Existence and characterization of characteristic points for a semilinear wave equation in one space dimension *

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Abstract: We consider the semilinear wave equation with power nonlinearity in one space dimension. We first show the existence of a blow-up solution with a characteristic point. Then, we consider an arbitrary blow-up solution $u(x,t)$, the graph $x \to T(x)$ of its blow-up points and $\mathcal{S}$ the set of all characteristic points, and show that the $\mathcal{S}$ has an empty interior. Finally, given $x_0 \in \mathcal{S}$, we show that $T(x)$ is hat shaped near x_0, and that in selfsimilar variables, the solution decomposes into a decoupled sum of (at least 2) solitons (with alternate signs).

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1 Introduction

1.1 Known results and the case of non characteristic points

We consider the one dimensional semilinear wave equation

$$\left\{ \begin{array}{l} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{array} \right. \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup\limits_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
We will also consider the following equation for $p > 1$,

$$\left\{ \begin{array}{l} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p}, \\ u(0) = u_0 \text{ and } u_t(0) = u_1. \end{array} \right. \tag{2}$$

The Cauchy problem for equations (1) and (2) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$ (see Ginibre, Soffer and

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Velo [6]). The existence of blow-up solutions for equation (1) follows from Levine [9]. More blow-up results can be found in Caffarelli and Friedman [5], [4], Alinhac [1], [2] and Kichenassamy and Litman [7], [8].

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(x, T(x))\}$ such that u cannot be extended beyond the set called the maximal influence domain of u:

$$D = \{(x,t) \mid t < T(x)\}. \tag{3}$$

$\bar{T} = \inf_{x\in\mathbb{R}} T(x)$ and Γ are called the blow-up time and the blow-up curve of u. x_0 is a non characteristic point if

$$\text{there are } \delta_0 \in (0,1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{4}$$

where $\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x-\bar{x}|\}$. We denote by $\mathcal{R}$ (resp. $\mathcal{S}$) the set of non characteristic (resp. characteristic) points.

Following our earlier work ([10]-[12]), we aim at describing the blow-up behavior for *any* blow-up solution, especially Γ and the solution near Γ.

Given some (x_0, T_0) such that $0 < T_0 \leq T(x_0)$, a natural tool is to introduce the following self-similar change of variables:

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{5}$$

If $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. The function $w = w_{x_0,T_0}$ satisfies the following equation for all $y \in B = B(0,1)$ and $s \geq -\log T_0$:

$$\partial^2_{ss} w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1}w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s} w \tag{6}$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho}\partial_y \left(\rho(1-y^2)\partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{7}$$

The Lyapunov functional for equation (6)

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2}\left(\partial_s w\right)^2 + \frac{1}{2}\left(\partial_y w\right)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1}\right) \rho dy \tag{8}$$

is defined for $(w, \partial_s w) \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ q \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2 (1-y^2) + q_2^2\right) \rho dy < +\infty \right\}. \tag{9}$$

We will note

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1} \left(r'^2(1-y^2) + r^2\right) \rho dy < +\infty\}. \tag{10}$$

We also introduce for all $|d| < 1$ the following stationary solutions of (6) defined by

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \tag{11}$$

In [12] and [13], we established the following results:

(Blow-up behavior for $x_0 \in \mathcal{R}$, *see Corollary 4 in [12], Theorem 1 (and the following remark) and Lemma 2.2 in [13]).*

(i) The set of non characteristic points $\mathcal{R}$ is non empty and open.

(ii) **(Selfsimilar blow-up profile for** $x_0 \in \mathcal{R}$**)** *There exist positive μ_0 and C_0 such that if $x_0 \in \mathcal{R}$, then there exist $\delta_0(x_0) > 0$, $d(x_0) \in (-1,1)$, $|\theta(x_0)| = 1$, $s_0(x_0) \geq -\log T(x_0)$ such that for all $s \geq s_0$:*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0),.) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)}, \tag{12}$$

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0),.) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y|<1+\delta_0)} \to 0 \text{ as } s \to \infty. \tag{13}$$

Moreover, $E(w_{x_0}(s)) \to E(\kappa_0)$ as $s \to \infty$.

(iii) The function $T(x)$ is C^1 on $\mathcal{R}$ and for all $x_0 \in \mathcal{R}$, $T'(x_0) = d(x_0) \in (-1,1)$. Moreover, $\theta(x_0)$ is constant on connected components of $\mathcal{R}$.

1.2 Existence of characteristic points

For characteristic points, the only available result about existence or non existence is due to Caffarelli and Friedman [5] and [4] who proved (using the maximum principle) the non existence of characteristic points for equation (1):
- under conditions on initial data that ensure that for all $x \in \mathbb{R}$ and $t \geq 0$, $u \geq 0$ and $\partial_t u \geq (1+\delta_0)|\partial_x u|$ for some $\delta_0 > 0$,
- for $p \geq 3$ with $u_0 \geq 0$, $u_1 \geq 0$ and $(u_0, u_1) \in C^4 \times C^3(\mathbb{R})$.

¿From this example, it was generally conjectured by most people that there were no blow-up solutions for equation (1) with characteristic points: *for all (u_0, u_1) which lead to blow-up, $\mathcal{R} = \mathbb{R}$.*
Our first result is to disprove this fact. Existence of characteristic points is seen as a consequence of two facts:
- on the one hand, the study of the blow-up profile at a non characteristic point,
- on the other hand, connectedness arguments related to the sign of the blow-up profile.

To state our results, let us consider $u(x,t)$ a blow-up solution of equation (1) (take for example initial data satisfying $\int_{\mathbb{R}} \left(\frac{1}{2}|\partial_x u_0|^2 + \frac{1}{2}u_1^2 - \frac{1}{p+1}|u_0|^{p+1}\right) dx < 0$, which gives blow-up by Levine [9]). The first result follows from the study near a regular point, that ensures the existence of an explicit signed profile.

Proposition 1 *If the initial data (u_0, u_1) is odd and $u(x,t)$ blows up in finite time, then $0 \in \mathcal{S}$.*

The second one follows from the continuity of the profile on the connected components of $\mathcal{R}$ (see Theorem 1 in [13]).

Theorem 2 (Existence and generic stability of characteristic points)
(i) **(Existence)** *Let $a_1 < a_2$ two non characteristic points such that*

$$w_{a_i}(s) \to \theta(a_i)\kappa(d_{a_i}, \cdot) \text{ as } s \to \infty \text{ with } \theta(a_1)\theta(a_2) = -1$$

for some d_{a_i} in $(-1,1)$, in the sense (12). Then, there exists a characteristic point $c \in (a_1, a_2)$.
(ii) **(Stability)** *There exists $\epsilon_0 > 0$ such that if $\|(\tilde{u}_0, \tilde{u}_1) - (u_0, u_1)\|_{\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}} \le \epsilon_0$, then, $\tilde{u}(x,t)$ the solution of equation (1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ blows up and has a characteristic point $\tilde{c} \in [a_1, a_2]$.*

Remark: It is enough to take (u_0, u_1) with large plateaus of opposite signs to guarantee that $u(x,t)$ blows up satisfying the hypotheses of this theorem.

Since a solution in one space dimension is also a solution in higher dimensions, we get from the finite speed of propagation the following existence result in N dimensions:

Corollary 3 (Existence of characteristic points in higher dimensions) *Consider $\tilde{u}(x_1, t)$ a blow-up solution of (1) in one space dimension with a characteristic point. Then, for R large enough, initial data (u_0, u_1) such that $u_i(x) = \tilde{u}_i(x_1)$ for $|x| < R$, the solution $u(x,t)$ of equation (1) with initial data (u_0, u_1) blows up and has a characteristic point.*

1.3 Non existence results for characteristic points

In this section, we give sufficient conditions under which no characteristic point can occur. Our analysis in fact relates the fact that x_0 is a characteristic point to sign changes of the solution in a neighborhood of $(x_0, T(x_0))$. We claim the following:

Theorem 4 *Consider $u(x,t)$ a blow-up solution of (1) such that $u(x,t) \ge 0$ for all $x \in (a_0, b_0)$ and $t_0 \le t < T(x)$ for some real a_0, b_0 and $t_0 \ge 0$. Then, $(a_0, b_0) \subset \mathcal{R}$.*

Remark: This result can be seen as a generalization of the result of Caffarelli and Friedman, with no restriction on initial data. Indeed, from our result, taking nonnegative initial data suffices to exclude the occurrence of characteristic points.

Considering the equation (2), we get the following twin result of Theorem 4:

Theorem 4' *The set of characteristic points is empty for any blow-up solution of equation (2).*

1.4 Shape of the blow-up set near characteristic points and properties of $\mathcal{S}$

We have the following theorem, which is the main result of our analysis:

Theorem 5 (The interior of $\mathcal{S}$ is empty) *Consider $u(x,t)$ a blow-up solution of (1). The set of characteristic points $\mathcal{S}$ has an empty interior.*

Now, for $x_0 \in \mathcal{S}$, we are able to give the precise behavior of the solution near $(x_0, T(x_0))$:

Proposition 6 (Description of the behavior of w_{x_0} where x_0 is characteristic) *Consider $u(x,t)$ a blow-up solution of (1) and $x_0 \in \mathcal{S}$. Then, it holds that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(x_0)} e_i^* \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to k(x_0) E(\kappa_0) \tag{14}$$

as $s \to \infty$, for some

$$k(x_0) \geq 2, \quad e_i^* = e_1^*(-1)^{i+1}$$

and continuous $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k$. Moreover, we have $\zeta_1(s) \leq -C_0 \log s$ and $\zeta_{k(x_0)}(s) \geq C_0 \log s$ for s large enough for some $C_0 > 0$, and $\zeta_{i+1}(s) - \zeta_i(s) \to \infty$ as $s \to \infty$.

Remark: In [12], we proved a much weaker version of this result, with (14) valid just with $k(x_0) \geq 0$ and no information on the signs of e_i^*, $\zeta_1(s)$ and $\zeta_k(s)$. Note that eliminating the case $k(x_0) = 0$ is the most difficult part in our analysis. In some sense, we put in relation the notion of charactersitic point at x_0 and the notion of decomposition of w_{x_0} in a decoupling sum of (at least 2) $\pm\kappa(d_i(s))$. This result can be seen as a result of decomposition up to dispersion into sum of decoupling solitons in dispersive problems. Accroding to the value of $k(x_0)$, this sum appears to have a *multipole* nature (dipole if $k(x_0) = 2$, tripole if $k(x_0) = 3$,....).
Remark: In subsection 4.4, we derive from Proposition 6 some estimates on various norms of the solution at blow-up.

We also have the following consequences in the original variables:

Proposition 7 (Description of $T(x)$ for x near x_0)
(i) If $x_0 \in \mathcal{S}$, then we have for some $\delta_0 > 0$, $C_0 > 0$ and $\gamma > 0$,

$$\text{if } 0 < |x - x_0| \leq \delta_0, \text{ then } T(x_0) - |x - x_0| < T(x) \leq T(x_0) - |x - x_0| + \frac{C_0 |x - x_0|}{|\log(x - x_0)|^\gamma}. \tag{15}$$

(ii) If $x_0 \in \mathcal{S}$, then $T(x)$ is right and left differentiable at x_0, with

$$T_l'(x_0) = 1 \text{ and } T_r'(x_0) = -1.$$

(iii) For all $t \in [T(x_0) - \tau_0, T(x_0))$ for some $\tau_0 > 0$, there exist $z_1(t) < ... < z_k(t)$ continuous in t such that

$$e_1^*(-1)^{i+1} u(z_i(t), t) > 0$$

and $z_i(t) \to x_0$ as $t \to T(x_0)$.

Remark: From (iii), we have the existence of zero lines $x_1(t) < ... < x_{k-1}(t)$ (not necessarily continuous in t) such that $u(x_i(t), t) = 0$ and $x_i(t) \to x_0$ as $t \to T(x_0)$.

The paper is organized as follows. Section 2 is devoted to the proofs of Proposition 1 and Theorem 2 (note that Corollary 3 follows straightforwardly from Theorem 2 and the finite speed of propagation). In Section 3, we consider a characteristic point and study the equation in selfsimilar variables. As for Section 4, it is devoted to the proof of Theorems 5, 4 and 4', as well as Propositions 6 and 7.

2 Existence and stability of characteristic points

Here in this section, we consider $u(x,t)$ a blow-up solution of equation (1). As mentioned in the introduction, we prove in this section the existence of characteristic points (Proposition 1 and Theorem 2).

Proof of Proposition 1: Assuming that (u_0, u_1) is odd, we would like to prove that $0 \in \mathcal{S}$. Arguing by contradiction, we assume that $0 \in \mathcal{R}$.
On the one hand, using the result of [12] stated in (13), we see that for some $d(0) \in (-1,1)$,

$$\|w_0(s) - \kappa(d(0),\cdot)\|_{L^\infty(-1,1)} \le C\|w_0(s) - \kappa(d(0),\cdot)\|_{H^1(-1,1)} \to 0 \text{ as } s \to \infty.$$

In particular,

$$|w_0(0,s)| \to \kappa(d(0),0) > 0 \text{ as } s \to \infty. \tag{16}$$

On the other hand, since the initial data is odd, the same holds for the solution, in particular, $u(0,t) = 0$ for all $t \in [0, T(0))$, hence $w_0(0,s) = 0$ for all $s \ge -\log T(0)$, which contradicts (16). This concludes the proof of Proposition 1. ∎
Remark: We don't need to know that for $x_0 \in \mathcal{R}$, w_{x_0} converges to a particular profile to derive this result. It is enough to know that w_{x_0} approaches the set $\{\theta(x_0)\kappa(d,\cdot) \mid |d| < 1-\eta\}$ for some $\eta > 0$, which is a much weaker result.

We now turn to the proof of Theorem 2. It is a consequence of three results from our earlier work:

- the continuity with respect to initial data of the blow-up time at $x_0 \in \mathcal{R}$.

Proposition 2.1 (Continuity with respect to initial data at $x_0 \in \mathcal{R}$) *There exists $A_0 > 0$ such that $\tilde{T}(x_0) \to T(x_0)$ as $(\tilde{u}_0, \tilde{u}_1) \to (u_0, u_1)$ in $H^1 \times L^2(|x| < A_0)$, where $\tilde{T}(x_0)$ is the blow-up time of $\tilde{u}(x,t)$ at $x = x_0$, the solution of equation (1) with initial data $(\tilde{u}_0, \tilde{u}_1)$.*

Proof: This is a direct consequence of the Liouville Theorem and its applications given in [13]. See Appendix A for a sketch of the proof. ∎

- the continuity of the blow-up profile on $\mathcal{R}$ proved in Theorem 1 in [13] (in particular, the fact that $\theta(x_0)$ given in (12) is constant on the connected components of $\mathcal{R}$).

- the following trapping result from [12]:

Proposition 2.2 (See Theorem 3 in [12] and its proof) *There exists $\epsilon_0 > 0$ such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (6) such that*

$$\forall s \ge s^*, \;\; E(w(s)) \ge E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \le \epsilon^* \tag{17}$$

for some $d^* = -\tanh \xi^*$, $\omega^* = \pm 1$ and $\epsilon^* \in (0, \epsilon_0]$, then there exists $d_\infty = -\tanh \xi_\infty$ such that

$$|\xi_\infty - \xi^*| \le C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0. \tag{18}$$

Let us use these results to prove Theorem 2.

Proof of Theorem 2: We consider $a_1 < a_2$ two non characteristic points such that $w_{a_i}(s) \to \theta(a_i)\kappa(d_{a_i},\cdot)$ with $\theta(a_1)\theta(a_2) = -1$ for some d_{a_i} in $(-1,1)$, in the sense (12). Up to changing u in $-u$, we can assume that $\theta(a_1) = 1$ and $\theta(a_2) = -1$. We aim at proving that $(a_1, a_2) \cap \mathcal{S} \neq \emptyset$ and the stability of such a property with respect to initial data.

(i) If we assume by contradiction that $[a_1, a_2] \subset \mathcal{R}$, then the continuity of $\theta(x_0)$ where $x_0 \in [a_1, a_2]$ implies that $\theta(x_0)$ is constant on $[a_1, a_2]$. This is a contradiction, since $\theta(a_1) = 1$ and $\theta(a_2) = -1$.

(ii) By hypothesis and estimate (13), there is $\delta_0 > 0$ and $s_0 \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} w_{a_i}(s_0) \\ \partial_s w_{a_i}(s_0) \end{pmatrix} - \theta(a_i) \begin{pmatrix} \kappa(d_{a_i},\cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y|<1+\delta_0)} \leq \frac{\epsilon_0}{2}$$

where ϵ_0 is defined in Proposition 2.2. From the continuity with respect to initial data for equation (1) at the fixed time $T(a_i) - e^{-s_0}$, we see there exists $\eta(\epsilon_0) > 0$ such that if

$$\|(\tilde{u}_0, \tilde{u}_1) - (u_0, u_1)\|_{\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}} \leq \eta,$$

then $\tilde{u}(x,t)$ the solution of equation (1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ is such that $\tilde{w}_{a_i}(y, s_0)$ is defined for all $|y| < 1 + \delta_0/2$ and

$$\left\| \begin{pmatrix} \tilde{w}_{a_i}(s_0) \\ \partial_s \tilde{w}_{a_i}(s_0) \end{pmatrix} - \theta(a_i) \begin{pmatrix} \kappa(d_{a_i},\cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y|<1+\delta_0/2)} \leq \frac{3}{4}\epsilon_0,$$

where $\tilde{w}_{a_i}$ is the selfsimilar version defined from $\tilde{u}(x,t)$ by (5).
¿From Proposition 2.1, we have $\tilde{T}(a_i) \to T(a_i)$ as $\eta \to 0$, where $\tilde{T}(a_i)$ is the blow-up time of $\tilde{u}(t)$ at a_i. We then have for η small enough,

$$\left\| \begin{pmatrix} \tilde{w}_{a_i}(s_0) \\ \partial_s \tilde{w}_{a_i}(s_0) \end{pmatrix} - \theta(a_i) \begin{pmatrix} \kappa(d_{a_i},\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon_0. \tag{19}$$

Two cases then arise (by the way, we will prove later in Proposition 6 that the Lyapunov functional stays above $2E(\kappa_0)$ at a characteristic point, which means by (19) that a_i and a_2 are non characteristic points for η small enough, but we cannot use Proposition 6 for the moment):
- If a_1 or a_2 is a characteristic point of $\tilde{u}(t)$, then the proof is finished.
- Otherwise, (12) holds for $\tilde{w}_{a_i}$ from the fact that the point is non characteristic. Thus, from the monotonicity of $E(\tilde{w}_{a_i}(s))$, (17) holds with $\omega^* = \theta(a_i)$. Applying Proposition 2.2, we see that $\tilde{w}_{a_i}(s) \to \theta(a_i)\kappa(\tilde{d}_{a_i},\cdot)$ as $s \to \infty$, for some $\tilde{d}_{a_i} \in (-1,1)$. Noting that $\theta(a_1) = 1$ and $\theta(a_2) = -1$, we apply (i) to get the result. This concludes the proof of Theorem 2. ■

3 Refined behavior for w_{x_0} where x_0 is characteristic

In this section, we consider $x_0 \in \mathcal{S}$. We know from [13] that

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} e_i \begin{pmatrix} \kappa(d_i(s),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty \tag{20}$$

for some $k(x_0) \geq 0$, $e_i = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k(x_0)$ with

$$\zeta_1(s) < ... < \zeta_{k(x_0)}(s) \text{ and } \zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ for all } i = 1, ..., k-1. \quad (21)$$

Since $w_{x_0}(s)$ is convergent when $k(x_0) \leq 1$ (to 0 when $k(x_0) = 0$ and to some $\kappa(d_\infty)$ by Proposition 2.2), we focus throughout this section on the case

$$k(x_0) \geq 2.$$

For simplicity in the notations, we forget the dependence of w_{x_0} and $k(x_0)$ on x_0.
This section is organized as follows. In Subsection 3.1, assuming an ODE on the solitons' center, we find their behavior. Then, in Subsection 3.2, we study equation (6) around the solitons' sum and derive in Subsection 3.3 the ODE satisfied by the solitons' center. Finally, we prove in Subsection 3.4 the hat property near characteristic points.

3.1 Time behavior of the solitons' centers

We will prove the following:

Proposition 3.1 (Refined behavior of w_{x_0} where $x_0 \in \mathcal{S}$) *Assuming that $k \geq 2$, there exists another set of parameters (still denoted by $\zeta_1(s)$, ... $\zeta_k(s)$) such that (20) and (21) hold and:*
(i) For all $i = 1, ..., k$, $e_i = (-1)^{i+1} e_1$.
(ii) For some $C_0 \in \mathbb{R}$, we have for s large,

$$\zeta_1(s) \leq -C_0 \log s \text{ and } \zeta_k(s) \geq C_0 \log s.$$

Remark: If we knew that in the following proposition $R_i = o\left(\sum_{j=i-1}^{i+1} e^{-\beta|\zeta_{j+1}-\zeta_j|}\right)$ (where R_i is defined below in Proposition 3.2), then, we would get $\zeta_1(s) \geq -C_1 \log s$ and $\zeta_k(s) \leq C_1 \log s$. Indeed, we write from Proposition 3.2 (and (i) of this proposition)

$$\zeta_{i+1}'(s) - \zeta_i'(s) \leq 3e^{-\beta(\zeta_{i+1}(s)-\zeta_i(s))},$$

which yields by integration $0 \leq \zeta_{i+1}(s) - \zeta_i(s) \leq C \log s$ and then, $0 \leq \zeta_k(s) - \zeta_1(s) \leq C \log s$. Since $\zeta_k(s) - \zeta_1(s) = |\zeta_1| + |\zeta_k|$ from (ii) of this proposition, this yields the desired bound.
The following ODE system satisfied by $(\zeta_i(s))$ is crucial in our proof:

Proposition 3.2 (Equations satisfied by the solitons' centers) *Assuming that $k \geq 2$, there exists another set of parameters (still denoted by $\zeta_1(s)$, ... $\zeta_k(s)$) such that (20) and (21) hold and for all $i = 1, ..., k$:*

$$\frac{1}{c_1}\zeta_i' = -e_{i-1}e_i e^{-\frac{2}{p-1}|\zeta_i - \zeta_{i-1}|} + e_i e_{i+1} e^{-\frac{2}{p-1}|\zeta_{i+1}-\zeta_i|} + R_i \quad (22)$$

$$\text{with } R_i = o(J) \text{ as } s \to \infty \text{ and } J(s) = \sum_{j=1}^{k-1} e^{-\frac{2}{p-1}|\zeta_{j+1}(s)-\zeta_j(s)|}, \quad (23)$$

for some $c_1 > 0$, with the convention $e_0 = e_{k+1} = 0$.

