Hasse-Arf theorem

K c.d.v.f res fd F char $F = p > 0$

Thm (Hasse-Arf) Version 0.
L cyclic extn of K. E res fd.
If E/F separable, conductor of L/K is an integer.

Reformulation
$\chi : G = \mathrm{Gal}(L/K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ faithful char.

$\chi \in X_K = \mathrm{Hom}(G_K^{ab}, \mathbb{Q}/\mathbb{Z})$

fil. on X_K (Kato)

$\{\ ,\ \} : X_K \times K^\times \longrightarrow \mathrm{Br}\, K$

$K_1 = \mathcal{O}_K[T]^\wedge_{m_K \mathcal{O}_K[T]} \otimes_{\mathcal{O}_K} K$ $n \geq 0$

$\mathrm{Fil}_n X_K = \{ \chi \in X_K \mid \{\chi, 1 + m_{K_1}^{n+1}\} = 0 \}$
increasing fil.

$X_K = \bigcup \mathrm{Fil}_n X_K$. $\mathrm{Fil}_0 =$ tame

$\mathrm{Sw}\, \chi = \min\{ n \mid \chi \in \mathrm{Fil}_n \}$ integer

Thm (H-A) L/K cyclic.
version 1 Kato
E/F sep $\Rightarrow$ $\mathrm{Sw}\,\chi = \mathrm{cond}$.

Conductor
L/K Gal ext. of c.d.v.f
not nec. cyclic or res. ext sep

$G = \mathrm{Gal}(L/K)$ lower numbering ram. gp

$G_i = \mathrm{Ker}\,(G \to \mathrm{Aut}(L^\times/(1+m_L^i)))$

$G_1 = P =$ p-Sylow of $I = \mathrm{Ker}(G \to \mathrm{Aut}(E/F))$
$\bigcap G_i = 1$
$\sigma \in G, \neq 1, \; i_G(\sigma) = \max\{i \mid \sigma \in G_i\}$

cond $= 0 \Leftrightarrow$ tame
L/K cyclic.

$$\text{conductor} = \frac{1}{e_{L/K}}\left(\sum_{\sigma \in P, \neq 1} i_G(\sigma) + \max\{i_G(\sigma) \mid \sigma \in G, \neq 1\}\right)$$

Thm (H-A) L/K cyclic.
version 2 Then $\mathrm{Sw}\,\chi \leq \mathrm{cond}$
& if E/F sep $\Rightarrow$ $=$ holds //

Proof of $\leq$: $L^\times \underset{\overset{\uparrow}{\text{compute}}}{\xrightarrow{N}} K^\times \xrightarrow{\{\chi,\cdot\}} \mathrm{Br}\,K \to \mathrm{Br}\,L$ exact

Question: Condition for $=$

Wild ramification

$P \subset Gal(L/K)$ not nec. cyclic

P p-gp nilpotent

log mono $\Longleftarrow$

$$\begin{tikzcd} \mathrm{Spec}\, O_L \arrow[r, "\text{exact closed imm}"] & Q \arrow[d] \\ & \mathrm{Spec}\, O_K \end{tikzcd}$$

Q ← log smooth rel $\underline{\text{dim } 1}$

$K \subset M \subset L$

M: max tamely ramified ext

Def We say L is a log monogenic ext'n of K if there exist a unif t of M & s of L s.t. $u = s^{e_{L/M}}/t$ is a generator of a purely insep ext E of F_M. //

E/F sep $\Rightarrow$ L/K log mono

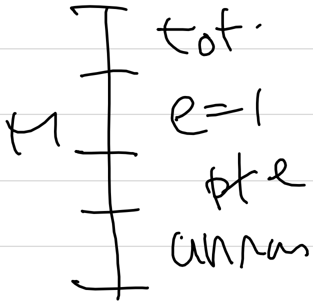
Prop Let $\overset{L/K \text{ log monogenic}}{\vee}$ $\underset{\text{max tame}}{M} \subset L' \subset L$.

If $[L':M] \leq [E:F_M]$

$\Rightarrow e_{L'/M} = 1$

If $[L':M] \geq [E:F_M]$

$\Rightarrow E' = E$.



