonlinearities
- Some elements of proof

Stability of two soliton collision (general nonlinearity)

THM 1. [Martel and Merle, 2007]

Assume $0 < c_2 < c_0(c_1) \ll c_1$. *Let* $U(t)$ *be s.t.*

$$\lim_{t\to-\infty} \left\| U(t) - (Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t)) \right\|_{H^1(\mathbb{R})} \to 0$$

There exist $c_1^+ \underset{c_2\sim 0}{\sim} c_1$, $c_2^+ \underset{c_2\sim 0}{\sim} c_2$, *s.t.*

$$c_1^+ \geq c_1, \quad c_2^+ \leq c_2$$

$$w^+(t,x) = U(t,x) - (Q_{c_1^+}(x - \rho_1(t)) + Q_{c_2^+}(x - \rho_2(t)))$$

$$\lim_{+\infty} \|w^+(t)\|_{H^1(x \geq \frac{c_2}{10} t)} = 0, \quad \sup_t \|w^+(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}}$$

Comments on THM 1

- **The two solitons are preserved through the collision**

$$\sup_t \|w^+(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}} \quad \text{and} \quad \|Q_{c_2}\|_{H^1} \sim K c_2^{\frac{1}{p-1}-\frac{1}{4}}$$

- **Speed change is related to a dispersive residue**

$$\|w^+(t)\|_{H^1} \not\to 0 \text{ as } t \to +\infty \textbf{ iff } c_1^+ > c_1 \text{ and } c_2^+ < c_2$$

- Center of mass : $\lim_{+\infty} |\rho_j'(t) - c_j^+| = 0$
- Stability in H^1 of the behavior of $U(t)$ for all time

Detailed description for the quartic KdV equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}$$

Assume $0 < c \ll 1$. Let $U(t)$ be s.t. $(Q = Q_1)$

$$\lim_{t \to -\infty} \left\| U(t) - Q(. - t) - Q_c(. - ct) \right\|_{H^1(\mathbb{R})} \to 0$$

THM 2. [Martel and Merle, 2007]

$$c_1^+ - 1 \geq K\, c^{\frac{17}{6}}, \quad 1 - \frac{c_2^+}{c} \geq K\, c^{\frac{8}{3}}$$

$$0 < \mathbf{K\, c^{\frac{17}{12}}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2} \leq K' c^{\frac{11}{12}}, \quad t \textit{ large}$$

Comments on THM 2

- **Nonexistence of a pure 2-soliton solution in this regime**
- THM 2 is the first rigorous result describing an inelastic collision between two nonlinear objects
- The collision is almost elastic

$$\|w^+(t)\|_{L^2} \leq K\|Q_c\|_{L^2}^7$$

Generalized 2-soliton and explicit shifts (quartic)

THM 3. [Martel and Merle, 2007]

Assume $0 < c \ll 1$. *There exists a solution* $\varphi(t)$ *s.t.*

$$\varphi(-t,-x) = \varphi(t,x)$$

$$w^-(t,x) = \varphi(t,x) - Q(x - t + \tfrac{1}{2}\Delta) - Q_c(x - ct + \tfrac{1}{2}\Delta_c)$$

$$w^+(t,x) = \varphi(t,x) - Q(x - t - \tfrac{1}{2}\Delta) - Q_c(x - ct - \tfrac{1}{2}\Delta_c)$$

$$\lim_{-\infty} \|w^-(t)\|_{H^1(x\leq\frac{1}{10}ct)} = 0, \quad \lim_{+\infty} \|w^+(t)\|_{H^1(x\geq\frac{1}{10}ct)} = 0$$

$$K\, c^{\frac{17}{12}} \leq \|w_x^\pm(t)\|_{L^2} + c^{\frac{1}{2}}\|w^\pm(t)\|_{L^2} \leq K' c^{\frac{17}{12}}, \quad \pm t \text{ large}$$

$$\Delta \underset{c\sim 0}{\sim} -\frac{K_1}{c^{1/6}} < 0, \quad \Delta_c \underset{c\sim 0}{\sim} -K_2 < 0$$

Comments on THM 3

- The solution $\varphi(t)$ is a generalization of multi-soliton in the nonintegrable situation.
  We can obtain $\varphi(t)$ **at any order of** c.
- Speeds at $t = \pm\infty$ are identical.
  The shift Δ on Q becomes negative infinite as $c \to 0$.
  The shift Δ_c on Q_c is negative and of size 1
- $\varphi(t)$ is not unique but lower bound on the defect is universal.
- From critical Cauchy theory [Tao, 2006], we conjecture that w^+ is pure dispersion.
- THM 3 extends to general nonlinearity $f(u)$, except the lower bound on $w^+(t)$ ($\Delta \sim K_1 \int Q_c$).