Proof: See subsection 3.2. ■

Let us now give the proof of Proposition 3.1.

Proof of Proposition 3.1: Given some $s_0 \in \mathbb{R}$, we first define for all $s \geq s_0$, $J_0(s)$ and $j_0(s) \in \{1, ..., k-1\}$,

$$J_0(s) \equiv \max_{i=1,..k-1} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds' = \int_{s_0}^{s} e^{-\beta|\zeta_{j_0(s)+1}(s')-\zeta_{j_0(s)}(s')|}ds' \quad (24)$$

where $\beta = \frac{2}{p-1}$. Then, we claim that

$$J_0(s) \to \infty \text{ as } s \to \infty. \quad (25)$$

Indeed, we write from (22) and (24) $|\zeta_i(s) - \zeta_i(s_0)| \leq C \sum_{i=1}^{k-1} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds' \leq CJ_0(s)$ and (21) implies (25).
Integrating equation (22), this yields as $s \to \infty$ for all $i = 1, ..., k$:

$$\frac{\zeta_1(s) + ... + \zeta_i(s)}{i} = e_i e_{i+1} \frac{c_1}{i} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds' + o\left(J_0(s)\right), \quad (26)$$

$$\frac{\zeta_i(s) + ... + \zeta_k(s)}{k-i+1} = -e_i e_{i-1} \frac{c_1}{k-i+1} \int_{s_0}^{s} e^{-\beta|\zeta_i(s')-\zeta_{i-1}(s')|}ds' + o\left(J_0(s)\right). \quad (27)$$

(i) Using (21), we write for s large,

$$\frac{\zeta_1(s) + ... + \zeta_{j_0(s)}(s)}{j_0(s)} < \frac{\zeta_{j_0(s)+1}(s) + ... + \zeta_k(s)}{k - j_0(s)},$$

$$\text{if } i < j_0(s), \quad \frac{\zeta_1(s) + ... + \zeta_i(s)}{i} < \frac{\zeta_1(s) + ... + \zeta_{j_0(s)}(s)}{j_0(s)},$$

$$\text{if } i > j_0(s), \quad \frac{\zeta_{j_0(s)+1}(s) + ... + \zeta_k(s)}{k - j_0(s)} < \frac{\zeta_{i+1}(s) + ... + \zeta_k(s)}{k-i}.$$

Then, using (26), (27) and (24), we write for s large,

$$e_{j_0(s)} e_{j_0(s)+1} \frac{c_1}{j_0(s)} J_0(s) \leq -e_{j_0(s)} e_{j_0(s)+1} \frac{c_1}{k - j_0(s)} J_0(s) + o(J_0(s)),$$

if $\quad i < j_0(s)$,

$$e_i e_{i+1} \frac{c_1}{i} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds' \leq e_{j_0(s)} e_{j_0(s)+1} \frac{c_1}{k - j_0(s)} J_0(s) + o(J_0(s)), \quad (28)$$

if $\quad i > j_0(s)$,

$$-\frac{c_1 e_{j_0(s)} e_{j_0(s)+1}}{k - j_0(s)} J_0(s) + o(J_0(s)) \leq -e_i e_{i+1} \frac{c_1}{k-i} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds'. \quad (29)$$

Therefore, for s large, $J_0(s) \left(e_{j_0(s)} e_{j_0(s)+1} (\frac{c_1}{j_0(s)} + \frac{c_1}{k - j_0(s)}) + o(1) \right) \leq 0$, hence,

$$e_{j_0(s)} e_{j_0(s)+1} = -1.$$

Then, (28) and (29) write together with (24)

$$\forall i, \quad \frac{1}{2k} J_0(s) \leq -e_i e_{i+1} \int_{s_0}^{s} e^{-\beta|\zeta_{i+1}(s')-\zeta_i(s')|}ds' \leq J_0(s),$$

which gives for all i and s large,

$$e_i e_{i+1} = -1 \text{ and } \frac{J_0(s)}{C_0} \leq \int_{s_0}^{s} e^{-\beta|\zeta_{j+1}(s')-\zeta_j(s')|} ds' \leq J_0(s) \to \infty. \tag{30}$$

Using a finite induction, we get $e_i = (-1)^{i+1} e_1$.

(ii) Using Proposition 3.2, (i) and (30), we see that for all $i = 1, ..., k-1$ and s large,

$$\frac{1}{c_1}(\zeta_{i+1}(s) - \zeta_i(s)) \leq 3 \int_{s_0}^{s} e^{-\beta(\zeta_{i+1}(s')-\zeta_i(s'))} ds', \tag{31}$$

$$\frac{1}{c_1}\zeta_1(s) \leq -\frac{1}{2} \int_{s_0}^{s} e^{-\beta(\zeta_2(s')-\zeta_1(s'))} ds', \tag{32}$$

$$\frac{1}{c_1}\zeta_k(s) \geq \frac{1}{2} \int_{s_0}^{s} e^{-\beta(\zeta_k(s')-\zeta_{k-1}(s'))} ds'. \tag{33}$$

Since we get by integrating (31), $\int_{s_0}^{s} e^{-\beta(\zeta_{i+1}(s')-\zeta_i(s'))} ds' \geq C \log s$, the conclusion then follows from (32) and (33). This concludes the proof of Proposition 3.1. ■

3.2 Refinement of (20) for $k \geq 2$

Note that the case $k = 1$ has been already treated in [12] giving rise to estimate (12). As announced in the beginning of the section, we assume that $k \geq 2$ and claim the following:

Proposition 3.3 (Size of q in terms of the distance between solitons) *There exists another set of parameters (still denoted by $\zeta_1(s)$, ... $\zeta_k(s)$) such that (20) and (21) hold and for some $s^* \in \mathbb{R}$ and for all $s \geq s^*$,*

$$\|q(s)\|_{\mathcal{H}} \leq C \sum_{i=1}^{k-1} h(\zeta_{i+1}(s) - \zeta_i(s)), \tag{34}$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} w \\ \partial_s w \end{pmatrix} - \sum_{i=1}^{k} e_i \begin{pmatrix} \kappa(d_i) \\ 0 \end{pmatrix},$$

and

$$h(\zeta) = e^{-\frac{p}{p-1}\zeta} \text{ if } p < 2, \ \ h(\zeta) = e^{-2\zeta}\sqrt{\zeta} \text{ if } p = 2 \text{ and } h(\zeta) = e^{-\frac{2}{p-1}\zeta} \text{ if } p > 2. \tag{35}$$

Before proving the estimate, we need to use a modulation technique to slightly change the $\zeta_i(s)$ in order to guarantee some orthogonality conditions. In order to do so, we need to introduce for $\lambda = 0$ or 1, for any $d \in (-1, 1)$ and $r \in \mathcal{H}$,

$$\pi_\lambda^d(r) = \phi(W_\lambda(d), r) \tag{36}$$

where:

$$\phi(q, r) = \phi\left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right) = \int_{-1}^{1} \left(q_1 r_1 + q_1' r_1' (1 - y^2) + q_2 r_2\right) \rho dy, \tag{37}$$

$W_\lambda(d,y) = (W_{\lambda,1}(d,y), W_{\lambda,2}(d,y))$,

$$W_{1,2}(d,y)(y) = c_1(d)\frac{1-y^2}{(1+dy)^{\frac{2}{p-1}+1}},\ \ W_{0,2}(d,y) = c_0\frac{y+d}{1+dy}\kappa(d,y), \tag{38}$$

with $0 < c_1(d) \le C(1-d^2)^{\frac{1}{p-1}}$, $c_0 > 0$,
and $W_{\lambda,1}(d,y) \in \mathcal{H}_0$ is uniquely determined as the solution of

$$-\mathcal{L}r + r = \left(\lambda - \frac{p+3}{p-1}\right) W_{\lambda,2}(d) - 2y\partial_y W_{\lambda,2}(d) + \frac{8}{p-1}\frac{W_{\lambda,2}(d)}{1-y^2} \tag{39}$$

(in [12], we defined $W_{0,2}(d,y)$ by $\frac{c_0(d)(y+d)}{(1+dy)^{\frac{2}{p-1}+1}}$ with

$$1 = c_0(d)(1-d^2)^{\frac{1}{p-1}}\frac{4}{p-1}\int_{-1}^{1}\frac{(y+d)^2}{(1+dy)^{\frac{4}{p-1}+2}}\frac{\rho}{1-y^2}dy.$$

Setting $y = \tanh\xi$, we compute the integral and get $c_0(d) = c_0'(1-d^2)^{\frac{1}{p-1}}$. Using (11), we get (38)). Recall from Lemma 4.4 page 85 in [12] that

$$\forall d \in (-1,1),\ \ \|W_\lambda(d)\|_{\mathcal{H}} \le C. \tag{40}$$

We now have the following:

Lemma 3.4 (Modulation technique) *Assume that $k \ge 2$.*
(i) **(Choice of the modulation parameters)** *There exist other values of the parameters (still denoted by $d_i(s)$) of class C^1, such that $\zeta_{i+1}(s) - \zeta_i(s) \to \infty$ as $s \to \infty$ where $d_i(s) = -\tanh\zeta_i(s)$,*

$$\|q(s)\|_{\mathcal{H}} \to 0 \text{ and } \pi_0^{d_i(s)}(q) = 0 \text{ for all } i = 1,..,k, \tag{41}$$

where π_0^d and q are defined in (36) and (34) respectively.
(ii) **(Equation on q)** *For s large, we have*

$$\frac{\partial}{\partial s}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = L\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \sum_{j=1}^{k} e_j d_j'(s)\begin{pmatrix} \partial_d\kappa(d_j(s),y) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} 0 \\ f(q_1) \end{pmatrix} \tag{42}$$

$$\begin{aligned} \text{where } L\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_2 \\ \mathcal{L}q_1 + \psi q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \\ \psi(y,s) &= p|K(y,s)|^{p-1} - \tfrac{2(p+1)}{(p-1)^2},\ \ K(y,s) = \textstyle\sum_{j=1}^k e_j\kappa(d_j(s),y), \\ f(q_1) &= |K+q_1|^{p-1}(K+q_1) - |K|^{p-1}K - p|K|^{p-1}q_1, \\ R &= |K|^{p-1}K - \textstyle\sum_{j=1}^k e_j\kappa(d_j)^p. \end{aligned}$$

Remark: From the modulation technique, it is clear that the distance between old and new parameter $\zeta_i(s)$ goes to zero as $s \to \infty$.
Proof: See the proof of Proposition 5.1 in [12] where the case $k = 1$ is treated. There is no difficulty in adapting the proof to $k \geq 2$. ■

In the following, we will show that Proposition 3.3 holds with the set of parameters $\zeta_1(s)$,...,$\zeta_k(s)$ given by the modulation technique of Lemma 3.4. Before giving the proof, we start by reformulating the problem.

Let us first remark that equation (42) can be localized near each soliton's center which allows us to view it as a perturbation of the case of one soliton already treated in [12]. For this, given $i = 1, ..., k$, we need to expand the linear operator of equation (42) as

$$L(q) = L_{d_i(s)}(q) + (0, V_i(y,s)q_1) \text{ with}$$

$$\begin{aligned} L_d\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_2 \\ \mathcal{L}q_1 + (p\kappa(d_i(s),y)^{p-1} - \frac{2(p+1)}{(p-1)^2})q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \\ V_i(y,s) &= p|K(y,s)|^{p-1} - p\kappa(d_i(s),y)^{p-1}. \end{aligned} \tag{43}$$

Since the solitons' sum is decoupled (remember from (i) of Lemma 3.4 that

$$\xi_{i+1} - \xi_i \to \infty \text{ as } s \to \infty), \tag{44}$$

the properties of $L_{d_i(s)}$ will be essential in our analysis.
¿From section 4 in [12], we know that for any $d \in (-1,1)$, the operator L_d has 1 and 0 as eigenvalues, the rest of the eigenvalues are negative. More precisely, introducing

$$F_1(d,y) = (1-d^2)^{\frac{p}{p-1}} \begin{pmatrix} (1+dy)^{-\frac{2}{p-1}-1} \\ (1+dy)^{-\frac{2}{p-1}-1} \end{pmatrix}, \quad F_0(d,y) = (1-d^2)^{\frac{1}{p-1}} \begin{pmatrix} \dfrac{y+d}{(1+dy)^{\frac{2}{p-1}+1}} \\ 0 \end{pmatrix}, \tag{45}$$

we have

$$L_d(F_\lambda(d)) = \lambda F_\lambda(d) \text{ and } \|F_1(d)\|_{\mathcal{H}} + \|F_0(d)\|_{\mathcal{H}} \leq C. \tag{46}$$

The projection on $F_\lambda(d)$ is defined in (36) by $\pi^d_\lambda(r) = \phi\left(W_\lambda(d), r\right)$. Of course,

$$L^*_d W_\lambda(d) = \lambda W_\lambda(d) \tag{47}$$

where L^*_d is the conjugate of L_d with respect to the inner product ϕ, and the choice of the constants $c_1(d)$ and c_0 guarantees the orthogonality condition

$$\pi^d_\lambda(F_\mu(d)) = \phi(W_\lambda(d), F_\mu(d)) = \delta_{\lambda,\mu}. \tag{48}$$

In the following, we give a decomposition of the solution which is well adapted to the proof:

Lemma 3.5 (Decomposition of q) *If we introduce for all r and $\mathbf{r}$ in $\mathcal{H}$ the operator $\pi_-(r) \equiv r_-(y,s)$ defined by*

$$r(y,s) = \sum_{i=1}^{k}\left(\pi_1^{d_i(s)}(r)F_1(d_i(s),y) + \pi_0^{d_i(s)}(r)F_0(d_i(s),y)\right) + \pi_-(r) \tag{49}$$

and the bilinear form

$$\varphi(r,\mathbf{r}) = \int_{-1}^{1} \left(r_1'\mathbf{r}_1'(1-y^2) - \psi r_1\mathbf{r}_1 + r_2\mathbf{r}_2\right)\rho dy \tag{50}$$

where $\psi(y,s)$ is defined in (42), then:
(i) for s large enough and for all r and $\mathbf{r}$ in $\mathcal{H}$, we have

$$|\varphi(r,\mathbf{r})| \le C\|r\|_{\mathcal{H}}\|\mathbf{r}\|_{\mathcal{H}}, \tag{51}$$

(ii) for some $C_0 > 0$ and for all s large enough, we have:

$$q(y,s) = \sum_{i=1}^{k}\alpha_1^i(s)F_1(d_i(s),y) + q_-(y,s), \tag{52}$$

$$\frac{1}{C_0}\|q_-(s)\|_{\mathcal{H}}^2 - C_0\bar{J}(s)^2\|q(s)\|_{\mathcal{H}}^2 \le A_-(s) \le C_0\|q_-(s)\|_{\mathcal{H}}^2, \tag{53}$$

$$\frac{1}{C_0}\|q(s)\|_{\mathcal{H}}^2 \le \sum_{i=1}^{k}\left(\alpha_1^i(s)\right)^2 + A_-(s) \le C_0\|q(s)\|_{\mathcal{H}}^2 \tag{54}$$

where

$$\bar{J}(s) = \sum_{j=1}^{k-1}(\zeta_{j+1}-\zeta_j)e^{-\frac{2}{p-1}(\zeta_{j+1}-\zeta_j)},\ \ \alpha_\lambda^i(s) = \pi_\lambda^{d_i(s)}(q(s)) \text{ and } A_-(s) = \varphi(q_-(s),q_-(s)). \tag{55}$$

Remark: Note that the choice of $d_i(s)$ made in (41) guarantees that for s large enough,

$$\forall i = 1,...,k,\ \ \alpha_0^i(s) \equiv {\alpha_0^i}'(s) \equiv 0. \tag{56}$$

Moreover, we see from (53) that $A_-(s)$ is nearly positive and nearly equivalent to $\|q_-\|_{\mathcal{H}}^2$.
Remark: The operator π_- depends on the time variable s. In [12], we had only one soliton, and we decomposed q as follows:

$$q(y,s) = \pi_1^d(q)F_1(d,y) + \pi_0^d(q)F_0(d,y) + \pi_-^d(q), \tag{57}$$

where we had only one $d(s)$ (note that this decomposition is in fact a definition of the operator π_-^d). Here, due to (44), we have a decoupling effect, in the sense that $\pi_\lambda^{d_j(s)}(q)$ for $j \neq i$ cannot be "seen" when y is close to $-d_i(s)$, the "center" of the soliton $\kappa(d_i(s),y)$. Hence, $\pi_-(q)$ is more or less $\pi_-^{d_i(s)}(q)$ and we are reduced to the situation of one soliton already treated in [12]. This idea will be essential in our proof since given some $i = 1,...,k$, we have two types of terms in equation (42):
- terms involving the soliton $\kappa(d_i(s),y)$ for which we refer the reader to [12],
- interaction terms involving a different soliton $\kappa(d_j(s),y)$ which we treat in details.
Proof of Lemma 3.5: See Appendix B. ■

In order to prove Proposition 3.3, we project equation (42) according to the decomposition (49). More precisely, we have the following:

Lemma 3.6 *For s large enough, the following holds:*
(i) **(Control of the positive modes and the modulation parameters)**

$$\forall i=1,...,k,\quad \left|{\alpha_1^i}'(s)-\alpha_1^i(s)\right|+|\zeta_i'(s)|\le C\|q(s)\|_{\mathcal{H}}^2+CJ(s) \tag{58}$$

where $J(s)$ is defined in (23).
(ii) **(Control of the negative part)**

$$\begin{aligned}\left(R_-+\frac{1}{2}A_-\right)' &\le -\frac{3}{p-1}\int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2}dy+o\left(\|q(s)\|_{\mathcal{H}}^2\right)+C\sum_{m=1}^{k-1}\left(h(\zeta_{m+1}-\zeta_m)\right)^2\\ &+ CJ(s)\sqrt{|A_-(s)|}\end{aligned} \tag{59}$$

for some $R_-(s)$ satisfying

$$|R_-(s)|\le C\|q(s)\|_{\mathcal{H}}^{\bar p+1} \tag{60}$$

where $\bar p=\min(p,2)$ and $h(s)$ is defined in (35).
(iii) **(An additional relation)**

$$\frac{d}{ds}\int_{-1}^1 q_1q_2\rho\le -\frac{4}{5}A_-+C\sum_{m=1}^{k-1}h(\zeta_{m+1}-\zeta_m)^2+C\int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2}+C\sum_{i=1}^k\left(\alpha_1^i\right)^2. \tag{61}$$

Proof: See Appendix C. ■

With Lemma 3.6, we are ready to prove Proposition 3.3.

Proof of Proposition 3.3: We proceed as in section 5.3 page 113 in [12], though the situation is a bit different because of the presence of the forcing terms $J(s)$ and $\sum_{i=1}^k h(\zeta_{i+1}-\zeta_i)^2$ in the differential inequalities in Lemma 3.6.

If we introduce

$$a(s)=\sum_{i=1}^k\alpha_1^i(s)^2,\quad b(s)=A_-(s)+2R_-(s)\text{ and }H(s)=\sum_{m=1}^{k-1}h(\zeta_{m+1}-\zeta_m)^2 \tag{62}$$

where h is defined in (35), then we see from (i) of Lemma 3.4 and (60) that

$$a(s)+b(s)+H(s)\to 0\text{ as }s\to\infty. \tag{63}$$

Moreover, we see from (60) and (54) that $|b-A_-|\le\frac{1}{1000}\left(A_-+\sum_{i=1}^k(\alpha_1^i)^2\right)$ for s large enough, hence

$$\frac{99}{100}A_--\frac{1}{100}a\le b\le\frac{101}{100}A_-+\frac{1}{100}a. \tag{64}$$

Therefore, since $J(s)\le H(s)$ by (23) and (35), we have for s large,

$$\forall\epsilon>0,\quad CJ\sqrt{|A_-|}\le\epsilon(a+b)+\frac{C}{\epsilon}H(s).$$

Using (62), (64) and (63), we rewrite estimates (54) and Lemma 3.6 with the new variables, in the following:

Corollary 3.7 (Equations in the new framework) *There exists $K_0 \geq 1$ such that for all $\epsilon > 0$, there exists $s_0(\epsilon) \in \mathbb{R}$ such that for all $s \geq s_0(\epsilon)$, the following holds:*
(i) **(Size of the solution)**

$$\frac{1}{K_0}(a+b) \leq \|q\|_{\mathcal{H}}^2 \leq K_0(a+b), \tag{65}$$

$$\left|\int_{-1}^{1} q_1 q_2 \rho dy\right| \leq K_0(a+b). \tag{66}$$

(ii) **(Equations)**

$$\begin{aligned}
\frac{3}{2}a - \epsilon b - K_0 H \leq a' &\leq \frac{5}{2}a + \epsilon b + K_0 H,\\
b' &\leq -\frac{6}{p-1}\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + \epsilon(a+b) + \frac{K_0}{\epsilon}H,\\
\frac{d}{ds}\int_{-1}^{1} q_1 q_2 \rho dy &\leq -\frac{3}{5}b + K_0\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + K_0 a + K_0 H,\\
|H'| &\leq \epsilon H.
\end{aligned} \tag{67}$$

We proceed in 2 steps:
- In Step 1, we show that a is controlled by $b+H$:.
- In Step 2, we show that b is controlled by H and conclude the proof using (65).

Step 1: a is controlled by $b+H$
We claim that for ϵ small enough, we have:

$$\forall s \geq s_0(\epsilon),\ \ a(s) \leq \epsilon b(s) + \frac{K_0}{\epsilon}H(s). \tag{68}$$

Indeed, from Corollary 3.7, we see that for all $s \geq s_0(\epsilon)$, we have

$$\begin{aligned}
a' &\geq \frac{3}{2}a - (\epsilon b + \frac{K_0}{\epsilon}H),\\
(\epsilon b + \frac{K_0}{\epsilon}H)' &\leq 2\epsilon(\epsilon b + \frac{K_0}{\epsilon}H) + \epsilon^2 a.
\end{aligned}$$

Introducing $\gamma_1(s) = a(s) - (\epsilon b(s) + \frac{K_0}{\epsilon}H(s))$, we see that for all $s \geq s_0(\epsilon)$,

$$\begin{aligned}
\gamma_1' = a' - (\epsilon b' + \frac{K_0}{\epsilon}H') &\geq \frac{3}{2}a - (\epsilon b + \frac{K_0}{\epsilon}H) - 2\epsilon(\epsilon b + \frac{K_0}{\epsilon}H) - \epsilon^2 a\\
&= (\frac{3}{2} - \epsilon^2 - 1 - 2\epsilon)a + (1+2\epsilon)\gamma_1 \geq \gamma_1
\end{aligned}$$

if ϵ is small enough. Since $\gamma_1(s) \to 0$ as $s \to \infty$ (see (63)), this implies $\gamma_1(s) \leq 0$, hence (68) follows.