Thm (H-A) version 3 $\quad L/K$ cyclic

Then $\quad \mathrm{Sw}\,\chi \leq \mathrm{cond}$ &

$= \iff L/K$ log. mono

classical $=$ $\quad \leq$ in gen'l $+$ $= \Leftarrow$ log mono

Conductor-discriminant formula.

$D_{L/K}^{-1} = \max\{I \mid \text{frac. ideal of } L. \text{ s.t. } Tr_{L/K} I \subset O_K\}$

$D_{L/K}^{\log\,-1} = \min\{J \mid \quad : \quad \text{s.t. } Tr_{L/K} J \supset O_K\}$

$\operatorname{ord} D_{L/K} \overset{\operatorname{ord}}{=} D^{\log}_{L/K} \cdot + e_{L/K} - 1$

Prop $\operatorname{ord} D^{l_y}_{L/K} \leq \sum_{\sigma \in P, \neq 1} i_G(\sigma)$

& $=$ $\Longleftrightarrow$ L/K log mono

Trace of diff fms

$$\mathrm{Tr} : \Omega^1_{\mathcal{O}_L} \longrightarrow \Omega^1_{\mathcal{O}_K}.$$

$$\underbrace{\mathcal{O}_L}_{B} \simeq \underbrace{\overset{,A}{\mathcal{O}_K}[x_1, \ldots, x_n]}_{C} / (f_1, \ldots \underline{f_n})$$

$$N_{B/C} \longrightarrow \Omega^1_C \otimes_C B \longrightarrow \Omega^1_B \longrightarrow 0 \quad \text{exact}$$

$$0 \longrightarrow \Omega^1_A \otimes_A C \longrightarrow \Omega^1_C \longrightarrow \Omega^1_{C/A} \longrightarrow 0 \quad \text{split exact}$$

$$\Omega^1_B \otimes_B \Lambda^n N_{B/C} \longrightarrow \Omega^{n+1}{}_C \otimes_C B$$

$$\Omega^{n+1}{}_C \longrightarrow \Omega^1_A \otimes_A \Omega^n{}_{C/A}$$

$$\Omega^1_B \longrightarrow \underbrace{\underset{\displaystyle \mathrm{Hom}_A(B, A)}{\mathrm{Hom}_B(\Lambda^n N_{B/C}, \Omega^n_{C/A} \otimes_C B)} \otimes_A \Omega^1_A}_{\displaystyle \mathrm{Hom}_A(B, \Omega^1_A)}$$

$$\mathrm{Tr}_{B/A}\, \Omega^1_B \longrightarrow \Omega^1_A$$

$\mathrm{Tr} : \Omega^1_{O_L} \to \Omega^1_{O_K}$

variant

$\Omega^1_{O_L}(\log) \to \Omega^1_{O_K}(\log)$

$\mathrm{Tr}_{L/K} : \Omega^1_E(\log) \to \Omega^1_F(\log)$

$$0 \to \Omega^1_F \to \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \to 0$$

$(,\ d\log t \longmapsto 1$

$\Omega^1_{O_K}(\log) \otimes_{O_K} F$

Thm (H-A) version 4

L/K cyclic Then $\mathrm{Sw}\, X \leq \mathrm{cond}\, X$

& TFAE

(1) = holds

(2) L/K log mono

(3) $\mathrm{Tr}_{L/K} : \Omega^1_E(\log) \to \Omega^1_F(\log)$

is non-zero

Application L/K log mono Gal $G = \mathrm{Gal}(L/K)$

V rep'n of G $n = \mathrm{Sw}\, V \in \mathbb{N}$ ism of F-v.sp of dim 1

$\mathrm{rsw}\, V \ : \ m_K^n / m_K^{n+1} \longrightarrow Z_{F\backslash E}(\log)^{\otimes d}$

$d = \dim(V / V^P)$

$Z_{F\backslash E}(\log) = \mathrm{Ker}(\Omega^1_F(\log) \to \Omega^1_E(\log))$