Classification of the nonlinearities

We go back to the general framework: for $p = 2, 3$, $m \geq p + 1$,

$$f(u) = u^p + f_1(u), \quad \lim_{u \to 0} \left| \frac{f_1(u)}{u^p} \right| = 0, \quad f_1^{(m)}(0) \neq 0.$$

Considering small solutions, after scaling, we reduce to

$$f(u) = u^p + \varepsilon u^m + f_2(u), \quad \lim_{u \to 0} \left| \frac{f_2(u)}{u^m} \right| = 0.$$

THM 4. [Muñoz, 2009]

Assume $0 < \varepsilon \ll 1$, $0 < c \ll 1$.

$$0 < \mathbf{K} \varepsilon\, \mathbf{c}^{\frac{2}{p-1} + \frac{3}{4}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2}, \quad t \textit{ large}$$

Non existence of a pure 2-soliton solution in this regime.

Sketch of the method (quartic case)

Proofs are based on both algebraic computations (during the interaction) and asymptotic analysis. Define $T_c = c^{-\frac{1}{2}-\frac{1}{100}}$.

- Asymptotic arguments for $|t| > T_c$

  For $|t| > T_c$, we expect the solitons to be decoupled
  We apply refinements of previous stability and asymptotic stability arguments ([Martel and Merle, 2001]).
  Monotonicity of localized L^2 quantities, Viriel type identities.

- **Construction of an explicit approximate solution**

  Algebraic computations relevant in the collision region $|t| < T_c$

- Justification of the algebra on $[-T_c, T_c]$

  Stability arguments using a modified Hamiltonian structure (refinement of [Weinstein, 1986])

Approximate solution at all order for $|t| < T_c$

Let

$$y_c = x + (1-c)t, \quad y = x - \alpha(y_c)$$

$$v(t,x) = Q(y) + Q_c(y_c) + W(t,x)$$

$$\alpha'(s) = \sum_{\substack{1\leq k\leq k_0\\ 0\leq \ell\leq \ell_0}} a_{k,\ell}\, c^\ell Q_c^k(s)$$

$$W(t,x) = \sum_{\substack{1\leq k\leq k_0\\ 0\leq \ell\leq \ell_0}} c^\ell \left(Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y) \right)$$

where $(a_{k,\ell}, A_{k,\ell},\ B_{k,\ell})$ are to be determined so that

$$\|\partial_t v + \partial_x(\partial_x^2 v - v - v^4)\|_{L^2} \leq K c^{N(k_0,\ell_0)}$$

$N(k_0,\ell_0) \to +\infty$ as $k_0, \ell_0 \to +\infty$
(Introduction of parameters $(a_{k,\ell})$ is related to the shift of Q)

Model system

For each (k, ℓ), we obtain the system

$$(\Omega_{k,\ell}) \quad \begin{cases} (\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2Q^4)' = F_{k,\ell} \\ (\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - 4Q^3 A_{k,\ell} = G_{k,\ell} \end{cases}$$

where $F_{k,\ell}$ and $G_{k,\ell}$ are given in terms of $(a_{k',\ell'}, A_{k',\ell'}, B_{k',\ell'})$, for $k' \leq k$, $\ell' \leq \ell$, with either $k' < k$ or $\ell' < \ell$ and where $\mathcal{L}A = -A'' + A - 4Q^3 A$.

System $(\Omega_{k,\ell})$ can be solved when $F_{k,\ell}$ and $G_{k,\ell}$ have certain parity properties (no uniqueness: two free parameters).

We obtain $A_{k,\ell}$, $B_{k,\ell}$ which are localized functions plus polynomial.