Step 2: b is controlled by H
We claim that in order to conclude, it is enough to prove for some $K_1 > 0$ that

$$\forall s \geq s_0(\epsilon),\ \ f(s) \leq K_1 H(s) \tag{69}$$

where

$$f = b + \eta \int_{-1}^{1} q_1 q_2 \rho dy \text{ and } \eta = \frac{1}{2} \min\left(\frac{1}{2K_0}, \frac{6}{(p-1)K_0}\right). \tag{70}$$

Indeed, using (66) and (68), and taking ϵ small enough, we get for all $s \geq s_0(\epsilon)$, $\left|\int_{-1}^{1} q_1 q_2 \rho dy\right| \leq 2K_0 b + \frac{K_0^2}{\epsilon} H$ and $|f - b| \leq 2K_0 \eta b + \eta \frac{K_0^2}{\epsilon} H \leq \frac{b}{2} + \eta \frac{K_0^2}{\epsilon} H$, hence

$$\frac{b}{2} - \eta \frac{K_0^2}{\epsilon} H \leq f \leq 2b + \eta \frac{K_0^2}{\epsilon} H. \tag{71}$$

Therefore, if (69) holds, then using (65), (71) and (68), we see that for some $K_2 > 0$ and for all $s \geq s_0(\epsilon)$, $\|q(s)\|_{\mathcal{H}}^2 \leq K_0(a(s) + b(s)) \leq K_2 H(s)$ which is the desired conclusion of Proposition 3.3. It remains to prove (69).
Using Corollary 3.7, (68), (70) and the fact that $K_0 \geq 1$, and taking ϵ small enough, we get for all $s \geq s_0(\epsilon)$:

$$b' \leq -\frac{6}{p-1} \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + 2\epsilon b + 2\frac{K_0}{\epsilon} H, \tag{72}$$

$$\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy \leq -\frac{2}{5} b + K_0 \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + 2\frac{K_0^2}{\epsilon} H, \tag{73}$$

$$\begin{aligned} f' &\leq -(\frac{2}{5}\eta - 2\epsilon) b - \left(\frac{6}{p-1} - K_0 \eta\right) \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + (2\frac{K_0}{\epsilon} + 2\frac{K_0^2}{\epsilon}\eta) H \\ &\leq -\frac{\eta}{4} b + 3\frac{K_0}{\epsilon} H \leq -\frac{\eta}{8} f + 4\frac{K_0}{\epsilon} H. \end{aligned} \tag{74}$$

If $\gamma_2(s) = f(s) - \frac{64K_0}{\eta\epsilon} H(s)$, then we write from (67) and (74), for all $s \geq s_0(\epsilon)$,

$$\gamma_2' = f' - \frac{64K_0}{\eta\epsilon} H' \leq -\frac{\eta}{8} f + 4\frac{K_0}{\epsilon} H + \frac{64K_0}{\eta\epsilon} \epsilon H = -\frac{\eta}{8}\gamma_2 + \frac{K_3}{\epsilon} H \leq -\frac{\eta}{8}\gamma_2$$

because $K_3 = -\frac{64K_0}{\eta}\frac{\eta}{8} + \frac{64K_0}{\eta}\epsilon + 4K_0 = -4K_0 + \frac{64K_0}{\eta}\epsilon \leq 0$ if ϵ is small enough. Therefore, for all $s \geq s_0(\epsilon)$, $\gamma_2(s) \leq e^{-\frac{\eta}{8}(s-s_0)}\gamma_2(s_0)$, hence

$$f(s) \leq \frac{64K_0}{\eta\epsilon} H(s) + e^{-\frac{\eta}{8}(s-s_0)} |\gamma_2(s_0)|. \tag{75}$$

Since we have from (67) and (62),

$$H(s) \geq e^{-\epsilon(s-s_0)} H(s_0) \text{ and } H(s_0) > 0, \tag{76}$$

taking $\epsilon \leq \frac{\eta}{8}$, we see that (69) follows from (75) and (76). This concludes the proof of Proposition 3.3. ■

3.3 An ODE system satisfied by the solitons' centers

With Proposition 3.3, we are ready to prove Proposition 3.2 now. The proof consists in refining the projection of equation (42) with the projector π_0^d (36), already performed in the proof of (i) of Lemma 3.6 (see Part 1 page 39).

Proof of Proposition 3.2: Using (152), (153), (157), (158), (161), the differential inequality (58) on ζ_i and the fact that $\alpha_0^i(s) \equiv {\alpha_0^i}'(s) \equiv 0$ (see (56)), we write:

$$\left| -e_i \frac{2\kappa_0}{p-1} \zeta_i'(s) + \pi_0^{d_i(s)} \begin{pmatrix} 0 \\ R \end{pmatrix} \right| \le C \|q(s)\|_{\mathcal{H}}^2 + o(J(s)) \text{ as } s \to \infty. \tag{77}$$

Since we have from Proposition 3.3,

$$\|q(s)\|_{\mathcal{H}}^2 \le C \sum_{i=1}^{k-1} (h(\zeta_{i+1} - \zeta_i))^2 = o(J(s)) \text{ as } s \to \infty \tag{78}$$

where $h(\zeta)$ is defined in (35), it is clear that if one proves that for some $c_1' > 0$,

$$\frac{1}{c_1'} \pi_0^{d_i(s)} \begin{pmatrix} 0 \\ R \end{pmatrix} = -e_{i-1} e^{-\frac{2}{p-1}|\zeta_i(s) - \zeta_{i-1}(s)|} + e_{i+1} e^{-\frac{2}{p-1}|\zeta_{i+1}(s) - \zeta_i(s)|} + o(J) \tag{79}$$

as $s \to \infty$ (with the convention $\zeta_0 = -\infty$ and $\zeta_{k+1} = +\infty$), then, Proposition 3.2 immediately follows from (77) and (78). It remains to prove (79).

Proof of (79): We claim first that

$$\begin{aligned} &\left| R - \sum_{j=1}^{k} p\kappa(d_j(s))^{p-1} 1_{\{y_{j-1}(s) < y < y_j(s)\}} \sum_{l \ne j} e_l \kappa(d_l(s)) \right| \\ \le\ & C \sum_{j=1}^{k} \kappa(d_j(s))^{p-\bar{p}} 1_{\{y_{j-1}(s) < y < y_j(s)\}} \sum_{l \ne j} \kappa(d_l(s))^{\bar{p}} \end{aligned} \tag{80}$$

where y_i are the solitons' separators defined in (199).
Indeed, let us take $y \in (y_{j-1}(s), y_j(s))$ and set $X = (\sum_{l \ne j} e_l \kappa(d_l(s)))/e_j \kappa(d_j(s))$. From the fact that $\zeta_{j+1}(s) - \zeta_j(s) \to \infty$ and (199), we have $|X| \le 3$ hence

$$||1+X|^{p-1}(1+X) - 1 - pX| \le CX^2$$

and for $y \in (y_{j-1}(s), y_j(s))$ and s large,

$$\begin{aligned} &||K|^{p-1}K - e_j \kappa(d_j(s))^p - p\kappa(d_j(s))^{p-1} \sum_{l \ne j} e_l \kappa(d_l(s))| \\ \le\ & C\kappa(d_j(s))^{p-2} \sum_{l \ne j} \kappa(d_l(s))^2 \end{aligned}$$

Since for $y \in (y_{j-1}(s), y_j(s))$, $|\sum_{l \ne j} e_l \kappa(d_l(s))^p| \le \sum_{l \ne j} \kappa(d_l(s))^p$ and $\kappa(d_j(s)) \ge \kappa(d_l(s))$ if $l \ne j$, this concludes the proof of (80).
Now, we prove (79). Using (80), (36), (38) and the notations of Lemma E.1, we write

$$\left| \pi_0^{d_i} \begin{pmatrix} 0 \\ R \end{pmatrix} - pc_0 \sum_{j=1}^{k} \sum_{l \ne j} A_{i,j,l} \right| \le C \sum_{j=1}^{k} \sum_{l \ne j} B_{i,j,l}.$$

Since we have from (iii) and (iv) of Lemma E.1, $A_{i,j,l} + B_{i,j,l} = o(J)$ except for $A_{i,i,l}$ with $l = i \pm 1$ where we have $A_{i,i,i\pm1} \sim c_1''' e^{-\frac{2}{p-1}|\zeta_i - \zeta_{i\pm1}|}$, we get (79). Since Proposition 3.2 follows form (79) and (77), this concludes the proof of Proposition 3.2 too. ■

3.4 The blow-up set is hat-shaped near $x_0 \in \mathcal{S}$ when $k(x_0) \geq 2$

We derive here the following consequence of Proposition 3.1:

Proposition 3.8 (Existence of signed lines and the hat property near $x_0 \in \mathcal{S}$) *If $x_0 \in \mathcal{S}$ with $k(x_0) \geq 2$, then:*
(i) For all $j = 1, .., k$,

$$u(z_j(t), t) \sim e_j^* \kappa_0 \cosh^{\frac{2}{p-1}} \zeta_j(s)(T(x_0) - t)^{-\frac{2}{p-1}} \text{ as } t \to T(x_0),$$

where $t \mapsto z_j(t)$ is continuous and defined by

$$z_j(t) = x_0 + (T(x_0) - t) \tanh \zeta_j(s) \text{ with } s = -\log(T(x_0) - t). \quad (81)$$

(ii) We have for some $\delta_0 > 0$ and $\gamma > 0$,

$$\text{if } |x - x_0| \leq \delta_0, \text{ then } T(x_0) - |x - x_0| \leq T(x) \leq T(x_0) - |x - x_0| + \frac{|x - x_0|}{|\log(x - x_0)|^\gamma}. \quad (82)$$

Remark: The point $z_j(t)$ corresponds in the original variables to the center of the j-th soliton in the description (20).
Remark: In the next section, we prove that for all $x_0 \in \mathcal{S}$, we have $k(x_0) \geq 2$. Thus, the result will hold for all $x_0 \in \mathcal{S}$.
Proof: It follows from Proposition 3.1 and the following:

Lemma 3.9 (Upper blow-up bound for equation (1)) *For all $x_0 \in \mathbb{R}$ and $t \in [\frac{3}{4}T(x_0), T(x_0))$,*

$$\sup_{|x - x_0| < \frac{T(x_0)-t}{2}} |u(x,t)| \leq K(T(x_0) - t)^{-\frac{2}{p-1}}$$

where K depends only on p and an upper bound on $T(x_0)$ and $1/T(x_0)$.

Proof: Use Theorem 1 in [11] and the Sobolev injection. ■

Proof of Proposition 3.8:
(i) Using Proposition 3.1 and (i) of Lemma B.1, we see that we have

$$\sup_{|y|<1} \left| (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) - \sum_{i=1}^{k(x_0)} e_i^* (1 - y^2)^{\frac{1}{p-1}} \kappa(d_i, y) \right| \to 0 \text{ as } s \to \infty. \quad (83)$$

Since we have

$$\kappa(d_i(s), y)(1 - y^2)^{\frac{1}{p-1}} = \kappa_0 \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i(s)) \text{ if } y = \tanh \xi \quad (84)$$

and $\zeta_{i+1}(s) - \zeta_i(s) \to \infty$ as $s \to \infty$, we apply (83) with $y = -d_j(s) = \tanh \zeta_j(s)$ to get

$$(1 - d_j(s)^2)^{\frac{1}{p-1}} w_{x_0}(-d_j(s), s) \to e_j^* \kappa_0 \text{ as } s \to \infty. \quad (85)$$

Since $(1 - d_j(s)^2)^{-\frac{1}{p-1}} = \cosh^{\frac{2}{p-1}} \zeta_j(s)$, $e_j^* = e_1^*(-1)^{j+1}$ and $s \mapsto d_j(s) \in (-1, 1)$ is continuous, using the selfsimilar transformation (5), we see that (i) follows.

(ii) Note that the left-hand inequality follows from the fact that $x \mapsto T(x)$ is 1-Lipschitz, which is a consequence of the finite speed of propagation. For the right-hand inequality, we give the proof only for $x > x_0$ since the case $x < x_0$ follows in the same way. The key idea is to derive lower and upper bounds for $|u(z_k(t), t)|$ where $z_k(t)$ defined in (81) is the "center" of the k-th soliton.
Using (i) with $j = k(x_0)$, we see that for all $t \in [t_1, T(x_0))$ for some $t_1 < T(x_0)$,

$$|u(z_k(t), t)| \geq \frac{\kappa_0}{2} \cosh^{\frac{2}{p-1}} \zeta_k(s)(T(x_0) - t)^{-\frac{2}{p-1}} \tag{86}$$

on the one hand. On the other hand, using Lemma 3.9 and the continuity of $x \mapsto T(x)$, we see that for all for some $t_0 \in [t_1, T(x_0))$ and $C_1 > 0$, we have

$$\forall t \in [t_0, T(x_0)), \ \ |u(z_k(t), t)| \leq C_1(T(z_k(t)) - t)^{-\frac{2}{p-1}}.$$

Therefore, it follows from (86) that

$$\forall t \in [t_0, T(x_0)), \ \ T(z_k(t)) - t \leq C_0 \frac{(T(x_0) - t)}{\cosh \zeta_k(s)},$$

hence

$$T(z_k(t)) \leq T(x_0) - (T(x_0) - t)\left(1 - \frac{C_0}{\cosh \zeta_k(s)}\right) \text{ where } s = -\log(T(x_0) - t). \tag{87}$$

Recall from Proposition 3.1 and Lemma 3.2 that for some $\gamma > 0$ and for s large,

$$\zeta_k(s) \geq \gamma \log s, \ \ \cosh \zeta_k(s) \geq \frac{s^\gamma}{2}, \ \ 1 > \tanh \zeta_k(s) \text{ and } \zeta_k'(s) \to 0 \tag{88}$$

as $s \to \infty$. Therefore, we see from (81) that

$$z_k(t) \to x_0 \text{ and } z_k'(t) = -\tanh \zeta_k(s) + \zeta_k'(s) \cosh^{-2} \zeta_k(s) \to -1 \text{ as } t \to T(x_0).$$

Therefore, the map $t \mapsto z_k(t)$ is one to one from $[t_0, T(x_0))$ to $(x_0, x_0 + \delta_0]$ for some $\delta_0 > 0$ and we can make the change of variables

$$x = z_k(t) = x_0 + (T(x_0) - t) \tanh \zeta_k(s) \text{ where } s = -\log(T(x_0) - t). \tag{89}$$

Since we have from (89) and (88)

$$(T(x_0) - t)\left(1 - \frac{C_0}{\cosh \zeta_k(s)}\right) = (x - x_0)\frac{\left(1 - \frac{C_0}{\cosh \zeta_k(s)}\right)}{\tanh \zeta_k(s)} \geq (x - x_0)(1 - \frac{C}{s^\gamma}) \tag{90}$$

and $s = -\log(T(x_0) - t) \sim -\log(x - x_0)$ as $x \to x_0$, (82) follows from (87) and (90). This concludes the proof of Proposition 3.8. ■

4 Properties of $\mathcal{S}$

We proceed in 3 subsections. We first prove that the interior of $\mathcal{S}$ is empty (Theorem 5. Then, we give the proofs of Theorems 4 and 4', as well as Propositions 6 and 7. In the last subsection, we give various estimates on norms of the solution at blow-up, near a characteristic point.

4.1 Soliton characterization on $\mathcal{S}$

This subsection is devoted to the proof of the following result (which directly implies Theorem 5):

Proposition 4.1
(i) The interior of $\mathcal{S}$ is empty.
(ii) For all $x_0 \in \mathcal{S}$, $k(x_0) \geq 2$.

Before proving this proposition, let us first state the following Lemmas:

Lemma 4.2 (Characterization of the interior of $\mathcal{S}$) *For any $x_1 < x_2$, the following statements are equivalent:*
(a) $(x_1, x_2) \in \mathcal{S}$.
(b) There exists $x^* \in [x_1, x_2]$ such that for all $x \in [x_1, x_2]$, $T(x) = T(x^*) - |x - x^*|$.

Lemma 4.3 *Consider $x_1 < x_2$ such that $e \equiv \frac{T(x_2)-T(x_1)}{x_2-x_1} = \pm 1$. Then,*
(i) for all $x \in [x_1, x_2]$, $T(x) = T(x_1) + e(x - x_1)$,
(ii) $(x_1, x_2) \in \mathcal{S}$.

Lemma 4.4 (Boundary properties of $\mathcal{S}$)
(i) For all $x_0 \in \partial\mathcal{S}$, $k(x_0) \neq 0$.
(ii) Consider $x_0 \in \partial\mathcal{S}$ with $k(x_0) = 1$. If there exists a sequence $x_n \in \mathcal{R}$ converging from the left (resp. the right) to x_0, then x_0 is left-non-characteristic (resp. right-non-characteristic).

Remark: We mean by x_0 is left-non-characteristic (resp. right-non-characteristic) that it satisfies condition (4) only for $x < x_0$ (resp. for $x > x_0$).
Remark: It is not possible to prove by a direct argument that $k(x_0) \geq 1$ when x_0 is arbitrary in $\mathcal{S}$. We need to prove it first for $x_0 \in \partial\mathcal{S}$ and then prove that the interior is empty. See the derivation of Proposition 4.1 from Lemma 4.4.

We now give the proofs of Lemmas 4.2, 4.3 and 4.4.

Proof of Lemma 4.2:
(a) $\Longrightarrow$ (b): Let us introduce $x^* \in [x_1, x_2]$ such that

$$T(x^*) = \max_{x_1 \leq x \leq x_2} T(x).$$

We claim that $T(x)$ is nondecreasing on $[x_1, x^*]$, and nonincreasing on $[x^*, x_2]$. Indeed, let us prove the first fact, the second being similar. If for some $x' \leq x''$ in $[x_1, x^*]$, we have $T(x') > T(x'')$, then $\min_{x' \leq x \leq x^*} T(x) \leq T(x'') < T(x') \leq T(x^*)$. Therefore, this minimum is achieved at a point $\tilde{x}$ different from x' and x^*, hence

$$\tilde{x} \in (x', x^*) \subset (x_1, x_2).$$

In other words, $\tilde{x}$ is a local minimum, hence non characteristic, which is in contradiction with (a).
The result clearly follows if we prove that

$$\forall x \in (x_1, x^*), \qquad T(x) = T(x_1) + (x - x_1), \tag{91}$$

$$\forall x \in (x^*, x_2), \qquad T(x) = T(x_2) - (x - x_2). \tag{92}$$

We only prove (91) since (92) follows similarly.
Assume by contradiction that for some $x' \in (x_1, x^*)$, we have

$$\frac{T(x') - T(x_1)}{x' - x_1} = m_0 \notin \{-1, 1\}. \tag{93}$$

Then, since $x \mapsto T(x)$ is 1-Lipschitz and nondecreasing, it follows that $0 \leq m_0 < 1$. Considering a family of lines of slope $\frac{1+m_0}{2}$ growing from below, we find $\lambda_0 \in \mathbb{R}$ and $x_0 \in [x_1, x']$ such that

$$\forall x \in [x_1, x'], \;\; T(x) \geq \frac{(1+m_0)}{2}(x - x_1) + \lambda_0 \text{ and } T(x_0) = \frac{(1+m_0)}{2}(x_0 - x_1) + \lambda_0. \tag{94}$$

If $x_0 \in (x_1, x')$, then for all $x \in [x_1, x']$, $T(x) \geq \frac{(1+m_0)}{2}(x - x_0) + T(x_0)$, hence x_0 is non characteristic (the cone of slope $\frac{1+m_0}{2}$ is convenient).
If $x_0 = x'$, then since $T(x)$ is non decreasing on (x_1, x^*), it follows that x_0 is again non characteristic. In these two cases, we have a contradiction with the fact that $(x_1, x_2) \in \mathcal{S}$.
If $x_0 = x_1$, then we have from (94), $T(x_1) = \lambda_0$ and $T(x') \geq \frac{(1+m_0)}{2}(x' - x_1) + T(x_1)$, in contradiction with (93).
Thus, (91) holds. Since (92) follows similarly, (b) follows too.

(b) $\Longrightarrow$ (a): For any $x \in (x_1, x_2)$, the left-slope of $x \mapsto T(x)$ is 1 or -1, hence, by definition, $x \in \mathcal{S}$ and (a) follows. This concludes the proof of Lemma 4.2. ■

We now give the proof of Lemma 4.3:
Proof of Lemma 4.3: Up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that $x_1 < x_2$ and

$$e \equiv \frac{T(x_2) - T(x_1)}{x_2 - x_1} = 1. \tag{95}$$

(i) If $x \in (x_1, x_2)$, we use the fact that $x \mapsto T(x)$ together with (95) to write:

$$\begin{aligned} T(x) &\leq T(x_1) + (x - x_1), \\ T(x) &\geq T(x_2) - (x_2 - x) = T(x_1) + (x - x_1) \end{aligned}$$

and (i) follows.
(ii) It follows from (i), just by applying the fact that (b) implies (a) in Lemma 4.2 (take $x^* = x_2$). This concludes the proof of Lemma 4.3. ■

We now give the proof of Lemma 4.4.

Proof of Lemma 4.4: Consider $x_0 \in \partial\mathcal{S}$. Up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that for some sequence

$$x_n \in \mathcal{R}, \text{ we have } x_n < x_0 \text{ and } x_n \to x_0 \text{ as } n \to \infty. \tag{96}$$

Therefore, we have

$$\forall x < x_0, \;\; T(x) > T(x_0) - (x_0 - x). \tag{97}$$

Indeed, note first form the fact that $x \mapsto T(x)$ is 1-Lipschitz that for all $x < x_0$, $T(x) \geq T(x_0) - (x_0 - x)$. By contradiction, if for some $\hat{x} < x_0$, we have $T(\hat{x}) = T(x_0) - (x_0 - \hat{x})$, then we see from Lemma 4.3 that $(\hat{x}, x_0) \subset \mathcal{S}$, in contradiction with (96). Thus, (97) holds.