Recomposition at $t = \pm T_c$ and identification of a defect

We find $A_{1,0}$, $A_{2,0} \in L^2$ and

$$B_{1,0}(x) = -b_{1,0}\frac{Q'(x)}{Q(x)} + \tilde{B}_{1,0}(x), \quad \tilde{B}_{1,0} \in L^2, \quad b_{1,0} \neq 0$$

$$B_{2,0}(x) = -b_{2,0}\frac{Q'(x)}{Q(x)} + \tilde{B}_{2,0}(x), \quad \tilde{B}_{2,0} \in L^2, \quad b_{2,0} \neq 0$$

For $t = +T_c$, we have $y_c << y$, the two solitons are decoupled and

$$\begin{aligned} v(T_c, x) &\sim Q(y) + Q_c(y_c) - b_{1,0}Q_c'(y_c) - b_{2,0}(Q_c^2)'(y_c) + \dots \\ &\sim Q(y) + Q_c(y_c - b_{1,0}) - b_{2,0}(Q_c^2)'(y_c - b_{1,0}) + \dots \end{aligned}$$

But the term $-b_{2,0}(Q_c^2)'$ is a defect of size $\|(Q_c^2)'\|_{L^2} = Kc^{\frac{11}{12}}$.
We cannot recompose $v(T_c, x)$ as the sum of two solitons at this order (Q_c'' is related to Q_c^4 and $Q_c^{(3)}$ is related to $(Q_c^4)'$)

Nonexistence of a pure 2-soliton (quartic case)

- By contradiction, assume that there exists a pure 2-soliton $U(t)$:

$$\left\| U(t) - Q(.-t-x_{1,\pm}) - Q_c(.-ct-x_{2,\pm}) \right\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty$$

By stability, after time and space translations, $\exists T_+,\ \delta_+$

$$\| U(-T_c, .-T_c) - Q(.+\tfrac{1}{2}\Delta) - Q_c(.-(1-c)T_c+\tfrac{1}{2}\Delta_c) \|_{H^1} \leq Kc$$

$$\| U(T_+, .-\delta_+) - Q(.-\tfrac{1}{2}\Delta) - Q_c(.+(1-c)T_c-\tfrac{1}{2}\Delta_c) \|_{H^1} \leq Kc$$

A priori no relation between T_+ and T_c.

- From the algebra, there exists a nonsymmetric approx. solution

$$\left\| \tilde{v}(-T_c) - Q(.+\tfrac{1}{2}\Delta) - Q_c(.-(1-c)T_c+\tfrac{1}{2}\Delta_c) \right\|_{H^1} \leq Kc$$

$$\left\| \tilde{v}(T_c) - Q(.-\tfrac{1}{2}\Delta) - Q_c(.+(1-c)T_c-\tfrac{1}{2}\Delta_c) + \mathbf{2b_{2,0}(Q_c^2)'} \right\|_{H^1} \leq Kc$$

By stability analysis on $[-T_c, T_c]$ applied on $\tilde{v}$, $\exists\delta$ s.t.

$$\left\|U(T_c) - \tilde{v}(T_c, . - \delta)\right\|_{H^1} \leq Kc$$

- $T_+ \sim T_c$ by stability of 2 soliton structure

$$\left\|U(T_c) - Q(. - \rho_1) - Q_c(. - \rho_2)\right\|_{H^1} \leq Kc^{\frac{11}{12}}, \quad \rho_1 - \rho_2 \sim T_c,$$

$$\Rightarrow \quad \forall t > T_c, \left\|U(t) - Q(. - \rho_1(t)) - Q_c(. - \rho_2(t))\right\|_{H^1} \leq Kc^{\frac{5}{12}}$$

with $\rho_1(t) - \rho_2(t) \sim (1 - c)t$

- Contradiction follows from

$$\|U(T_+) - Q(. - \tilde{\rho}_1) - Q_c(. - \tilde{\rho}_2)\|_{H^1} \leq Kc$$

$$\left\|U(T_+) - Q(. - \rho_1) - Q_c(. - \rho_2) + \mathbf{2b_{2,0}(Q_c^2)'(. - \rho_2)}\right\|_{H^1} \leq Kc$$

Case of the BBM equation [Martel, Merle and Mizumachi]

For the BBM equation, we study the collision of a soliton of speed $c_0 > 1$ with a small soliton of speed $c > 1$ close to 1.

After renormalization, it is equivalent to study the collision of Q by a small soliton $R_\sigma \sim Q_\sigma$, $\sigma = c - 1 > 0$ small, for the equation

$$(1 - \lambda \partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^2) = 0, \quad \lambda = \frac{c_0 - 1}{c_0} \in (0, 1)$$

Similar analysis can be done, and shows the existence of a nonzero defect.