To prove (i) and (ii), we proceed by contradiction. Assume then that

either $k(x_0) = 0$ *(***case 1***),*
or $k(x_0) = 1$ *and* x_0 *is not left-non-characteristic (***case 2***).*

Using (20) when $k(x_0) = 0$ and Proposition 2.2 when $k(x_0) = 1$, we see that

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} w_\infty \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty, \tag{98}$$

$$\text{where} \quad w_\infty(y) = 0 \text{ if } k(x_0) = 0 \text{ and } w_\infty(y) = e^*\kappa(d(x_0), y) \text{ if } k(x_0) = 1, \tag{99}$$

for some $e^* = \pm 1$ and $d(x_0) \in (-1, 1)$. Now, we claim the following continuity result:

Claim 4.5 *For all* $\epsilon_0 > 0$*, there exists* $\tilde{t} < T(x_0)$ *and* $\tilde{x} < x_0$ *such that for all* $x' \in (\tilde{x}, x_0)$*,*

$$\left\| \begin{pmatrix} w_{x'}(\tilde{s}_0(x')) \\ \partial_s w_{x'}(\tilde{s}_0(x')) \end{pmatrix} - \begin{pmatrix} w_\infty \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \le \epsilon_0 \tag{100}$$

where $\tilde{s}_0(x') = -\log(T(x') - \tilde{t})$*.*

Proof: See Appendix D. ■

Let us first use this lemma to find a contradiction.

Case 1: $k(x_0) = 0$. Consider some $\epsilon_0 > 0$ (to be fixed small enough later). Using this claim, (99) and (96), we see that for some $\tilde{t} < T(x_0)$ and for n large enough, we have

$$x_n \in \mathcal{R} \text{ and } \left\| \begin{pmatrix} w_{x_n}(\tilde{s}_0(x')) \\ \partial_s w_{x_n}(\tilde{s}_0(x')) \end{pmatrix} \right\|_{\mathcal{H}} \le \epsilon_0.$$

Using the continuity of $E(w)$ in $\mathcal{H}$ (which is a consequence of Lemma B.1), we see that $E(w_{x_n}(\tilde{s}_0(x'))) \le C\epsilon_0 \le \frac{1}{2}E(\kappa_0)$ (if ϵ_0 is small enough), on the one hand. On the other hand, since $x_n \in \mathcal{R}$, we know from the limit and the monotonicity of $E(w_{x_n}(s))$ stated in page 3 that $E(w_{x_n}(s)) \ge E(\kappa_0) > 0$, which is a contradiction.

Case 2: $k(x_0) = 1$ **and** x_0 **is not left-non-characteristic**. Since x_0 is not left-non-characteristic, we see from (97) that there exists a sequence $\hat{x}_n$ such that

$$\hat{x}_n < x_0, \;\; \hat{x}_n \to x_0 \text{ and } \hat{m}_n \equiv \frac{T(\hat{x}_n) - T(x_0)}{\hat{x}_n - x_0} \in [1 - \frac{1}{n}, 1). \tag{101}$$

Considering a family of lines of slope $\frac{1+\hat{m}_n}{2}$, we can select one such that

$$\forall x \in [\hat{x}_n, x_0], \;\; T(x) \ge \frac{(1+\hat{m}_n)}{2}(x - \hat{x}_n) + \lambda_n \text{ and } T(\tilde{x}_n) = \frac{(1+\hat{m}_n)}{2}(\tilde{x}_n - \hat{x}_n) + \lambda_n \tag{102}$$

for some $\lambda_n \in \mathbb{R}$ and $\tilde{x}_n \in [\hat{x}_n, x_0]$.
If $\tilde{x}_n = x_0$, then for all $x \in [\hat{x}_n, x_0]$, $T(x) \ge \frac{(1+\hat{m}_n)}{2}(x - x_0) + T(x_0)$, which is in contradiction with the fact that x_0 is left-non-characteristic.
If $\tilde{x}_n = \hat{x}_n$, then we have from (102), $T(\hat{x}_n) = \lambda_n$ and $T(x_0) \ge \frac{1+\hat{m}_n}{2}(x_0 - \hat{x}_n) + T(\hat{x}_n)$, in

contradiction with (101).
If $\tilde{x}_n \in (\hat{x}_n, x_0)$, then $\tilde{x}_n \in \mathcal{R}$ (the cone of slope $\frac{1+\hat{m}_n}{2}$ is convenient). Since $\tilde{x}_n \to x_0$ and $\tilde{x}_n \in \mathcal{R}$, we see from Claim 4.5 that for some $\tilde{t} < T(x_0)$ and n large enough, we have

$$\left\| \begin{pmatrix} w_{\tilde{x}_n}(\tilde{s}_0(\tilde{x}_n)) \\ \partial_s w_{\tilde{x}_n}(\tilde{s}_0(\tilde{x}_n)) \end{pmatrix} - e^* \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^* \tag{103}$$

where ϵ^* is introduced in Proposition 2.2. Since the energy barrier follows from the fact that $\tilde{x}_n \in \mathcal{R}$, Proposition 2.2 applies and we have for n large enough,

$$(w_{\tilde{x}_n}(s), \partial_s w_{\tilde{x}_n}(s)) \to e^*(\kappa(d_n), 0) \text{ in } \mathcal{H} \text{ as } s \to \infty,$$

where $|d_n - d(x_0)| < \eta_0$ for some $\eta_0 > 0$ small enough so that $|d(x_0) \pm \eta_0| < 1$. The use of the geometrical interpretation of d_n is crucial for the conclusion. Indeed, from (103) and the regularity result of [12] cited in page 3, we see that $x \mapsto T(x)$ is differentiable at $x = \tilde{x}_n$ and that

$$T'(\tilde{x}_n) = d_n \leq d(x_0) + \eta_0 < 1$$

on the one hand. On the other hand, using (102) and (101), we see that

$$T'(\tilde{x}_n) = \frac{1 + \hat{m}_n}{2} \to 1 \text{ as } n \to \infty$$

which is a contradiction. This concludes the proof of Lemma 4.4. ■

Now, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1:
(i) Let us assume by contradiction that $\mathcal{S}$ contains some non empty interval (a', b'). Since $\mathcal{S} \neq \mathbb{R}$ by the result of [12] cited in page 3, by maximizing this interval and up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that:

$$(a, b) \subset \mathcal{S} \text{ with } a \in \partial\mathcal{S}, \;\; b > a$$

and, either $b \in \partial\mathcal{S}$ or $b = +\infty$. If b is finite, then up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that $T(b) \geq T(a)$. Using Lemma 4.2 and the fact that $T(x) \geq 0$, we see that for some $\tilde{b} < b$, we have

$$\forall x \in (a, \tilde{b}), \;\; T(x) = T(a) + (x - a). \tag{104}$$

We consider three cases and find a contradiction in each case.
- If $k(a) = 0$, then a contradiction occurs from (i) of Lemma 4.4.
- If $k(a) = 1$, then from the fact that $a \in \partial\mathcal{S}$, there exists a sequence $x_n \in \mathcal{R}$ such that $x_n \to a$ as $n \to \infty$. Since $(a, b) \subset \mathcal{S}$, it follows that $x_n < a$ for n large enough. Therefore, applying (ii) of Lemma 4.4, we see that a is left-non-characteristic. Since it is clearly right-non-characteristic by (104), a is in fact non-characteristic, which contradicts the fact that $a \in \partial\mathcal{S} \subset \mathcal{S}$ (note that $\mathcal{S}$ is closed since its complementary set $\mathcal{R}$ is open by the result of [13] cite in page 3).
- If $k(a) \geq 2$, then the hat property stated in Proposition 3.1 is in contradiction with (104).
Thus, (i) of Proposition 4.1 follows.

(ii) Consider $x_0 \in \mathcal{S}$. From (i), we have $x_0 \in \partial\mathcal{S}$. Using (i) of Lemma 4.4, we see that $k(x_0) \geq 1$. The result follows if we rule out the case $k(x_0) = 1$.
Assume by contradiction that $k(x_0) = 1$. Since the interior of $\mathcal{S}$ is empty, we can construct 2 sequences x_n and y_n in $\mathcal{R}$, such that $x_n \to x_0$ from the left, and $y_n \to x_0$ from the right. Applying (ii) of Lemma 4.4, we see that x_0 is in fact left-non-characteristic and right-non-characteristic, hence non characteristic. This contradicts the fact that $x_0 \in \mathcal{S}$. Thus, (ii) follows. This concludes the proof of Proposition 4.1 ■

4.2 On characteristic points for equation (1)

We prove Propositions 6 and 7, as well as Theorem 4 here.

Proof of Proposition 6: Consider $u(x,t)$ a solution of equation (1) and $x_0 \in \mathcal{S}$. Using (ii) of Proposition 4.1, we see that $k(x_0) \geq 2$. Therefore, Proposition 3.1 and Proposition 3.8 apply and directly give the conclusion of Proposition 6.

Proof of Proposition 7: Consider $u(x,t)$ a solution of equation (1) and $x_0 \in \mathcal{S}$. Using (ii) of Proposition 4.1, we see that $k(x_0) \geq 2$. Therefore, Proposition 3.1 and Proposition 3.8 apply and give the conclusion of Proposition 7, except for the strict inequality in (15), which we prove now.
Assume by contradiction that for some $x_1 < x_0$, we have equality in the left-hand inequality of (15). Then, we see from Lemma 4.3 that $(x_1, x_0) \subset \mathcal{S}$, which contradicts the fact that the interior of $\mathcal{S}$ is empty (see (i) of Proposition 4.1). Thus, (15) follows and Proposition 7 follows too. ■

Proof of Theorem 4: Consider $u(x,t)$ a solution of equation (1) that blows up on a graph $x \mapsto T(x)$ such that for some $a_0 < b_0$ and some $t_0 \geq 0$, we have

$$\forall x \in (a_0, b_0) \text{ and } t \in [t_0, T(x)),\ \ u(x,t) \geq 0. \tag{105}$$

We would like to prove that $(a_0, b_0) \subset \mathcal{R}$. Proceeding by contradiction, we assume that there exists $x_0 \in (a_0, b_0) \cap \mathcal{S}$. Using Proposition 7, we see that for some $e_1 = \pm 1$ and $t_1 \in [t_0, T(x_0))$, there are continuous $t \mapsto z_i(t)$ where $i = 1$ and 2 such that $z_i(t) \to x_0$ as $t \to T(x_0)$ and

$$\forall t \in [t_1, T(x_0)),\ \ e_1 u(z_1(t), t) > 0 \text{ and } e_1 u(z_2(t), t) < 0.$$

Therefore, u changes sign in $(a_0, b_0) \times [t_1, T(x_0))$, which is in contradiction with (105). Thus, $(a_0, b_0) \subset \mathcal{R}$ and Theorem 4 follows. ■

4.3 Non existence of characteristic points for equation (2)

This subsection is dedicated to the study of equation (2) which we recall here:

$$\begin{cases} \partial_{tt}^2 u = \partial_{xx}^2 u + |u|^p, \\ u(0) = u_0 \text{ and } u_t(0) = u_1. \end{cases} \tag{106}$$

We take $p > 1$ and $(u_0, u_1) \in \mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$. Our aim is to prove Theorem 4', which asserts that the set of characteristic points is empty, for any blow-up solution of (106). To do so, we need to perform for equation (106), an almost identical analysis to what we did for equation (1), in our previous papers, including this last one. Therefore, we only give the main steps and stress only the novelties.

Consider $u(x,t)$ a solution of equation (2) that blows up on some graph $x \mapsto T(x)$. As for equation (1), we denote the set of non characteristic points by $\mathcal{R}$ and the set of characteristic points by $\mathcal{S}$. Our aim is to show that $\mathcal{S} = \emptyset$.
Given $x_0 \in \mathbb{R}$ and $T_0 \in (0, T(x_0)]$, we define w_{x_0,T_0} as in (5) by

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{107}$$

If $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. The function $w = w_{x_0,T_0}$ satisfies the following equation for all $y \in B = B(0,1)$ and $s \geq -\log T_0$:

$$\partial^2_{ss} w = \mathcal{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^p - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s} w. \tag{108}$$

The Lyapunov functional for this equation is defined in $\mathcal{H}$ (9) and is given by

$$\tilde{E}(w(s)) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^p w \right) \rho dy$$

and satisfies

$$\frac{d}{ds} \tilde{E}(w(s)) = -\frac{4}{p-1} \int_{-1}^{1} (\partial_s w(y,s))^2 \frac{\rho}{1-y^2} dy.$$

We first give the following lower bound:

Lemma 4.6 (A lower bound on solutions of (2)) *For all $R > 0$, there exists $M(R) > 0$ such that for all $x \in (-R, R)$ and $t \in [0, T(x))$, we have $u(x,t) \geq -M(R)$.*

Proof: Using Duhamel's formula, we write for all $x \in \mathbb{R}$ and $t \in [0, T(x))$,

$$u(x,t) = S(t)u_0(x) + S_1(t)u_1(x) + \int_0^t (S_1(t-\tau)|u(\tau)^p|)(x) d\tau \tag{109}$$

where

$$S(t)h(x) = \frac{1}{2}(h(x+t) + h(x-t)) \text{ and } S_1(t)h(x) = \frac{1}{2}\int_{x-t}^{x+t} h(x')dx'.$$

Take $R > 0$ and introduce $R_0 = R + \max_{|x| \leq R} T(x)$. Since we have by hypothesis,

$$u_0 \in H^1(-R_0, R_0) \subset L^\infty(-R_0, R_0) \text{ and } u_1 \in L^2(-R_0, R_0),$$

we use the continuity of $x \mapsto T(x)$ to write from (109):

$$u(x,t) \geq S(t)u_0(x) + S_1(t)u_1(x) \geq -\|u_0\|_{L^\infty(-R_0,R_0)} - \sqrt{R_0}\|u_1\|_{L^2(-R_0,R_0)},$$

which concludes the proof of Lemma 4.6. ■

Using Lemma 4.6, we get the following consequences:

Claim 4.7 (A lower bound on solutions of (108)) *If $x_0 \in \mathbb{R}$, and $T_0 \in (0, T(x_0)]$, then:*
(i) For all $y \in (-1,1)$ and $s \geq -\log T_0$, we have $w(y,s) \geq -M_0 e^{-\frac{2s}{p-1}}$, where $M_0 = M(|x_0| + T(x_0))$.
(ii) For all $s \geq -\log T_0$,

$$\begin{aligned} \frac{1}{2}\int_{-1}^{1} |w|^{p+1}\rho dy - CM_0^{p+1}e^{-\frac{2(p+1)s}{p-1}} &\leq \int_{-1}^{1}|w|^p w\rho dy \leq \int_{-1}^{1}|w|^{p+1}\rho dy, \\ (1-2M_0e^{-\frac{2s}{p-1}})\int_{-1}^{1}|w|^{p+1}\rho dy - CM_0e^{-\frac{2s}{p-1}} &\leq \int_{-1}^{1}|w|^p w\rho dy. \end{aligned}$$

Proof:
(i) It follows straightforwardly from Lemma 4.6.
(ii) Consider $x \geq -\log T_0$. The right-hand side inequality is obvious. For the left-hand side inequality, we use (i) to write

$$|z|^p z - |z|^{p+1} \geq -\epsilon_0|z|^p \geq \max(-\frac{1}{2}z^{p+1} - 2^p\epsilon_0^{p+1}, -\epsilon_0(1+z^{p+1}))$$

where $z = w(y,s)$ and $\epsilon_0 = 2M_0e^{-\frac{2s}{p-1}}$. By integration, (ii) follows. ■

In the following, we give the following blow-up criterion for equation (2):

Claim 4.8 (A blow-up criterion for equation (2)) *Consider $W(y,s)$ a solution to equation (108) such that $W(y,s_0)$ is defined for all $|y|<1$ and $\tilde{E}(W(s_0)) < 0$ for some $s_0 \in R$. Then, $W(y,s)$ cannot exist for all $(y,s) \in (-1,1) \times [s_0,\infty)$.*

Proof: The proof is the same as the proof of Theorem 2 in Antonini and Merle [3] (of course, one need to use the Lyapunov functional $\tilde{E}(w)$). ■

Using the Lyapunov functional $\tilde{E}(w)$ together with the estimate in (ii) of Claim 4.7, one can adapt with no difficulty the analysis of our previous papers ([10], [11], [12] and [13], without forgetting the present paper) to equation (2), and get the same results, with the following new feature:
Due to the lower bound of (i) in Claim 4.7, only nonnegative objects appear in the limit at infinity of w_{x_0} when $x_0 \in \mathcal{R}$ (take $\theta(x_0) = 1$ in (12) and (13)) and in the asymptotic decomposition of w_{x_0} when $x_0 \in \mathcal{S}$ (take $e_i = 1$ for all $i = 1, ..., k$ in (20)). This is the main difference with the case of equation (1), where different signs may appear. More precisely, we have the following:

Claim 4.9 (Classification of nonnegative stationary solutions of equation (108)) *Consider $w \in \mathcal{H}_0$ a nonnegative stationary solution of (108). Then*

$$\text{either } w \equiv 0 \text{ or } w(y,s) = \kappa(d,y) \tag{110}$$

for some $d \in (-1,1)$, where $\kappa(d,y)$ is defined in (11).

Remark: It is easy to prove that all stationary solutions of (108) in $\mathcal{H}_0$ are in fact nonnegative, hence characterized by (110).
Proof: If $w \in \mathcal{H}_0$ a nonnegative stationary solution of (108), then it is also a stationary solution of (6). Using Proposition 1 of [13], we see that either $w \equiv 0$ or $w(y,s) = e\kappa(d,y)$ for some $d \in (-1,1)$ and $e = \pm 1$. Since w is nonnegative, we get $e = 1$. ■

Arguing as in [13] (note that the Liouville Theorem 2 and 2' of [13] hold for equation (1) and that only nonnegative solutions are possible), we get that the set $\mathcal{R}$ of non characteristic points is non empty, open and $x \mapsto T(x)$ is C^1 on $\mathcal{R}$ (see page 3 in this paper, see Theorem 1 and the following remark in [13]). In other words, the set $\mathcal{S}$ of characteristic points is closed and

$$\partial\mathcal{S} \subset \mathcal{S}.$$

As a consequence of the fact that only nonnegative solitons appear in the asymptotitc decomposition (20) of w_{x_0} when $x_0 \in \mathcal{S}$, we have the following result which is the main difference with equation (1):

Claim 4.10
(i) For all $x_0 \in \mathcal{S}$, $k(x_0) = 0$ or 1.
(ii) For all $x_0 \in \partial\mathcal{S}$, $k(x_0) = 1$.
(iii) Consider $x_0 \in \partial\mathcal{S}$. If there exists a sequence $x_n \in \mathcal{R}$ converging from the left (resp. the right) to x_0, then x_0 is left-non-characteristic (resp. right-non-characteristic).

Proof:
(i) Proceeding by contradiction, we assume that for some $x_0 \in \mathcal{S}$, we have $k(x_0) \geq 2$. As for equation (1), w_{x_0} can be decomposed as $s \to \infty$ as a sum of decoupled solitons (take

$$e_i = 1 \text{ for all } i = 1, ..., k, \tag{111}$$

in (20)), and we can show that (up to slightly changing the solitons' location), the solitons' centers satisfy the same ODE system (22) as in the case of equation (1).
Therefore, (i) of Proposition 3.1 holds and we have

$$\forall i = 1, ..., k, \ \ e_i = (-1)^{i+1} e_1.$$

Since $k(x_0) \geq 2$, this is in contradiction with (111). Thus, (i) holds.
(ii) Using Lemma 4.4, we see that for all $x_0 \in \partial\mathcal{S}$, $k(x_0) \neq 0$. Using (i), we get the conclusion.
(iii) Consider $x_0 \in \partial\mathcal{S}$. From (ii), we have $k(x_0) = 1$. Applying (ii) of Lemma 4.4, we get the result. This concludes the proof of Claim 4.10. ■

Now, we are ready to give the proof of Theorem 4'.

Proof of Theorem 4': Assume by contradiction that $\mathcal{S} \neq \emptyset$. Since $\mathcal{R} \neq \emptyset$ (see the remark following Theorem 1 in [13]), it follows that $\partial\mathcal{S} \neq \emptyset$. If $x_0 \in \partial\mathcal{S}$, then up to replacing $u(x,t)$ by $u(-x,t)$, we assume that there exists a sequence $x_n \in \mathcal{R} \to x_0$ from the left as $n \to \infty$. Applying Claim 4.10, we see that

$$k(x_0) = 1 \text{ and } x_0 \text{ is left-non-characteristic.} \tag{112}$$

Now, we consider 2 cases.
- If $[x_0,\infty)\subset\mathcal{S}$, then we have from Lemma 4.2 and the positivity of $T(x)$ that

$$\forall x\ge x_0, T(x)=T(x_0)+(x-x_0).$$

Therefore, x_0 is right-non-characteristic, hence non characteristic. Contradiction with the fact that $x_0\in\partial\mathcal{S}\subset\mathcal{S}$.
- Now, if $[x_0,\infty)\not\subset\mathcal{S}$, then we can define $x_1\ge x_0$ maximal such that

$$[x_0,x_1]\subset\mathcal{S}.$$

Since x_1 is maximal, it follows that $x_1\in\partial\mathcal{S}$ and that there exists a sequence $y_n\in\mathcal{R}\to x_1$ from the right as $n\to\infty$. Applying Claim 4.10, we see that

$$k(x_1)=1 \text{ and } x_1 \text{ is right-non-characteristic.} \tag{113}$$

If $x_1=x_0$, then x_0 is non characteristic by (112), which is a contradiction.
If $x_1>x_0$, then applying Lemma 4.2, we see that for some $x^*\in[x_0,x_1]$, we have

$$\forall x\in[x_0,x_1],\ \ T(x)=T(x^*)-|x-x^*|.$$

If $x^*>x_0$, then x_0 is right-non-characteristic, hence non characteristic by (112). Contradiction again.
If $x^*=x_0$, then x_1 is left-non-characteristic, hence non characteristic by (113). This is in contradiction with the fact that $x_1\in\partial\mathcal{S}\subset\mathcal{S}$.
This concludes the proof of Theorem 4'. ■

4.4 Estimates on various norms of the solution localized at characteristic points

Il faudrait rajouter des corrections logarithmiques sur $\|w_{x_0}\|_{L^\infty(-1,1)}$*, en utilisant le fait que* $\|\kappa(d_1(s))\|_{L^\infty(-1,1)}\sim|s|^{-\alpha}$*.....*

If $x_0\in\mathcal{R}$, we know from [11] (Theorem 1.8 page 1132) that

$$\forall s\ge-\log\frac{T(x_0)}{4},\ \ 0<\epsilon_0(N,p)\le\|w_{x_0}(s)\|_{H^1(-1,1)}+\|\partial_s w_{x_0}(s)\|_{L^2(-1,1)}\le K, \tag{114}$$

where K depends only on the norm of initial data, $\delta_0(x_0)$ defined in (4), and on an upper bound on $T(x_0)$ and $1/T(x_0)$. Since we have an upper and a lower bound in (114), this means that the blow-up rate of u near x_0 is of ODE type, i.e. given by $(T(x_0)-t)^{-\frac{2}{p-1}}$.

If $x_0\in\mathcal{S}$, we could only obtain an upper bound in $\mathcal{H}$ (see Proposition 3. page 66 in [12]):

$$\forall s\ge s_0,\ \ \|(w_{x_0}(s),\partial_s w_{x_0}(s))\|_{\mathcal{H}}+\|(w_{x_0}(s),\partial_s w_{x_0}(s))\|_{H^1\times L^2(-\frac12,\frac12)}\le K, \tag{115}$$

where $s_0\ge-\log T(x_0)$ and $C_0>0$. Note that (115) is valid also on $H^1\times L^2(-1+\eta,1-\eta)$, for any $\eta>0$ and that we couldn't obtain information on the whole interval $(-1,1)$ with no weights. Note also that no lower estimate could be derived in our previous work. As a matter of fact, Proposition 6 shows that asymptotically, the solution is a decoupled sum of solitons among them two go to the boundary ± 1. Thus, any norm on $(-1+\eta, 1-\eta)$ where $\eta > 0$ cannot be a good measure for the size of the solution at blow-up (to illustrate this, we give at the end of this subsection a counterexample where $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1+\eta, 1-\eta)} \to 0$ as $s \to \infty$, for any $\eta > 0$). We now claim the following:

Proposition 4.11 *The following assertions are equivalent:*
(a) $x_0 \in \mathcal{S}$.
(b) $\|w_{x_0}(s)\|_{L^{p+1}(-1,1)} \to \infty$ *as* $s \to \infty$.
(c) $\|w_{x_0}(s)\|_{H^1(-1,1)} \to \infty$ *as* $s \to \infty$.

Remark: From this Proposition, we see that near a characteristic point, the blow-up rate of $u(x,t)$ in H^1 is larger that the ODE rate $(T(x_0)-t)^{-\frac{2}{p-1}}$.
Remark: When $x_0 \in \mathcal{S}$, the behavior of $\|w(s)\|_{L^2(-1,1)}$ and $\|\partial_s w(s)\|_{L^2(-1,1)}$ as $s \to \infty$ should be less universal and depends on the power p.
Proof: Clearly, (b) implies (c) from the Sobolev injection and (c) implies (a) by (114). It remains then to prove that (a) implies (b) to conclude.
Let us take $x_0 \in \mathcal{S}$ and prove (b). We claim that it is enough to prove that for some $c_0 > 0$ and for all $\delta \in (0,1)$, there exists $s_0(\delta) \in \mathbb{R}$ such that

$$\forall s \geq s_0, \quad \int_{1-\delta}^{1} |w(y,s)|^{p+1} \rho(y) dy \geq c_0 > 0. \tag{116}$$

Indeed, if (116) is true, then we write for any $\delta \in (0,1)$ and $s \geq s_0(\delta)$,

$$\int_{-1}^{1} |w_{x_0}(y,s)|^{p+1} dy \geq \int_{1-\delta_0}^{1} |w_{x_0}(y,s)|^{p+1} dy \geq \frac{1}{\rho(\delta)} \int_{1-\delta_0}^{1} |w_{x_0}(y,s)|^{p+1} \rho(y) dy \geq \frac{c_0}{\rho(\delta)}.$$

Since $\frac{c_0}{\rho(\delta)} \to \infty$ as $\delta \to 0$, (b) follows. Let us prove (116) then.
Using Proposition 6 and Lemma B.1, we see that for some $e_1 = \pm 1$, we have

$$\|w_{x_0}(s) - e_1 \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s), \cdot)\|_{L^{p+1}_\rho} \to 0 \text{ as } s \to \infty,$$

where

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty, \text{ for all } i = 1, ..., k-1, \tag{117}$$

and

$$\zeta_k(s) \to \infty \text{ as } s \to \infty. \tag{118}$$

Therefore, given $\delta \in (0,1)$, we see that

$$\|w_{x_0}(s) - e_1 \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s), \cdot)\|_{L^{p+1}_\rho(1-\delta,1)} \to 0 \text{ as } s \to \infty. \tag{119}$$

Performing the change of variables $\xi = \tanh y$ and using (117) and (118), we write for δ small enough,

$$
\begin{aligned}
& \| \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s), \cdot) \|_{L^{p+1}_\rho(1-\delta,1)} = \left\| \sum_{i=1}^{k} (-1)^{i+1} \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s)) \right\|_{L^{p+1}(\xi > A(\delta))} \\
\geq\ & \frac{1}{2} \| \cosh^{-\frac{2}{p-1}}(\xi - \zeta_k(s)) \|_{L^{p+1}(\xi > A(\delta))} \\
\geq\ & \frac{1}{4} \| \cosh^{-\frac{2}{p-1}}(\xi - \zeta_k(s)) \|_{L^{p+1}(\mathbb{R})} = \frac{1}{4} \| \cosh^{-\frac{2}{p-1}}(\xi) \|_{L^{p+1}(\mathbb{R})} \equiv c_0
\end{aligned}
$$

where $A(\delta) = \tanh^{-1}(1-\delta)$. Therefore, using (119), we see that for s large enough, (116) holds. Since we already know that (116) implies (b), this concludes the proof of Proposition 4.11. ■

Remark: We give here an example (with odd initial data) where $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-\frac{1}{2}, \frac{1}{2})}$ is not bounded from below by a positive constant:

Claim 4.12 *If the initial data is odd and $u(x,t)$ blows up in finite time, then for any $\eta \in (0,1)$, $\|(w_0(s), \partial_s w_0(s))\|_{H^1 \times L^2(-1+\eta, 1-\eta)} \to 0$ as $s \to \infty$.*

Remark: It is enough to take odd initial data with large plateaus to guarantee that $u(x,t)$ blows up and satisfies the hypotheses of this claim.

Proof: We already know from Proposition 1 that $0 \in \mathcal{S}$. Using Proposition 6, we see that for some $k \geq 2$ and $e_1 = \pm 1$, we have

$$
\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - e_1 \begin{pmatrix} \sum_{i=1}^{k(x_0)} (-1)^{i+1} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty, \tag{120}
$$

where $d_i(s) = -\tanh \zeta_i(s)$ satisfy

$$
\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty, \text{ for all } i = 1, ..., k-1. \tag{121}
$$

Using Lemma B.1, we see that

$$
\sup_{|y|<1} \left| (1-y^2)^{\frac{1}{p-1}} w_0(y,s) - \sum_{i=1}^{k} e_i^* (1-y^2)^{\frac{1}{p-1}} \kappa(d_i(s), y) \right| \to 0 \text{ as } s \to \infty. \tag{122}
$$

We claim that the conclusion follows if we prove the following:

$$
\forall i = 1, ..., k, \ \ |d_i(s)| \to 1 \text{ as } s \to \infty, \tag{123}
$$

since we have from the definition (11) of $\kappa(d,y)$ that

$$
\forall \eta \in (0,1), \ \ \|\kappa(d,y)\|_{L^\infty(-1+\eta, 1-\eta)} \leq \|\kappa(d,y)\|_{H^1(-1+\eta,1-\eta)} \to 0 \text{ as } |d| \to 1. \tag{124}
$$

It remains to prove (123) to conclude.

Proof of (123): If for some $i_0 = 1, ..., k$ and $\delta_0 \in (0,1)$, we have $|d_i(s_n)| \leq \delta_0$ for some sequence $s_n \to \infty$ as $n \to \infty$, then up to extracting a subsequence, we can assume that $d_i(s_n) \to d_0 \in (-1,1)$ as $n \to \infty$. Using (121), we see that if $i \neq i_0$, then $|d_i(s_n)| \to 1$ as $n \to \infty$. Therefore, using (122) and (124), we see that

$$\sup_{|y|<\frac{1+d_0}{2}} |w_0(y,s) - \kappa(d_{i_0}, y)| \to 0 \text{ as } s \to \infty.$$

Since $\kappa(d_{i_0}, y) > 0$ if $|y| < \frac{1+d_0}{2}$, it follows that $w_0(y,s)$ can not be odd, and by the selfsimilar transformation (5), $u(x,t)$ cannot be odd neither, which is a contradiction. Thus, (123) holds and Claim 4.12 is proved. ■

A Continuity with respect to initial data of the blow-up time at a non characteristic point

This section is devoted to the proof of Proposition 2.1. The proof is more or less included in the arguments of the proof of Lemma 2.2 of [13]. We only give here a sketch of the proof (see [13] for more details).

Sketch of the proof of Proposition 2.1: We will prove the continuity in the norm $H^1 \times L^2(\mathbb{R})$ since the result with the norm $H^1 \times L^2(|x| < A_0)$ follows from the finite speed of propagation.
Using the continuity of $u(t_0)$ for $t_0 < T(x_0)$ with respect to initial data, it follows that $T(x_0)$ is lower semi-continuous as a function of initial data.
For the upper continuity, we consider $T_0 > T(x_0)$ to be taken close enough to $T(x_0)$ and aim at proving that $\tilde{u}(x,t)$ blows up in finite time $\tilde{T}(x_0) < T_0$, where $\tilde{u}(x,t)$ is the solution of equation (1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ close enough to (u_0, u_1).

Up to changing u in $-u$, we know from (13) that for some $d_0 \in (-1,1)$ and $\delta_0 > 0$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \kappa(d_0, .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y|<1+\delta_0)} \to 0 \text{ as } s \to \infty. \tag{125}$$

Consider $s_0 < 0$ to be fixed later and introduce $t_0 < T(x_0)$ such that $\frac{T(x_0)-t_0}{T_0-t_0} = 1 - e^{s_0}$. Using the selfsimilar transformation (5) and (125), we see that

$$w_{x_0,T_0}(y, -\log(T_0 - t_0)) = (1 - e^{s_0})^{-\frac{2}{p-1}} w_{x_0}\left(\frac{y}{1 - e^{s_0}}, -\log(T(x_0) - t_0)\right),$$

$$\left\| \begin{pmatrix} w_{x_0,T_0}(-\log(T_0 - t_0)) \\ \partial_s w_{x_0,T_0}(-\log(T_0 - t_0)) \end{pmatrix} - \begin{pmatrix} w_-(s_0) \\ \partial_s w_-(s_0) \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to 0 \tag{126}$$

as $T_0 \to T(x_0)$, where $w_-(y,s) = \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1-e^{s_0}+d_0 y)^{\frac{2}{p-1}}}$ is a particular solution of equation (6).
Since $E(w_-(s_0)) < 0$ for some $s_0 < 0$ from Appendix B in [13], we see from (126) that for T_0 close enough to $T(x_0)$ and t_0 defined above, we have

$$E(w_{x_0,T_0}(-\log(T_0 - t_0))) < 0. \tag{127}$$

Using the blow-up criterion of Antonini and Merle (see Theorem 2 in [3]), we see that w_{x_0,T_0} cannot be defined for all $(y,s) \in (-1,1) \times [-\log T_0, \infty)$, which means that u blows up in finite time and that $T(x_0) < T_0$. This yields the upper semi-continuity and concludes the proof of Proposition 2.1. ■

B Estimates on the quadratic form φ

This section is devoted to the proof of Lemma 3.5. We proceed in two subsections:
- in the first subsection, we give some preliminary results, in particular, we change the problem to the ξ variable, where $y = \tanh \xi$.
- in the second subsection, we give the proof of Lemma 3.5.

B.1 Preliminaries and formulation in the ξ variable with $y = \tanh \xi$

We first recall the following result from [12].

Claim B.1 *(i)* **(A Hardy-Sobolev type identity)** *For all $h \in \mathcal{H}_0$, it holds that*

$$\|h\|_{L^2_{\frac{\rho}{1-y^2}}} + \|h\|_{L^{p+1}_\rho} + \|h(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} \le C\|h\|_{\mathcal{H}_0}.$$

(ii) **(Boundedness of $\kappa(d,y)$ in several norms)** *For all $d \in (-1,1)$, it holds that*

$$\|\kappa(d,y)\|_{L^{p+1}_\rho} + \|\kappa(d,y)(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} \le C\|\kappa(d,y)\|_{\mathcal{H}_0} \le CE(\kappa_0).$$

Proof: For (i), see Lemma 2.2 page 51 in [12]. For (ii), use (i) and identity (49) page 59 in [12]. ■

To prove estimates about φ, we take advantage of the decoupling in the solitons' sum (see (44)) and use information we proved in [12] for the 1-soliton version of φ defined for all $d \in (-1,1)$, r and r' in $\mathcal{H}$ by

$$\varphi_d(r, \mathbf{r}) = \int_{-1}^1 \left(r_1' \mathbf{r}_1' (1-y^2) - \left(p\kappa(d)^{p-1} - \frac{2(p+1)}{(p-1)^2} \right) r_1 \mathbf{r}_1 + r_2 \mathbf{r}_2 \right) \rho dy \qquad (128)$$

and satisfying (see estimate (138) page 91 in [12]):

$$|\varphi_d(r, \mathbf{r})| \le C\|r\|_{\mathcal{H}} \|\mathbf{r}\|_{\mathcal{H}}. \qquad (129)$$

It happens that the proof is clearer in the ξ variable where

$$y = \tanh \xi.$$

More precisely, let us introduce the transformations

$$r(y) \mapsto \bar{T} r(\xi) = \bar{r}(\xi) = r(y)(1-y^2)^{\frac{1}{p-1}} \text{ and } r(y) \mapsto \hat{T} r(\xi) = \hat{r}(\xi) = r(y)(1-y^2)^{\frac{1}{p-1}+\frac{1}{2}}, \qquad (130)$$

and for $r = (r_1, r_2)$, the notation

$$\tilde{T}(r) = \tilde{r} = \begin{pmatrix} \bar{r}_1 \\ \hat{r}_2 \end{pmatrix} = \begin{pmatrix} \bar{T}(r_1) \\ \hat{T}(r_2) \end{pmatrix}.$$

In the following claim, we transform φ and φ_d in the new set of variables. Let us first introduce the quadratic forms (where $d \in (-1,1)$):

$$\bar{\varphi}_d(q, \mathbf{q}) = \int_{\mathbb{R}} \left(q_1' \mathbf{q}_1' + \beta_d(\xi) q_1 \mathbf{q}_1 + q_2 \mathbf{q}_2 \right) d\xi \qquad (131)$$

$$\bar{\varphi}(q, \mathbf{q}) = \int_{\mathbb{R}} \left(q_1' \mathbf{q}_1' + \beta(\xi, s) q_1 \mathbf{q}_1 + q_2 \mathbf{q}_2 \right) d\xi \qquad (132)$$

where (using (11))

$$\begin{aligned} \beta_d(\xi) &= \frac{4}{(p-1)^2} - p(\bar\kappa(d,y))^{p-1} = \frac{4}{(p-1)^2} - p\bar\kappa_0(\xi-\zeta)^{p-1} \text{ with } d=-\tanh\zeta, \\ \bar\kappa_0(\xi) &= \kappa_0 \cosh^{-\frac{2}{p-1}}(\xi), \qquad (133) \\ \beta(\xi,s) &= \frac{4}{(p-1)^2} - p|\bar K(\xi,s)|^{p-1} = \frac{4}{(p-1)^2} - p\left|\sum_{i=1}^k e_i\bar\kappa_0(\xi-\zeta_i(s))\right|^{p-1}. \qquad (134) \end{aligned}$$

In the following claim, we give the effect of the new transformation:

Claim B.2
(i) There exists $C_0>0$ such that for all $r\in\mathcal{H}$, we have

$$\frac{1}{C_0}\|r\|_{\mathcal{H}} \le \|\tilde r\|_{H^1\times L^2(\mathbb{R})} \le C_0\|r\|_{\mathcal{H}}.$$

(ii) If $r_1\in\mathcal{H}_0$, then $(1-y^2)\bar{\mathcal{T}}\left(\mathcal{L}r_1 - \frac{2(p+1)}{(p-1)^2}r_1\right) = \left(\partial_\xi^2\bar r_1 - \frac{4}{(p-1)^2}\bar r_1\right)$.
(iii) For all r, $\mathbf{r}$ in $\mathcal{H}$ and $d\in(-1,1)$, we have

$$\varphi(r,\mathbf{r}) = \bar\varphi(\tilde r,\tilde{\mathbf{r}}) \text{ and } \varphi_d(r,\mathbf{r}) = \bar\varphi_d(\tilde r,\tilde{\mathbf{r}})$$

where $\bar\varphi$ and $\bar\varphi_d$ are introduced in (132) and (131).

Proof:
(i) Consider $r=(r_1,r_2)\in\mathcal{H}$. Using (130), we first write

$$\int_{\mathbb{R}} \bar r_1(\xi)^2 d\xi = \int_{-1}^1 r_1(y)^2\frac{\rho(y)}{1-y^2}dy \text{ and } \int_{\mathbb{R}}\hat r_2(\xi)^2d\xi = \int_{-1}^1 r_2(y)^2\rho(y)dy. \qquad (135)$$

Using Lemma B.1, we obtain

$$\|r_1\|_{L^2_\rho}^2 \le \int_{\mathbb{R}} \bar r_1(\xi)^2 d\xi \le \|r\|_{\mathcal{H}}^2. \qquad (136)$$

Now, using again (130), we write

$$\partial_\xi\bar r_1(\xi) = \partial_y r_1(y)(1-y^2)^{\frac{1}{p-1}+1} - \frac{2y}{p-1}(1-y^2)^{\frac{1}{p-1}}r_1(y),$$

therefore,

$$\begin{aligned} |\partial_\xi\bar r_1|^2 &\le 2|\partial_y r_1|^2\rho(1-y^2)^2 + C|r_1|^2\rho, \\ |\partial_y r_1|^2\rho(1-y^2)^2 &\le 2|\partial_\xi\bar r_1|^2 + C|\bar r_1|^2. \end{aligned}$$

Integrating this and using Lemma B.1, we write

$$\begin{aligned} &\int_{\mathbb{R}}|\partial_\xi\bar r_1|^2d\xi \le 2\int_{-1}^1|\partial_y r_1|^2\rho(1-y^2)dy + C\int_{-1}^1 r_1^2\frac{\rho}{1-y^2}dy \le C\|r\|_{\mathcal{H}}^2, \\ &\int_{-1}^1|\partial_y r_1|^2\rho(1-y^2)dy \le 2\int_{\mathbb{R}}|\partial_\xi\bar r_1|^2d\xi + C\int_{\mathbb{R}}|\bar r_1|^2d\xi. \qquad (137) \end{aligned}$$

Gathering (135), (136) and (137), we conclude the proof of (i).

(ii) See page 60 in [12].

(iii) We only prove the estimate for φ since it is even easier for φ_d. Using the definitions (50), (7) and (42) of φ, $\mathcal{L}$ and ψ, integration by parts and the change of variables (130), we write

$$\begin{aligned}\varphi(r,\mathbf{r}) &= \int_{-1}^{1}\left[-\mathcal{L}r_1.\mathbf{r}_1 - \psi r_1\mathbf{r}_1 + r_2\mathbf{r}_2\right]\rho dy \\ &= \int_{-1}^{1}(-\mathcal{L}r_1 + \frac{2(p+1)}{(p-1)^2}r_1)\mathbf{r}_1\rho dy - p\int_{-1}^{1} r_1\mathbf{r}_1|K|^{p-1}\rho dy + \int_{-1}^{1} r_2\mathbf{r}_2\rho dy \\ &= \int_{\mathbb{R}}(1-y^2)\bar{T}(-\mathcal{L}r_1 + \frac{2(p+1)}{(p-1)^2}r_1)\bar{\mathbf{r}}_1 d\xi - p\int_{\mathbb{R}}\bar{r}_1\bar{\mathbf{r}}_1|\bar{K}|^{p-1}d\xi + \int_{\mathbb{R}}\hat{r}_2\hat{\mathbf{r}}_2 d\xi\end{aligned}$$

Using (ii) and integration by parts, we see that

$$\begin{aligned}\varphi(r,\mathbf{r}) &= -\int_{\mathbb{R}}\left(\partial_\xi^2\bar{r}_1 - \frac{4}{(p-1)^2}\bar{r}_1\right)\bar{\mathbf{r}}_1 d\xi - p\int_{\mathbb{R}}\bar{r}_1\bar{\mathbf{r}}_1|\bar{K}|^{p-1}d\xi + \int_{\mathbb{R}}\hat{r}_2\hat{\mathbf{r}}_2 d\xi \\ &= \bar{\varphi}(\tilde{r},\tilde{\mathbf{r}})\end{aligned}$$

where $\bar{\varphi}$ is introduced in (132). This concludes the proof of Claim B.2. ∎

As we said earlier, we take advantage of the decoupling in the solitons' sum. In the following claim, we give a localized estimate coming from an identity we proved in [12] for $\bar{\varphi}_d$, the 1-soliton version of φ defined in (128), then we derive a global estimate for $\bar{\varphi}$.

Claim B.3 (Identity for φ_d)
(i) There exist $\epsilon_0 > 0$ and $A_0 > 0$ such that for all $A > A_0$, $d \in (-1,1)$ and $q \in H^1\times L^2(\mathbb{R})$, we have

$$\bar{\varphi}_d(q\sqrt{\chi_{A,d}}, q\sqrt{\chi_{A,d}}) \geq \epsilon_0\|q\sqrt{\chi_{A,d}}\|^2_{H^1\times L^2} - \frac{\epsilon_0}{8k}\|q\|^2_{H^1\times L^2} - \sum_{\lambda=0}^{1}(\tilde{T}^{-1}(q))^2$$

where $\chi_{A,d}(\xi) = \chi_{1,0}(\frac{\xi-\zeta}{A})$, $\tanh\zeta = -d$ and $\chi_{1,0} \in C^\infty(\mathbb{R},[0,1])$ is even, decreasing for $\xi > 0$ with $\chi_{1,0}(\xi) = 1$ if $|\xi| < 1$ and $\chi_{1,0}(\xi) = 0$ if $|\xi| > 2$.
(ii) There exists $\epsilon_2 > 0$ such that for s large enough and for all $q \in H^1\times L^2$, we have

$$\bar{\varphi}(q,q) \geq \epsilon_2\|q\|^2_{H^1\times L^2} - \frac{1}{\epsilon_2}\sum_{i=1}^{k}\sum_{\lambda=1}^{2}\pi_\lambda^{d_i}(\tilde{T}^{-1}(q))^2$$

Proof:
(i) Consider some $d \in (-1,1)$ and $r \in \mathcal{H}$. On the one hand, we write from Proposition 4.7 page 90 in [12],

$$\varphi_d(r_{-,d}, r_{-,d}) \geq 2\epsilon_1\|r\|^2_{\mathcal{H}} - \frac{1}{\epsilon_1}\sum_{\lambda=0}^{1}|\pi_1^d(r)|^2 \tag{138}$$

for some $\epsilon_1 > 0$ where $r^d_- = \pi^d_-(r)$ defined in (57). On the other hand, using the continuity of φ_d stated in (129) and (46), we write

$$\begin{aligned}\varphi_d(r_{-,d}, r_{-,d}) &\leq \varphi_d(r,r) + C\sum_{\lambda=0}^{1} |\pi^d_\lambda(r)|^2 + C\|r\|_{\mathcal{H}} \sum_{\lambda=0}^{1} |\pi^d_\lambda(r)| \\ &\leq \varphi_d(r,r) + \frac{C}{\epsilon_1}\sum_{\lambda=0}^{1} |\pi^d_\lambda(r)|^2 + \epsilon_1 \|r\|_{\mathcal{H}}.\end{aligned}$$

Using (138), we see that

$$\varphi_d(r,r) \geq \epsilon_0 \|r\|^2_{\mathcal{H}} - \sum_{\lambda=0}^{1} \pi^d_\lambda(r)^2.$$

Using the ξ framework and Claim B.2, we get for all $q \in H^1 \times L^2(\mathbb{R})$,

$$\bar{\varphi}_d(q,q) \geq \epsilon_0 \|q\|^2_{H^1\times L^2(\mathbb{R})} - \sum_{\lambda=0}^{1} \pi^d_\lambda(\tilde{T}^{-1}(q))^2.$$

Now, we claim that (ii) follows from the fact that for all $d \in (-1,1)$ and $\lambda = 0$ or 1, we have

$$\forall u \in H^1 \times L^2(\mathbb{R}),\ \ |\pi^d_\lambda(\tilde{T}^{-1}(u))| \leq C\int \bar{\kappa}_0(\xi - \zeta)|(|u_1(\xi)| + |u_2(\xi)|)d\xi \text{ where } d = -\tanh\xi. \tag{139}$$

Indeed, consider $q \in H^1 \times L^2$, $d \in (-1,1)$, $A > 0$ and $\lambda = 0$ or 1. Taking

$$u = \tilde{T}^{-1}(q(1 - \sqrt{\chi_{A,d}})),$$

using the Cauchy-Schwartz inequality and performing the change of variables $z = \xi - \zeta$, we see that

$$\begin{aligned}|\pi^d_\lambda(\tilde{T}^{-1}(q(1-\sqrt{\chi_{A,d}})))| &\leq C\int \bar{\kappa}_0(\xi-\zeta)(1-\sqrt{\chi_{A,d}})(|q_1(\xi)| + |q_2(\xi)|)d\xi \\ &\leq C\left(\int \bar{\kappa}_0(z)^2(1-\sqrt{\chi_{A,0}})^2 dz\right)^{1/2} \|q\|_{H^1\times L^2}.\end{aligned}$$

Using Lebesgue's theorem, we find $A_0 > 0$ such that if $A \geq A_0$, then

$$|\pi^d_\lambda(\tilde{T}^{-1}(q(1-\sqrt{\chi_{A,d}})))| \leq \sqrt{\frac{\epsilon_0}{16k}}\|q\|_{H^1\times L^2}$$

(uniformly in $d \in (-1,1)$ of course). Since π^d_λ is linear, this gives

$$|\pi^d_\lambda(\tilde{T}^{-1}(q\sqrt{\chi_{A,d}}))|^2 \leq 2|\pi^d_\lambda(\tilde{T}^{-1}(q))|^2 + \frac{\epsilon_0}{8k}\|q\|^2_{H^1\times L^2}.$$

Using (i) with $q\sqrt{\chi_{A,d}}$, (ii) follows. It remains to prove (139) to finish the proof of Claim B.3.

Proof of (139): Consider $d \in (-1,1)$, $\lambda = 0$ or 1 and $u \in H^1 \times L^2$. If we introduce $r = \tilde{T}^{-1}(q)$ which is in $\mathcal{H}$ by (i) of Claim B.2, then we have from (36) and integration by parts

$$\pi_\lambda^d(r) = \int_{-1}^1 \left[(-\mathcal{L}W_{\lambda,1}(d) + W_{\lambda,1}(d))r_1 + W_{\lambda,2}(d)r_2\right]\rho(y)dy. \tag{140}$$

Since we have from (38), (39) and (11)

$$W_{\lambda,2}(d,y) \le C\kappa(d,y) \text{ and } |-\mathcal{L}W_{\lambda,1}(d,y) + W_{\lambda,1}(d,y)| \le C\frac{\kappa(d,y)}{1-y^2},$$

we get from (140) and the transformation (130)

$$\begin{aligned} |\pi_\lambda^d(r)| &\le C\int_{-1}^1 \kappa(d,y)|r_1(y)|\frac{\rho(y)}{1-y^2}dy + C\int_{-1}^1 \kappa(d,y)|r_2(y)|\rho(y)dy \\ &\le C\int \bar{\kappa}(d,\xi)|u_1(\xi)|d\xi + \int \hat{\kappa}(d,\xi)|u_2(\xi)|d\xi. \end{aligned}$$

Since we have from (130) and (133),

$$\hat{\kappa}(d,\xi) \le \bar{\kappa}(d,\xi) = \bar{\kappa}_0 \cosh^{-\frac{2}{p-1}}(\xi - \zeta) = \kappa_0(\xi-\zeta) \text{ with } d = -\tanh,$$

(139) follows. This concludes the proof of (i) in Claim B.3.

(ii) Introducing the notation

$$\chi_i = \chi_{A,d_i}(\xi) = \chi_{1,0}\left(\frac{\xi - \zeta_i}{A}\right)$$

and using (132), we write

$$\begin{aligned} \bar{\varphi}(q,q) &= \int (\partial_\xi q_1)^2 + \frac{2(p+1)}{(p-1)^2}\int q_1^2 + \int q_2^2 - p\int |\bar{K}|^{p-1}q_1^2 \\ &= \sum_{j=1}^k \left[\int (\partial_\xi q_1)^2\chi_j + \frac{2(p+1)}{(p-1)^2}\int q_1^2\chi_j + \int q_2^2\chi_j - p\int |\bar{K}|^{p-1}q_1^2\chi_j\right] \\ &+ \int (\partial_\xi q_1)^2(1 - \sum_{j=1}^k \chi_j) + \frac{2(p+1)}{(p-1)^2}\int q_1^2(1-\sum_{j=1}^k \chi_j) + \int q_2^2(1-\sum_{j=1}^k\chi_j) \\ &- p\int |\bar{K}|^{p-1}q_1^2(1-\sum_{j=1}^k\chi_j) \\ &= \sum_{j=1}^k \bar{\varphi}(q\sqrt{\chi_j}, q\sqrt{\chi_j}) + \bar{\varphi}\left(q\sqrt{1-\sum_{j=1}^k\chi_j}, q\sqrt{1-\sum_{j=1}^k\chi_j}\right) + I_1(s) \end{aligned} \tag{141}$$

where

$$\begin{aligned} I_1(s) &= -\sum_{j=1}^k \left\{\int q_1^2\left(\partial_\xi\sqrt{\chi_j}\right)^2 - 2\int q_1\partial_\xi q_1\sqrt{\chi_j}\partial_\xi\sqrt{\chi_j}\right\} \\ &- \int q_1^2\left(\partial_\xi\sqrt{1-\sum_{j=1}^k\chi_j}\right)^2 - 2\int q_1\partial_\xi q_1\sqrt{1-\sum_{j=1}^k\chi_j}\,\partial_\xi\sqrt{1-\sum_{j=1}^k\chi_j}. \end{aligned} \tag{142}$$

Using the definitions (132) and (131) of $\bar\varphi$ and $\bar\varphi_d$, we write

$$\bar\varphi\left(q\sqrt{\chi_j}, q\sqrt{\chi_j}\right) = \bar\varphi_{d_i(s)}\left(q\sqrt{\chi_j}, q\sqrt{\chi_j}\right) - I_2(s),$$
$$\bar\varphi\left(q\sqrt{1-\sum_{j=1}^k \chi_j}, q\sqrt{1-\sum_{j=1}^k \chi_j}\right) \ge c_0(p)\left\|q\sqrt{1-\sum_{j=1}^k \chi_j}\right\|^2_{H^1\times L^2} - I_3(s)$$

where $c_0(p) = \min\left(1, \frac{2(p+1)}{(p-1)^2}\right)$,

$$I_2(s) = p\int\left(|\bar K(\xi,s)|^{p-1} - \bar\kappa_0(\xi-\zeta_i(s))^{p-1}\right) q_1^2\chi_j \text{ and } I_3(s) = p\int |\bar K|^{p-1} q_1^2 (1-\sum_{j=1}^k \chi_j).$$

Since we have from Claim B.3 and (134)

$$|\partial_\xi \chi_j| \le C/A, \qquad \|\left(|\bar K(\xi,s)|^{p-1} - \bar\kappa_0(\xi-\zeta_i(s))^{p-1}\right)\chi_j\|_{L^\infty} \le C(A)J(s), \tag{143}$$
$$\text{and} \qquad \||\bar K|^{p-1}(1-\sum_{j=1}^k \chi_j)\|_{L^\infty} \le Ce^{-2A}$$

where $J(s)\to 0$ is defined in (23), it follows that for A and s large enough,

$$|I_1(s)| + |I_2(s)| + |I_3(s)| \le \frac{C}{A}\|q_1\|^2_{H^1}. \tag{144}$$

Therefore, using (141), (142), (143), (143), (144) and Claim B.3, we write for A and s large enough,

$$\begin{aligned}\bar\varphi(q,q) \ \ge\ & \epsilon_0\sum_{j=1}^k \|q\sqrt{\chi_j}\|^2_{H^1\times L^2} + c_0(p)\left\|q\sqrt{1-\sum_{j=1}^k\chi_j}\right\|^2_{H^1\times L^2} - \frac{\epsilon_0}{4}\|q\|^2_{H^1\times L^2} \\ & - \sum_{j=1}^k\sum_{\lambda=0}^1 |\pi_\lambda^{d_j(s)}(\tilde T^{-1}(q))|^2.\end{aligned} \tag{145}$$

Since (141) holds with $\bar\varphi$ replaced by the canonical inner product of $H^1\times L^2$, we use (144) to write

$$\|q\|^2_{H^1\times L^2} \le \sum_{j=1}^k \|q\sqrt{\chi_j}\|^2_{H^1\times L^2} + \left\|q\sqrt{1-\sum_{j=1}^k\chi_j}\right\|^2_{H^1\times L^2} + \frac{C}{A}\|q\|^2_{H^1\times L^2}$$

hence for A and s large enough,

$$\|q\|^2_{H^1\times L^2} \le 2\sum_{j=1}^k \|q\sqrt{\chi_j}\|^2_{H^1\times L^2} + 2\left\|q\sqrt{1-\sum_{j=1}^k\chi_j}\right\|^2_{H^1\times L^2}$$

and (ii) follows from (145). This concludes the proof of Claim B.3. ■

B.2 Proof of Lemma 3.5

Now we are ready to start the proof of Lemma 3.5.

Proof of Lemma 3.5:

(i) Since $\psi(y,s) = p|K(y,s)|^{p-1} - \frac{2(p+1)}{(p-1)^2}$ with $K(y,s) = \sum_{j=1}^k e_j\kappa(d_j(s),y)$ by (42), we split $\varphi(r,r')$ into 2 parts as follows:
- We first use the definition (9) of the norm in $\mathcal{H}$ to write

$$\left|\int_{-1}^1 \left(\partial_y r_1 \partial_y r_1'(1-y^2) + \frac{2(p+1)}{(p-1)^2} r_1 r_1' + r_2 r_2'\right)\rho dy\right| \le C\|r\|_{\mathcal{H}}\|r'\|_{\mathcal{H}}.$$

- Then, using Claim B.1, we write

$$\left|\int_{-1}^1 |K(s)|^{p-1} r_1 r_1' \rho dy\right| \le C\int_{-1}^1 \frac{|r_1||r_1'|}{1-y^2}\rho dy \le C\|r_1\|_{L^2_{\frac{\rho}{1-y^2}}}\|r_1'\|_{L^2_{\frac{\rho}{1-y^2}}} \le C\|r\|_{\mathcal{H}}\|r'\|_{\mathcal{H}}.$$

Using these two bounds gives the conclusion of (i).

(ii) *Proof of (52)*: It immediately follows from (49), (55) and (56).

Proof of (53): The right inequality follows from (i). For the left inequality, we use Claim B.2 to translate (ii) of Claim B.3 back to the y variable:
for some $\epsilon_2 > 0$, for s large enough and for all $r \in \mathcal{H}$,

$$\varphi(r,r) \ge \epsilon_2\|r\|_{\mathcal{H}}^2 - \frac{1}{\epsilon_2}\sum_{i=1}^k\sum_{\lambda=0}^1 |\pi_\lambda^{d_i(s)}(r)|^2. \tag{146}$$

Using (146) with $r(y) = q_-(y,s)$, we write

$$\varphi(q_-,q_-) \ge \epsilon_2\|q_-\|_{\mathcal{H}}^2 - \frac{1}{\epsilon_2}\sum_{i=1}^k\sum_{\lambda=0}^1 |\pi_\lambda^{d_i}(q_-)|^2. \tag{147}$$

Since $\pi_\lambda^{d_i}(F_1^{d_i}) = \delta_{\lambda,1}$ by (48), we use (52) to write

$$\pi_\lambda^{d_i}(q_-) = \pi_\lambda^{d_i}(q) - \sum_{j=1}^k \pi_1^{d_j}(q)\pi_\lambda^{d_i}(F_1^{d_j}) = \sum_{j\neq i}\pi_1^{d_j}(q)\pi_\lambda^{d_i}(F_1^{d_j}). \tag{148}$$

Using (55), (36) and (40), we see that

$$|\alpha_1^j| = |\pi_1^{d_j}(q)| = |\phi(W_\lambda(d_j),q)| \le \|W_\lambda(d_j)\|_{\mathcal{H}}\|q\|_{\mathcal{H}} \le C\|q\|_{\mathcal{H}}. \tag{149}$$

Using (36), integration by parts and the definition (7) of $\mathcal{L}$, we write

$$\begin{aligned}
\pi_\lambda^{d_i}(F_\mu(d_j)) &= \int_{-1}^1 \left(W_{\lambda,1}(d_i)\partial_d\kappa(d_j) + \partial_y W_{\lambda,1}(d_i)\partial_y F_{\mu,1}(d_j)(1-y^2)\right)\rho dy \\
&+ \int_{-1}^1 W_{\lambda,2}(d_i)F_{\mu,2}(d_j)\rho dy \\
&= \int_{-1}^1 \left(-\mathcal{L}W_{\lambda,1}(d_i) + W_{\lambda,1}(d_i)\right)F_{\mu,1}(d_j)\rho dy + \int_{-1}^1 W_{\lambda,2}(d_i)F_{\mu,2}(d_j)\rho dy.
\end{aligned}$$

Since we have from the definitions (11), (38), (39) and (45) of $\kappa(d,y)$, $W_\lambda(d,y)$ and $F_\mu(d,y)$, for all $(d,y)\in(-1,1)^2$,

$$|W_{\lambda,2}(d,y)|+|\mathcal{L}W_{\lambda,1}(d,y)-W_{\lambda,1}(d,y)|\le C\frac{\kappa(d,y)}{1-y^2}\text{ and }|F_{\mu,l}(d,y)|\le C\kappa(d,y),\quad(150)$$

we use (i) of Lemma E.1 to write for s large enough,

$$\left|\pi_\lambda^{d_i}(F_\mu(d_i))\right|\le C\int_{-1}^1\kappa(d_i)\kappa(d_j)\frac{\rho}{1-y^2}dy\le C|\zeta_i-\zeta_j|e^{-\frac{2}{p-1}|\zeta_i-\zeta_j|}\le C\bar J(s)\qquad(151)$$

by definition (55) of $\bar J$. Using (148) and (149), we see that for s large enough,

$$|\pi_\lambda^{d_i}(q_-)|\le C\bar J\|q\|_{\mathcal H}.$$

Using (147), we see that the left inequality in (53) follows.

Proof of (54): The right inequality follows from (149), (51) and (53). For the left inequality in (54), we write from the bilinearity of φ, (52), (51) and (46)

$$\begin{aligned}\varphi(q_-,q_-)&\ge\varphi(q,q)-C\sum_{i=1}^k|\alpha_1^i|^2-C\|q\|_{\mathcal H}\sum_{i=1}^k|\alpha_1^i|\\&\ge\varphi(q,q)-\frac{C}{\epsilon_2}\sum_{i=1}^k|\alpha_1^i|^2-\frac{\epsilon_2}{2}\|q\|_{\mathcal H}^2\end{aligned}$$

where $\epsilon_1>0$ is introduced in (146). Using (146) with $r=q$, we get the left inequality in (54). This concludes the proof of Lemma 3.5. ■.

C Projection of equation (42) on the different modes

We prove Lemma 3.6 here. We proceed in 3 parts to prove (i), (ii), and finally (iii).

Proof of (i): Projection of equation (42) on $F_1(d_i(s),\cdot)$ and $F_0(d_i(s),\cdot)$

We prove (i) of Lemma 3.6 here. Fixing some $i=1,...,k$ and projecting equation (42) with the projector π_λ^d (36) (where $\lambda=0$ or 1), we write (putting on top the main terms)

$$\begin{aligned}\pi_\lambda^{d_i(s)}(\partial_s q)&=\pi_\lambda^{d_i(s)}\left(L_{d_i(s)}(q)\right)-e_id_i'(s)\pi_\lambda^{d_i(s)}\begin{pmatrix}\partial_d\kappa(d_i(s),y)\\0\end{pmatrix}+\pi_\lambda^{d_i(s)}\begin{pmatrix}0\\R\end{pmatrix}\\&+\pi_\lambda^{d_i(s)}\begin{pmatrix}0\\f(q_1)\end{pmatrix}+\pi_\lambda^{d_i(s)}\begin{pmatrix}0\\V_i(y,s)q_1\end{pmatrix}-\sum_{j\neq i}^k e_jd_j'(s)\pi_\lambda^{d_i(s)}\begin{pmatrix}\partial_d\kappa(d_j(s),y)\\0\end{pmatrix}\end{aligned}\quad(152)$$

Note that we expand the operator $L(q)$ according to (43). In the following, we handle each term of (152) in order to finish the proof of (58).

- Using the analysis performed in Claim 5.3 page 104 and Step 1 page 105 in [12] for the case of one soliton ($k = 1$), we immediately get the following estimates:

$$\begin{aligned}
\left|\pi_\lambda^{d_i(s)}(\partial_s q) - {\alpha_1^i}'(s)\right| &\le \frac{C_0}{1-(d_i(s))^2}|d_i'(s)|\|q(s)\|_{\mathcal{H}} \le C_0|\zeta_i'(s)|\|q(s)\|_{\mathcal{H}},\\
\pi_\lambda^{d_i(s)}\left(L_{d_i(s)}(q)\right) &= \lambda \alpha_1^i(s),\\
d_i'(s)\pi_\lambda^{d_i(s)}\begin{pmatrix}\partial_d\kappa(d_i(s),y)\\ 0\end{pmatrix} &= -\frac{2\kappa_0}{(p-1)}\frac{d_i'(s)}{(1-(d_i(s))^2)} = \frac{2\kappa_0}{(p-1)}\zeta_i'(s)\delta_{\lambda,0},\\
|f(q_1)| &\le C\delta_{\{p\ge 2\}}|q_1|^p + C|K|^{p-2}|q_1|^2 \qquad (153)
\end{aligned}$$

(recall that $d_i(s) = -\tanh \zeta_i(s)$, hence $\zeta_i'(s) = -\frac{d_i'(s)}{1-d_i(s)^2}$).

- Since we have from the definitions (42), (11) and (38) of R, $\kappa(d,y)$ and $W_{\lambda,2}(d,y)$

$$|R(y,s)| \le C\sum_{j\ne i}\kappa(d_j,y)^p + \kappa(d_i,y)^{p-1}\kappa(d_j,y) \text{ and } |W_{\lambda,2}(d_i,y)| \le C\kappa(d_i,y), \qquad (154)$$

we use (36), (i) of Lemma E.1 and the definition (23) of $J(s)$ to write as $s \to \infty$

$$\begin{aligned}
\left|\pi_\lambda^{d_i}\begin{pmatrix}0\\ R\end{pmatrix}\right| = \left|\int W_{\lambda,2}(d_i)R\rho dy\right| &\le C\sum_{j\ne i}\int_{-1}^1 \kappa(d_i)\kappa(d_j)^p\rho dy + \int_{-1}^1\kappa(d_i)^p\kappa(d_j)\rho dy\\
&\le \sum_{j\ne i} e^{-\frac{2}{p-1}|\zeta_i-\zeta_j|} \le CJ. \qquad (155)
\end{aligned}$$

- Using (36), (154), (153) and the Hölder inequality, we write

$$\begin{aligned}
\left|\pi_\lambda^{d_i}\begin{pmatrix}0\\ f(q_1)\end{pmatrix}\right| &\le C\int_{-1}^1\kappa(d_i)|f(q_1)|\rho dy\\
&\le C\delta_{p\ge 2}\int_{-1}^1\kappa(d_i)|q_1|^p\rho dy + C\int_{-1}^1\kappa(d_i)|K|^{p-2}|q_1|^2\rho dy\\
&\le C\delta_{p\ge 2}\|\kappa(d_i)\|_{L_\rho^{p+1}}\|q_1\|^p_{L_\rho^{p+1}} + CJ_i\|q_1(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty}
\end{aligned}$$

where

$$J_i = \int_{-1}^1 \kappa(d_i)|K|^{p-2}dy. \qquad (156)$$

Using (v) of Lemma E.1 and Claim B.1, we see that

$$\left|\pi_\lambda^{d_i}\begin{pmatrix}0\\ f(q_1)\end{pmatrix}\right| \le C\int_{-1}^1\kappa(d_i)|f(q_1)|\rho dy \le C\delta_{p\ge 2}\|q\|^p_{\mathcal{H}} + C\|q\|^2_{\mathcal{H}} \le C\|q\|^2_{\mathcal{H}} \qquad (157)$$

where we use (41) in the last step.

- We claim that

$$\left|\pi_\lambda^{d_i}\begin{pmatrix}0\\ V_iq_1\end{pmatrix}\right| \le C\|q\|^2_{\mathcal{H}} + o(J) \text{ as } s\to\infty \qquad (158)$$

and

$$|V_i(y,s)| \le C1_{\{y_{i-1}<y<y_i\}}\sum_{l\neq i}\kappa(d_i,y)^{p-2}\kappa(d_l,y) + C\sum_{l\neq i}\kappa(d_l,y)^{p-1}1_{\{y_{l-1}<y<y_l\}} \qquad (159)$$

where $y_0 = -1$, $y_j = \tanh(\frac{\zeta_j+\zeta_{j+1}}{2})$ if $j = 1,..,k-1$, $y_k = 1$ and $\bar p = \min(p,2)$. In particular, we have

$$-1 = y_0 < -d_1 < y_1 < -d_2 < ... < y_j < -d_j < y_{j+1} < ... < -d_k < y_k = 1$$

and $\kappa(d_j(s), y_{j+1}(s)) = \kappa(d_{j+1}(s), y_{j+1}(s))$ for $j = 1,...,k-1$ (use (84) to see this).
We first prove (159) and then (158). To prove (159), using (44), we see that:

- if $y \in (y_{i-1}(s), y_i(s))$, then $|\sum_{l\neq i} e_l\kappa(d_l(s),y)| \le 3\kappa(d_i(s),y)$, hence $|V_i(y,s)| \le C\sum_{l\neq i}\kappa(d_i(s),y)^{p-2}\kappa(d_l(s),y)$;
- if $y \in (y_{j-1}(s), y_j(s))$ for some $j \neq i$, then for all $l = 1,...,k$, $\kappa(d_l(s),y) \le \kappa(d_j(s),y)$, hence $|V_i(y,s)| \le C\sum_{l=1}^k \kappa(d_l(s),y)^{p-1} \le C\kappa(d_j(s),y)^{p-1}$.

Thus, (159) follows. Now, we prove (158). Using (36), Claim B.1 and (159), we write

$$\begin{aligned}
&\left|\pi_\lambda^{d_i}\begin{pmatrix}0\\ V_iq_1\end{pmatrix}\right| \le C\int_{-1}^1 \kappa(d_i)\,|V_iq_1|\,\rho dy\\
\le\ & C\|q_1(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty}^2 + \left(\int_{-1}^1 \kappa(d_i)\,|V_i|\,(1-y^2)^{\frac{1}{p-1}}dy\right)^2\\
\le\ & C\|q\|_{\mathcal H}^2\\
+\ & \sum_{l\neq i}\left(\int_{y_{i-1}}^{y_i}\kappa(d_i)^{p-1}\kappa(d_l)(1-y^2)^{\frac{1}{p-1}}dy\right)^2 + \left(\int_{y_{l-1}}^{y_l}\kappa(d_i)^{p-1}\kappa(d_l)(1-y^2)^{\frac{1}{p-1}}dy\right)^2.
\end{aligned}$$

Using (ii) of Lemma E.1, (158) follows.

- Consider $j \neq i$. Since we have from the definitions (11) and (45) of $\kappa(d,y)$ and $F_0(d,y)$,

$$\begin{pmatrix}\partial_d\kappa(d,y)\\ 0\end{pmatrix} = -\frac{2\kappa_0}{(p-1)(1-d^2)}F_0(d,y), \qquad (160)$$

we use (151) and the fact that $d_j = -\tanh\zeta_j$ hence $\zeta_j' = -\frac{d_j'}{1-d_j^2}$, to write

$$\left|d_j'\pi_\lambda^{d_i}\begin{pmatrix}\partial_d\kappa(d_j)\\ 0\end{pmatrix}\right| \le \frac{C|d_j'|}{1-d_j^2}\left|\pi_\lambda^{d_i}(F_0(d_j)\right| \le C\bar J|\zeta_j'| \qquad (161)$$

where $\bar J(s)$ is defined in (55). Using (152), (153), (155), (157), (158), (161) and (56), we write for all $i = 1,..,k$ (starting with $\lambda = 0$ and then $\lambda = 1$),

$$\begin{aligned}
|\zeta_i'| &\le C|\zeta_i'|\|q\|_{\mathcal H} + CJ + C\|q\|_{\mathcal H}^2 + C\bar J\sum_{j\neq i}|\zeta_j'|, &(162)\\
\left|{\alpha_1^i}' - \alpha_1^i\right| &\le C|\zeta_i'|\|q\|_{\mathcal H} + CJ + C\|q\|_{\mathcal H}^2 + C\bar J\sum_{j\neq i}|\zeta_j'|, &(163)
\end{aligned}$$

Since $\|q\|_{\mathcal{H}} + J \to 0$ (see (i) of Lemma 3.4), summing up (162) in i, we get,

$$\sum_{i=1}^{k} |\zeta_i'| \le CJ + C\|q\|_{\mathcal{H}}^2.$$

Plugging this in (162), we get

$$\left|{\alpha_1^i}' - \alpha_1^i\right| \le CJ + C\|q\|_{\mathcal{H}}^2,$$

which closes the proof of (58). This concludes the proof of (i) of Lemma 3.6. ■

Proof of (ii): Differential inequality satisfied by $A_-(s)$
We proceed in 2 steps: we first project equation (42) with the projector π_- defined in (49), and then use that equation to write a differential inequality for $A_- = \varphi(q_-, q_-)$.

Step 2.1 : Projection of equation (42) with π_-
In this claim, we project equation (42) with the projector π_- defined in (49):

Claim C.1 (A partial differential inequality for q_-) *For s large enough, we have*

$$\begin{aligned}&\left\|\partial_s q_- - Lq_- - \sum_{i=1}^{k} \pi_1^{d_i}(q) \begin{pmatrix} 0 \\ V_i F_{1,1}(d_i) \end{pmatrix} - \begin{pmatrix} 0 \\ f(q_1) \end{pmatrix} - \begin{pmatrix} 0 \\ R \end{pmatrix}\right\|_{\mathcal{H}} \\ \le\ & CJ + C\|q\|_{\mathcal{H}}^2\end{aligned}$$

where $J(s)$ is defined in (23).

Proof: Applying the projector π_- defined in (49) to equation (42), we write

$$\pi_-(\partial_s q) = \pi_-(Lq) - \sum_{i=1}^{k} e_i d_i' \pi_- \begin{pmatrix} \partial_d \kappa(d_i) \\ 0 \end{pmatrix} + \pi_- \begin{pmatrix} 0 \\ f(q_1) \end{pmatrix} + \pi_- \begin{pmatrix} 0 \\ R \end{pmatrix}. \tag{164}$$

In the following, we will estimate each term appearing in this identity.

- Proceeding as for estimate (213) in [12] in the case of one soliton, one can straightforwardly control the left-hand term as follows:

$$\|\pi_-(\partial_s q) - \partial_s q_-\|_{\mathcal{H}} \le CJ\|q\|_{\mathcal{H}} + C\|q\|_{\mathcal{H}}^3. \tag{165}$$

- We claim that

$$\left\|\pi_-(Lq) - Lq_- - \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) \begin{pmatrix} 0 \\ V_i F_{1,1}(d_i(s), \cdot) \end{pmatrix}\right\|_{\mathcal{H}} \le C\|q(s)\|_{\mathcal{H}}^2 + o(J) \text{ as } s \to \infty. \tag{166}$$

Indeed, applying the operator L to (52) on the one hand, and using (49) with $r = Lq$ on the other hand, we write

$$\begin{aligned} Lq &= \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) L F_1(d_i(s), \cdot) + Lq_- \\ &= \sum_{i=1}^{k} \pi_1^{d_i(s)}(Lq) F_1(d_i(s), \cdot) + \sum_{i=1}^{k} \pi_0^{d_i(s)}(Lq) F_0(d_i(s), \cdot) + \pi_-(Lq). \end{aligned}$$

Therefore,

$$\pi_-(Lq)-Lq_- = \sum_{i=1}^{k}\pi_1^{d_i(s)}(q)LF_1(d_i(s),\cdot)-\pi_1^{d_i(s)}(Lq)F_1(d_i(s),\cdot)-\sum_{i=1}^{k}\pi_0^{d_i(s)}(Lq)F_0(d_i(s),\cdot). \tag{167}$$

Since we have from (36) and (47), $\pi_\lambda^d(L_d r) = \phi(W_\lambda(d,\cdot), L_d r) = \phi(L_d^* W_\lambda(d,\cdot), r) = \lambda\pi_\lambda^d(r)$, using this with (43) and (46) gives for $\lambda = 0$ or 1,

$$\begin{aligned} LF_\lambda(d_i(s),\cdot) &= L_{d_i(s)}F_\lambda(d_i(s),\cdot) + \begin{pmatrix} 0 \\ V_i F_{\lambda,1}(d_i(s),\cdot) \end{pmatrix} \\ &= \lambda F_\lambda(d_i(s),\cdot) + \begin{pmatrix} 0 \\ V_i F_{\lambda,1}(d_i(s),\cdot) \end{pmatrix} \qquad (168) \\ \pi_\lambda^{d_i(s)}(Lq) &= \pi_\lambda^{d_i(s)}(L_{d_i(s)}q) + \pi_\lambda^{d_i(s)}\begin{pmatrix} 0 \\ V_i q_1 \end{pmatrix} = \lambda\pi_\lambda^{d_i(s)}(q) + \pi_\lambda^{d_i(s)}\begin{pmatrix} 0 \\ V_i q_1 \end{pmatrix} \qquad (169) \end{aligned}$$

Using (167), (168) and (169) together with (158) and (46), we get (166).

- Using the definition (49) of the operator π_-, we see that

$$\begin{aligned} \pi_-\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix} &= \begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix} \qquad (170) \\ &- \sum_{j=1}^{k}\pi_1^{d_j}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix}F_1(d_j,\cdot) - \sum_{j=1}^{k}\pi_0^{d_j}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix}F_0(d_j,\cdot). \end{aligned}$$

Using (160), it follows from the orthogonality relation (48) that for $\lambda = 0$ or 1,

$$\pi_\lambda^{d_i}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix} = -\frac{2\kappa_0}{(p-1)(1-d^2)}\pi_\lambda^{d_i}(F_0(d_i,\cdot)) = \delta_{\lambda,0}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix}.$$

Therefore, it follows from (170) that

$$\pi_-\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix} = -\sum_{j\neq i}\pi_1^{d_j}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix}F_1(d_j,\cdot) - \sum_{j\neq i}\pi_0^{d_j}\begin{pmatrix} \partial_d\kappa(d_i,\cdot) \\ 0 \end{pmatrix}F_0(d_j,\cdot).$$

Using (161), (46) and (58), we see that

$$\left\| d_i'(s)\pi_-\begin{pmatrix} \partial_d\kappa(d_i(s),\cdot) \\ 0 \end{pmatrix}\right\|_{\mathcal{H}} \le C\sqrt{J(s)}|\zeta_i'(s)| \le C\sqrt{J(s)}\left(\|q(s)\|_{\mathcal{H}}^2 + J(s)\right). \tag{171}$$

- From definition (49) of the operator π_-, (46), (157) and (155), we have

$$\begin{aligned} \left\|\pi_-\begin{pmatrix} 0 \\ f(q_1) \end{pmatrix} - \begin{pmatrix} 0 \\ f(q_1) \end{pmatrix}\right\|_{\mathcal{H}} &\le C\sum_{\lambda=1,2;\ i=1}^{k}\left|\pi_\lambda^{d_i(s)}\begin{pmatrix} 0 \\ f(q_1) \end{pmatrix}\right| \le C\|q(s)\|_{\mathcal{H}}^2, \qquad (172) \\ \left\|\pi_-\begin{pmatrix} 0 \\ R \end{pmatrix} - \begin{pmatrix} 0 \\ R \end{pmatrix}\right\|_{\mathcal{H}} &\le C\sum_{\lambda=1,2;\ i=1}^{k}\left|\pi_\lambda^{d_i(s)}\begin{pmatrix} 0 \\ R \end{pmatrix}\right| \le CJ(s). \qquad (173) \end{aligned}$$

Using (164), (165), (166), (171), (172) and (173) closes the proof of Claim C.1. ■

Step 2.2: A differential inequality on $A_-(s)$

By definition (55) of $\alpha_-(s)$, it holds that

$$A'_-(s) = \varphi(\partial_s q_-, q_-) - \frac{p(p-1)}{2}\sum_{i=1}^{k} e_i d'_i I_i \tag{174}$$

with

$$I_i = \int_{-1}^{1} \partial_d \kappa(d_i)|K|^{p-2}\left(q_{-,1}\right)^2 \rho dy \text{ and } K = \sum_{j=1}^{k} e_j \kappa(d_j).$$

Using (150), (i) of Claim B.1 and the definition (9) of the norm in $\mathcal{H}$, we see that

$$|I_i| \le \frac{C}{1-d_i^2}\int_{-1}^{1}\kappa(d_i)|K|^{p-2}dy\|q_{-,1}(1-y^2)^{\frac{1}{p-1}}\|^2_{L^\infty} \le \frac{C}{1-d_i^2} J_i\|q_-\|^2_{\mathcal{H}} \tag{175}$$

with J_i defined in (156). Using (174), (175), (v) of Lemma E.1, (58) and (52)

$$\left|\frac{1}{2}A'_-(s) - \varphi(\partial_s q_-, q_-)\right| \le C\|q\|^2_{\mathcal{H}}\left(\|q\|^2_{\mathcal{H}} + J\right). \tag{176}$$

Since $\|q_-\|_{\mathcal{H}} \le C\min\left(\|q\|_{\mathcal{H}}, \bar{J}\|q\|_{\mathcal{H}} + \sqrt{|A_-|}\right)$ from (52) and (53), we use (51) and Claim C.1 to estimate $\varphi(\partial_s q_-, q_-)$ in the following:

$$\begin{aligned}
&\left|\varphi(\partial_s q_-, q_-) - \varphi(Lq_-, q_-) - \int_{-1}^{1} q_{-,2} f(q_1)\rho dy - \int_{-1}^{1} q_{-,2} G\rho(y) dy\right| \\
\le\;& C\|q_-\|_{\mathcal{H}}\left(J + \|q\|^2_{\mathcal{H}}\right) \le CJ\sqrt{|A_-|} + CJ\bar{J} + C\|q\|^3_{\mathcal{H}} \\
\le\;& CJ\sqrt{|A_-|} + C\|q\|^3_{\mathcal{H}} + C\sum_{m=1}^{k-1}\left(h(\zeta_{m+1}-\zeta_m)\right)^2
\end{aligned} \tag{177}$$

where h is defined in (35) and

$$G(y,s) = \sum_{i=1}^{k}\alpha_1^i(s)V_i(y,s)F_{1,1}(d_i(s),y) + R(y,s). \tag{178}$$

In the following, we estimate every term of (177) in order to finish the proof of (59).

- Arguing as in page 107 of [12], we write

$$\varphi(Lq_-, q_-) = -\frac{4}{p-1}\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2}dy. \tag{179}$$

- Since we have from the definitions (11) and (45) of $\kappa(d,y)$ and $F_1(d,y)$,

$$F_{1,1}(d,y) = F_{1,2}(d,y) \le C\kappa(d,y), \tag{180}$$

using (52) and (157), we write

$$\left|\int_{-1}^{1} q_{-,2} f(q_1)\rho dy - \int_{-1}^{1} q_2 f(q_1)\rho dy\right| \le C\sum_{i=1}^{k} |\alpha_1^i| \int_{-1}^{1} \kappa(d_i)|f(q_1)|\rho dy \le C\|q\|_{\mathcal{H}}^3. \quad (181)$$

If we introduce

$$\mathcal{F}(q_1) = \int_0^{q_1} f(\xi)d\xi = \frac{|K+q_1|^{p+1}}{p+1} - \frac{|K|^{p+1}}{p+1} - |K|^{p-1}Kq_1 - \frac{p}{2}|K|^{p-1}q_1^2,$$

then it is easy to see that

$$|\mathcal{F}(q_1)| \le C|q_1|^{p+1} + C\delta_{\{p\ge 2\}}|K|^{p-2}|q_1|^3. \quad (182)$$

Introducing $R_- = -\int_{-1}^{1} \mathcal{F}(q_1)\rho dy$ and using equation (42), we write

$$R_-' + \int_{-1}^{1} q_2 f(q_1)\rho dy = R_-' + \int_{-1}^{1} \partial_s q_1 f(q_1)\rho dy + \sum_{i=1}^{k} d_i' \int_{-1}^{1} \partial_d\kappa(d_i) f(q_1)\rho dy \quad (183)$$

$$= \sum_{i=1}^{k} d_i' \int_{-1}^{1} \left(\partial_d\kappa(d_i)f(q_1) - \partial_{d_i}\mathcal{F}(q_1)\right)\rho dy = \frac{p(p-1)}{2}\sum_{i=1}^{k} d_i' \int_{-1}^{1} \partial_d\kappa(d_i)|K|^{p-2}q_1^2\rho dy.$$

Therefore, using (181) and (183), arguing as for (175), using (v) of Lemma E.1 and (58), we write

$$\left|\int_{-1}^{1} q_{2,-} f(q_1)\rho dy + R_-'\right| \le C\|q\|_{\mathcal{H}}^3 + C\sum_{i=1}^{k} \frac{|d_i'|}{1-d_i^2} J_i\|q\|_{\mathcal{H}}^2 \le C\left(\|q\|_{\mathcal{H}}^3 + J\|q\|_{\mathcal{H}}^2\right) \quad (184)$$

Note that from (182), the Hölder inequality and Claim B.1, we have

$$\begin{aligned}\left|\int_{-1}^{1} \mathcal{F}(q_1)\rho dy\right| &\le C\int_{-1}^{1} |q_1|^{p+1}\rho dy + C\delta_{\{p\ge 2\}}\int_{-1}^{1} |K|^{p-2}|q_1|^3\rho dy \\ &\le C\|q\|_{\mathcal{H}}^{p+1} + C\delta_{\{p\ge 2\}}\left(\int_{-1}^{1} |q_1|^{p+1}\rho dy\right)^{\frac{3}{p+1}}\left(\int_{-1}^{1} |K|^{p+1}\rho dy\right)^{\frac{p-2}{p+1}} \\ &\le C\|q\|_{\mathcal{H}}^{p+1} + C\delta_{\{p\ge 2\}}\|q\|_{\mathcal{H}}^3 \le C\|q\|_{\mathcal{H}}^{\bar{p}+1} \quad (185)\end{aligned}$$

where $\bar{p} = \min(p, 2)$.

- Using the Cauchy-Schwartz inequality, we write

$$\left|\int_{-1}^{1} q_{-,2} G\rho dy\right| \le \frac{1}{p-1}\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + C\int_{-1}^{1} G^2\rho(1-y^2)dy. \quad (186)$$

¿From the definition (178) of G, we need to handle R and $V_iF_{1,1}$. We start by R first. We claim that

$$|R| \le C\sum_{j=1}^{k} \kappa(d_j(s),y)^{p-1} 1_{\{y_{j-1}(s)<y<y_j(s)\}} \sum_{l\ne j} \kappa(d_l(s),y) \quad (187)$$

where

$$y_0 = -1, \;\; y_j = \tanh\left(\frac{\zeta_j + \zeta_{j+1}}{2}\right) \text{ if } j = 1,..,k-1 \text{ and } y_k = 1. \tag{188}$$

In particular, we have

$$-1 = y_0 < -d_1 < y_1 < -d_2 < ... < y_j < -d_j < y_{j+1} < ... < -d_k < y_k = 1$$

and $\kappa(d_j(s), y_{j+1}(s)) = \kappa(d_{j+1}(s), y_{j+1}(s))$ for $j = 1, ..., k-1$ (to see this, just use the fact that $\kappa(d, y)(1-y^2)^{\frac{1}{p-1}} = \kappa_0 \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)$ if $y = \tanh \xi$).
To prove (187), we take $y \in (y_{j-1}(s), y_j(s))$ and set $X = (\sum_{l \neq j} e_l \kappa(d_l(s), y))/e_j \kappa(d_j(s), y)$. From the fact that $\zeta_{j+1}(s) - \zeta_j(s) \to \infty$, we have $|X| \leq 2$ hence

$$||1+X|^{p-1}(1+X) - 1| \leq C|X|$$

and for $y \in (y_{j-1}(s), y_j(s))$ and s large,

$$||K|^{p-1}K - e_j \kappa(d_j(s), y)^p| \leq C\kappa(d_j(s), y)^{p-1} \sum_{l \neq j} \kappa(d_l(s), y).$$

Since for all $y \in (y_{j-1}(s), y_j(s))$, $\kappa(d_j(s), y) \geq \kappa(d_l(s), y)$ if $l \neq j$, this concludes the proof of (187).
Using (187), we see that

$$\int_{-1}^{1} R^2 \rho(1-y^2) dy \leq C \sum_{j=1}^{k} \sum_{l \neq j} \int_{y_{j-1}}^{y_j} \kappa(d_j)^{2(p-1)} \kappa(d_l)^2 \rho(1-y^2) dy \leq C \sum_{m=1}^{k-1} h(\zeta_{m+1} - \zeta_m)^2 \tag{189}$$

where h is defined in (35).
Now, we handle $V_i F_{1,1}$. Using (159), (180) and (i) of Lemma E.1, we see that

$$\int_{-1}^{1} (V_i F_{1,1}(d_i))^2 \rho(1-y^2) dy \leq C \sum_{j \neq i} \int_{-1}^{1} \kappa(d_i)^2 \kappa(d_j)^{2(p-1)} \rho(1-y^2) dy$$

$$+ \quad C\delta_{\{p \geq 2\}} \sum_{j \neq i} \int_{-1}^{1} \kappa(d_i)^{2(p-1)} \kappa(d_j)^2 \rho(1-y^2) dy \to 0 \text{ as } s \to \infty.$$

Hence, using (54), we see that

$$\left(\alpha_1^i\right)^2 \int_{-1}^{1} (V_i F_{1,1}(d_i))^2 \rho(1-y^2) dy = o\left(\|q\|_{\mathcal{H}}^2\right). \tag{190}$$

Gathering (176), (177), (179), (184), (186), (178), (189) and (190), we get to the conclusion of (59). Note that the estimate for $R_-(s)$ is given in (185).

Proof of (iii): An additional estimate

We prove estimate (61) here. The proof is the same as in the case of one soliton treated in [12], except for the term involving the interaction term $R(y, s)$ (42). Therefore, arguing exactly as in pages 111 and 112 of [12], we write

$$\frac{d}{ds} \int q_1 q_2 \rho dy \leq -\frac{9}{10}\alpha_-^2 + CJ^2 + C\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + C\sum_{i=1}^{k} |\alpha_1^i|^2 + \int_{-1}^{1} q_1 R \rho dy.$$

Since we have from the Cauchy-Schwartz inequality, (i) of Claim B.1, (52) and (189)

$$\left|\int_{-1}^{1} q_1 R\rho dy\right| \le \left(\int_{-1}^{1} q_1^2\frac{\rho}{1-y^2}dy\right)^{\frac12}\left(\int_{-1}^{1} R^2(1-y^2)\rho dy\right)^{\frac12}$$
$$\le C\|q\|_{\mathcal{H}}\left(\int_{-1}^{1} R^2(1-y^2)\rho dy\right)^{\frac12} \le \frac{1}{10}\left(\alpha_-^2+\sum_{i=1}^{k}(\alpha_1^i)^2\right)+C\sum_{i=1}^{k-1}h(\zeta_{i+1}-\zeta_i)^2$$

where h is defined in (35), this concludes the proof of (61) and the proof of Lemma 3.6. ■

D A continuity result in the selfsimilar variable

We prove Claim 4.5 here. Consider $\epsilon_0>0$ and from (98), fix $\tilde t$ close enough to $T(x_0)$ so that

$$\|w_{x_0}(s_0)-w_\infty\|_{L^2_\rho}\le \epsilon_0 \text{ where } s_0=-\log(T(x_0)-\tilde t). \tag{191}$$

Note from (97) and the continuity of $x\mapsto T(x)$ that $u(x,\tilde t)$ is well defined for all $x\in[\bar x, x_0+(T(x_0)-\tilde t))$ for some $\bar x<x_0-(T(x_0)-\tilde t)$. Therefore, using the selfsimilar transformation (5), we see that

$$w(\cdot,s_0)\in L^2(\bar y,0) \text{ where } \bar y=\frac{\bar x-x_0}{T(x_0)-\tilde t}<-1. \tag{192}$$

We aim at proving that for x' close enough to x_0, we have

$$\left\|\begin{pmatrix} w_{x'}(\tilde s_0(x'))\\ \partial_s w_{x'}(\tilde s_0(x'))\end{pmatrix}-\begin{pmatrix} w_\infty\\ 0\end{pmatrix}\right\|_{\mathcal{H}}\le 6\epsilon_0 \text{ where } \tilde s_0(x')=-\log(T(x')-\tilde t). \tag{193}$$

For simplicity, we will only prove that

$$\|w_{x'}(\tilde s_0(x'))-w_\infty\|_{L^2_\rho}\le 2\epsilon_0, \tag{194}$$

provided that x_0-x' is small. The estimates involving $\partial_y w_{x'}(\tilde s_0(x'))$ and $\partial_s w_{x'}(\tilde s_0(x'))$ follow in the same way.
Using the selfsimilar transformation (6), we write

$$\forall \tilde y\in(-1,1),\;\; w_{x'}(\tilde y,\tilde s_0(x'))=\theta^{\frac{2}{p-1}}w_{x_0}(y,s_0) \text{ where } y=\tilde y\theta+\xi, \tag{195}$$

$$\theta=\frac{1}{1+e^{\tilde s_0(x')}(T(x_0)-T(x'))}\to 1 \text{ and } \xi=(x'-x_0)e^{\tilde s_0(x')}\theta\to 0 \text{ as } x'\to x_0. \tag{196}$$

Therefore, performing a change of variables, we write for x_0-x' small enough,

$$\|w_{x'}(\tilde s_0(x'))-w_\infty\|^2_{L^2_\rho}=\int_{-1}^{1}|w_{x'}(\tilde y,\tilde s_0(x'))-w_\infty(\tilde y)|^2\rho(\tilde y)d\tilde y$$
$$=\int_{-\theta+\xi}^{\theta+\xi}\left|\theta^{\frac{2}{p-1}}w_{x_0}(y,s_0)-w_\infty\left(\frac{y-\xi}{\theta}\right)\right|^2\rho\left(\frac{y-\xi}{\theta}\right)\frac{dy}{\theta}.$$

Since we have from (196), (193) and the fact that $x \mapsto T(x)$ is 1-Lipschitz,

$$\theta + \xi = \frac{1 + (x' - x_0)e^{\tilde{s}_0(x')}}{1 + e^{\tilde{s}_0(x')}(T(x_0) - T(x'))} = \frac{T(x') - \tilde{t} + x' - x_0}{T(x_0) - \tilde{t}} \leq 1,$$

it follows that

$$\|w_{x'}(\tilde{s}) - w_\infty\|^2_{L^2_\rho} = \int_{\bar{y}}^1 g(\theta, \xi, y)dy$$

where $\bar{y} < -1$ is defined in (192) and

$$g(\theta, \xi, y) = \frac{1_{\{-\theta+\xi<y<\theta+\xi\}}}{\theta} \left| \theta^{\frac{2}{p-1}} w_{x_0}(y, s_0) - w_\infty \left(\frac{y - \xi}{\theta} \right) \right|^2 \rho \left(\frac{y - \xi}{\theta} \right). \tag{197}$$

We claim that in order to conclude, it is enough to prove that for $x_0 - x'$ small enough,

$$\forall y \in (-\theta + \xi, \theta + \xi), \ g(\theta, \xi, y) \leq \tilde{g}(y) \text{ for some } \tilde{g} \in L^1(\bar{y}, 1). \tag{198}$$

Indeed, since we have from (196) that

$$\forall y \in (\bar{y}, 1), \ \ g(\theta, \xi, y) \to g(1, 0, y) \text{ as } x' \to x_0,$$

we use (198) to apply the Lebesgue Theorem and obtain that

$$\|w_{x'}(\tilde{s}_0(x')) - w_\infty\|^2_{L^2_\rho} = \int_{\bar{y}}^1 g(\theta, \xi, y)dy \to \int_{-1}^1 g(1, 0, y)dy = \|w_{x_0}(s_0) - w_\infty\|^2_{L^2_\rho}$$

as $x' \to x_0$. Using (191), we see that for $x_0 - x'$ small enough, (194) holds. It remains to prove (198) in order to conclude.
If $-\theta + \xi \leq y \leq 0$, then we have $\rho(\frac{y-\xi}{\theta}) \leq 1$. Using (197), (196), (192) and the definition (99) of w_∞, we write for $x_0 - x'$ small enough:

$$g(\theta, \xi, y) \leq C(|w_{x_0}(y, s_0)|^2 + \|w_\infty\|^2_{L^\infty(-1,1)}) \in L^1(\bar{y}, 0).$$

If $0 \leq y \leq \theta + \xi$, then we have from (196), $\rho(\frac{y-\xi}{\theta}) \leq C(1 - \frac{y-\xi}{\theta})^{\frac{2}{p-1}} = C(\frac{1-y+\xi}{\theta})^{\frac{2}{p-1}} \leq C(1 - y)^{\frac{2}{p-1}} \leq C\rho(y)$. Therefore, using (197) and (191), we write

$$g(\theta, \xi, y) \leq C(|w_{x_0}(y, s_0)|^2 + \|w_\infty\|^2_{L^\infty(-1,1)})\rho(y) \in L^1(0, 1).$$

Thus, (198) holds and so does (194).
Since the same technique works for $\left\| \partial_y w_{x'}(\tilde{s}_0(x')) - \frac{dw_\infty}{dy} \right\|_{L^2_{\rho(1-y^2)}}$ and $\|\partial_s w_{x'}(\tilde{s}_0(x'))\|_{L^2_\rho}$, estimate (193) follows in the same way. This concludes the proof of Claim 4.5. ■

E A table for integrals involving the solitons

In this section, we estimate integrals involving the solitons.
Recalling that $y = -d_i(s) = \tanh \zeta_i(s)$ is the center of the i-th soliton $\kappa(d_i(s), y)$, we introduce the following "separators" between the solitons:

$$y_0 = -1, \ \ y_j = \tanh \left(\frac{\zeta_j + \zeta_{j+1}}{2} \right) \text{ if } j = 1, .., k - 1 \text{ and } y_k = 1 \tag{199}$$

Note in particular that we have

$$-1 = y_0 < -d_1 < y_1 < -d_2 < ... < y_j < -d_j < y_{j+1} < ... < -d_k < y_k = 1$$

and $\kappa(d_j(s), y_{j+1}(s)) = \kappa(d_{j+1}(s), y_{j+1}(s))$ for $j = 1, ..., k-1$ (to see this, just use the fact that $\kappa(d, y)(1-y^2)^{\frac{1}{p-1}} = \kappa_0 \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)$ if $y = \tanh \xi$).
In the following lemma, we estimate various integrals involving the solitons $\kappa(d_i(s), y)$:

Lemma E.1 (A table of integrals involving the solitons) *We have the following estimates as $s \to \infty$:*

(i) If $i \neq j$, $\alpha > 0$, $\beta > 0$ and $I_1 = \displaystyle\int_{-1}^{1} \kappa(d_j)^\alpha \kappa(d_i)^\beta (1-y^2)^{\frac{\alpha+\beta}{p-1}-1} dy$, then:
for $\alpha = \beta$, $I_1 \sim C_0 |\zeta_i - \zeta_j| e^{-\frac{2\alpha}{p-1}|\zeta_i - \zeta_j|}$;
for $\alpha \neq \beta$, $I_1 \sim C_0 e^{-\frac{2}{p-1}\min(\alpha,\beta)|\zeta_i - \zeta_j|}$ for some $C_0 = C_0(\alpha, \beta) > 0$.

(ii) If $i \neq j$, $\alpha > 0$, $\beta > 0$ and $I_2 \equiv \displaystyle\int_{y_{j-1}}^{y_j} \kappa(d_j)^\alpha \kappa(d_i)^\beta (1-y^2)^{\frac{\alpha+\beta}{p-1}-1} dy$, then:
for $\alpha = \beta$, $I_2 \le C|\zeta_{j+1} - \zeta_j| e^{-\frac{2\beta}{p-1}|\zeta_{j+1} - \zeta_j|} + C|\zeta_{j-1} - \zeta_j| e^{-\frac{2\beta}{p-1}|\zeta_{j-1} - \zeta_j|}$;
for $\alpha > \beta$, $I_2 \le C e^{-\frac{2\beta}{p-1}|\zeta_{j+1} - \zeta_j|} + C e^{-\frac{2\beta}{p-1}|\zeta_{j-1} - \zeta_j|}$;
for $\beta > \alpha$, $I_2 \le C e^{-\frac{(\alpha+\beta)}{p-1}|\zeta_{j+1} - \zeta_j|} + C e^{-\frac{(\alpha+\beta)}{p-1}|\zeta_{j-1} - \zeta_j|}$.

(iii) Let $A_{i,j,l} = \displaystyle\int_{y_{j-1}}^{y_j} \frac{y + d_i}{1 + y d_i} \kappa(d_i) \kappa(d_j)^{p-1} \kappa(d_l) \rho dy$ with $l \neq j$.
If $i = j$, then $A_{i,i,l} \sim \operatorname{sgn}(l - j) c_1''' e^{-\frac{2}{p-1}|\zeta_l - \zeta_i|}$ for some $c_1''' > 0$.
If $j \neq i$, then $A_{i,j,l} = o(J)$ where J is defined in (23).

(iv) If $l \neq j$, then $B_{i,j,l} \equiv \displaystyle\int_{y_{j-1}}^{y_j} \kappa(d_i) \kappa(d_j)^{p-\bar{p}} \kappa(d_l)^{\bar{p}} \rho dy = o(J)$ (with $\bar{p} = \min(p, 2)$).

(v) For any $i = 1, .., k$, it holds that $J_i \equiv \displaystyle\int_{-1}^{1} \kappa(d_i) |K|^{p-2} dy \le C$ where $K(y, s)$ is defined in (42).

Proof: (i) With the change of variables $y = \tanh \xi$, we write

$$I_1 = \kappa_0^{\alpha+\beta} \int_{\mathbb{R}} \cosh^{-\frac{2\alpha}{p-1}}(\xi - \zeta_j) \cosh^{-\frac{2\beta}{p-1}}(\xi - \zeta_i) d\xi.$$

¿From symmetry, we can assume that $\alpha \ge \beta$ and $\zeta_i > \zeta_j$. Using the change of variables $z = \xi - \zeta_j$, we write

$$I_1 = \kappa_0^{\alpha+\beta} \int_{\mathbb{R}} \cosh^{-\frac{2\alpha}{p-1}}(z) \cosh^{-\frac{2\beta}{p-1}}(z + \zeta_j - \zeta_i) dz.$$

When $\alpha > \beta$, we get from Lebesgue's Theorem $I_1 \sim C e^{-\frac{2\beta}{p-1}(\zeta_i - \zeta_j)}$.
When $\alpha = \beta$, we write from symmetry and Lebesgue's Theorem

$$I_1 = 2\kappa_0^{\alpha+\beta} \int_{-\infty}^{\frac{\zeta_i - \zeta_j}{2}} \cosh^{-\frac{2\beta}{p-1}}(z) \cosh^{-\frac{2\beta}{p-1}}(z + \zeta_j - \zeta_i) dz \sim C(\zeta_i - \zeta_j) e^{-\frac{2\beta}{p-1}(\zeta_i - \zeta_j)}.$$

(ii) Since $I_2 \le I_1$ and $|\zeta_j - \zeta_i| \ge \min(|\zeta_{j+1} - \zeta_j|, |\zeta_j - \zeta_{j-1}|)$, the result follows from (i) if $\alpha \ge \beta$. When $\alpha < \beta$, we assume that $\zeta_i > \zeta_j$, the other case being parallel. Using the change of variables $y = \tanh \xi$ then $z = \xi - \zeta_j$, we write

$$\begin{aligned} I_2 &= \kappa_0^{\alpha+\beta} \int_{-\frac{(\zeta_j-\zeta_{j-1})}{2}}^{\frac{(\zeta_{j+1}-\zeta_j)}{2}} \cosh^{-\frac{2\alpha}{p-1}}(z) \cosh^{-\frac{2\beta}{p-1}}(z + \zeta_j - \zeta_i) dz \\ &\sim e^{-\frac{2\beta}{p-1}(\zeta_i-\zeta_j)} \int_{-\infty}^{\frac{(\zeta_{j+1}-\zeta_j)}{2}} \cosh^{-\frac{2\alpha}{p-1}}(z) e^{\frac{2\beta}{p-1}z} dz \\ &\sim Ce^{-\frac{2\beta}{p-1}(\zeta_i-\zeta_j)} e^{\frac{2(\beta-\alpha)}{p-1}.\frac{\zeta_{j+1}-\zeta_j}{2}} \le Ce^{-\frac{(\alpha+\beta)}{p-1}(\zeta_{j+1}-\zeta_j)} \end{aligned}$$

since $\zeta_i - \zeta_j \ge \zeta_{j+1} - \zeta_j$, which yields the result.

(iii) If $i = j$, then using the change of variables $y = \tanh \xi$, we write

$$\begin{aligned} A_{i,i,l} &= \kappa_0^{p+1} \int_{\frac{\zeta_{i-1}+\zeta_i}{2}}^{\frac{\zeta_i+\zeta_{i+1}}{2}} \cosh^{-\frac{2p}{p-1}}(\xi - \zeta_i) \tanh(\xi - \zeta_i) \cosh^{-\frac{2}{p-1}}(\xi - \zeta_l) d\xi \\ &= \kappa_0^{p+1} \int_{-\frac{(\zeta_i-\zeta_{i-1})}{2}}^{\frac{(\zeta_{i+1}-\zeta_i)}{2}} \cosh^{-\frac{2p}{p-1}}(z) \tanh(z) \cosh^{-\frac{2}{p-1}}(z + \zeta_i - \zeta_l) dz \sim \theta c_1" e^{-\frac{2}{p-1}|\zeta_l-\zeta_i|} \end{aligned}$$

as $s \to \infty$, where $\theta = \mathrm{sgn}(l - j)$, with

$$\begin{aligned} c_1" &= \frac{\kappa_0^{p+1}}{2^{\frac{2}{p-1}}} \int_{\mathbb{R}} \cosh^{-\frac{2p}{p-1}}(z) \tanh(z) e^{\frac{2z}{p-1}} dz \\ &= \frac{\kappa_0^{p+1}}{2^{\frac{2}{p-1}}} \int_0^\infty \cosh^{-\frac{2p}{p-1}}(z) \tanh(z) (e^{\frac{2z}{p-1}} - e^{-\frac{2z}{p-1}}) dz > 0. \end{aligned}$$

Now, if $j \ne i$, then we have from the Cauchy-Schwartz inequality, (ii) and the definition (23) of $J(s)$,

$$A_{i,j,l} \le \left(\int_{y_{j-1}}^{y_j} \kappa(d_j)^{p-1} \kappa(d_i)^2 \rho dy \right)^{1/2} \left(\int_{y_{j-1}}^{y_j} \kappa(d_j)^{p-1} \kappa(d_l)^2 \rho dy \right)^{1/2} = o(J).$$

(iv) If $i = j$, then the result follows from (ii). If $i \ne j$, using the Hölder inequality with $P = \bar{p} + 1$ and $Q = \frac{\bar{p}+1}{\bar{p}}$, (ii) and the definition (23) of $J(s)$, we write

$$B_{i,j,l} \le \left(\int_{y_{j-1}}^{y_j} \kappa(d_i)^{\bar{p}+1} \kappa(d_j)^{p-\bar{p}} \rho dy \right)^{\frac{1}{\bar{p}+1}} \left(\int_{y_{j-1}}^{y_j} \kappa(d_l)^{\bar{p}+1} \kappa(d_j)^{p-\bar{p}} \rho dy \right)^{\frac{\bar{p}}{\bar{p}+1}} = o(J).$$

(v) Using the change of variables $y = \tanh \xi$, we write

$$J_i = \kappa_0^{p-1} \int_{\mathbb{R}} \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i) |\bar{K}(\xi, s)|^{p-2} d\xi \text{ where } \bar{K}(\xi, s) = \sum_{j=1}^{k} e_i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_j). \tag{200}$$

If $p \geq 2$, then $|\bar{K}(\xi,s)| \leq C$ and $|J_i(s)| \leq C$.
If $p < 2$ and the e_j are the same, then $|\bar{K}(\xi,s)| \geq \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)$ and $|J_i(s)| \leq \int_{\mathbb{R}} \cosh^{-2}(\xi - \zeta_i)d\xi \leq C$.
It remains to treat the delicate case where $p < 2$ with the e_j not all the same. Taking advantage of the decoupling in the sum of the solitons (see (44)), we write

$$J_i = \kappa_0^{p-1} \sum_{j=1}^{k} \int_{\theta_{j-1}+A}^{\theta_j+A} \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)|\bar{K}(\xi,s)|^{p-2}d\xi \tag{201}$$

where $\theta_0 = -\infty$, $\theta_j = \frac{\zeta_j+\zeta_{j+1}}{2}$ if $j = 1,..,k-1$, $\theta_k = \infty$ and $A = A(p)$ is fixed such that

$$e^{\frac{2A}{p-1}} \geq 2e^{-\frac{2A}{p-1}}. \tag{202}$$

This partition isolates each soliton in the definition of $\bar{K}(\xi,s)$. It is shifted by A since $\bar{K}(\xi,s)$ may be zero for some $z_j(s) \sim \theta_j(s)$ if $e_je_{j+1} = -1$, giving rise to a singularity in $|\bar{K}(\xi,s)|^{p-2}$, integrable though delicate to control.
Consider some $j = 1,...,k-1$.
If $e_j = e_{j+1}$, then we have from (44) and (202) for all $\xi \in (\theta_{j-1}+A, \theta_j+A)$, $|\bar{K}(\xi,s)| \geq C(A)\cosh^{-\frac{2}{p-1}}(\xi - \zeta_j)$ and $\cosh^{-\frac{2}{p-1}}(\xi - \zeta_i) \leq C(A)\cosh^{-\frac{2}{p-1}}(\xi - \zeta_j)$, hence

$$\int_{\theta_{j-1}+A}^{\theta_j+A} \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)|\bar{K}(\xi,s)|^{p-2}d\xi \leq C(A)\int_{\theta_{j-1}+A}^{\theta_j+A} \cosh^{-2}(\xi - \zeta_j)d\xi \leq C(A). \tag{203}$$

If $e_j = -e_{j+1}$, then $\bar{K}(z_j(s),s) = 0$ with $z_j(s) \sim \theta_j(s)$, which makes $|\bar{K}(\xi,s)|^{p-2}$ singular at $\xi = z_j(s)$. For this we split the integral over the interval $(\theta_{j-1}+A, \theta_j+A)$ into two parts, below and above $\theta_j - A$:
- the part on the interval $(\theta_{j-1}+A, \theta_j-A)$ is bounded by the same argument as in the case $e_j = e_{j+1}$;
- the part on the interval $(\theta_j - A, \theta_j + A)$. Since we have from the definition (200) of $\bar{K}(\xi,s)$

$$\partial_\xi \bar{K}(\xi,s) = -\frac{2}{p-1}\sum_{l=1}^{k} e_l \sinh(\xi - \zeta_l)\cosh^{-\frac{2}{p-1}-1}(\xi - \zeta_l),$$

it follows that for all $\xi \in (\theta_j - A, \theta_j + A)$, $|\partial_\xi \bar{K}(\xi,s)| \geq C(A)e^{-\frac{2}{p-1}(\theta_j - \zeta_j)} = C(A)e^{-\frac{\zeta_{j+1}-\zeta_j}{p-1}}$ for some $C(A) > 0$, hence

$$|\bar{K}(\xi,s)|^{p-2} = |\bar{K}(\xi,s) - \bar{K}(z_j(s),s)|^{p-2} \leq C(A)|\xi - z_j(s)|^{p-2}e^{-\frac{p-2}{p-1}(\zeta_{j+1}-\zeta_j)}.$$

Therefore, since for all $\xi \in (\theta_j - A, \theta_j + A)$, $\cosh^{-\frac{2}{p-1}}(\xi - \zeta_i) \leq C(A)\cosh^{-\frac{2}{p-1}}(\xi - \zeta_j) \leq C(A)\cosh^{-\frac{2}{p-1}}(\theta_j - \zeta_j) \leq C(A)e^{-\frac{\zeta_{j+1}-\zeta_j}{p-1}}$, it follows that

$$\begin{aligned}\int_{\theta_j-A}^{\theta_j+A} \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i)|\bar{K}(\xi,s)|^{p-2}d\xi &\leq C(A)e^{-(\zeta_{j+1}-\zeta_j)}\int_{\theta_j-A}^{\theta_j+A} |\xi - z_j(s)|^{p-2}d\xi \\ &\leq C(A)e^{-(\zeta_{j+1}-\zeta_j)} \end{aligned} \tag{204}$$

because $z_j(s) \sim \theta_j(s)$ as $s \to \infty$. Therefore, (v) follows from (201), (203) and (204).

■

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