Hasse - Arf theorem

K c.d.v.f res fd F $\mathrm{char}\, F = p > 0$

Thm (Hasse-Arf) Version 0.
L cyclic ext'n of K. E res fd.
If E/F separable, conductor of L/K is an integer.

Reformulation
$\chi : G = \mathrm{Gal}(L/K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ faithful char.

$\chi \in X_K = \mathrm{Hom}(G_K^{ab}, \mathbb{Q}/\mathbb{Z})$

fil. on X_K (Kato)

$\{\ ,\ \} : X_K \times K^\times \longrightarrow \mathrm{Br}\, K$

$K_1 = \mathcal{O}_K[T]^\wedge_{m_K \mathcal{O}_K[T]} \otimes_{\mathcal{O}_K} K$ $n \geq 0$

$\mathrm{Fil}_n X_K = \{ \chi \in X_K \mid \{\chi, 1 + m_{K_1}^{n+1}\} = 0 \}$
increasing $F = \ell$.

$X_K = \bigcup \mathrm{Fil}_n X_K$. $\mathrm{Fil}_0 = \text{tame}$

$\mathrm{Sw}\, \chi = \min\{ n \mid \chi \in \mathrm{Fil}_n \}$ integer

Thm (H-A) L/K cyclic.
version 1 Kato
E/F sep $\Rightarrow$ $\mathrm{Sw}\,\chi = \mathrm{cond}$.

Conductor
L/K Gal ext. of c.d.v.f
not nec. cyclic or res. ext sep

$G = \mathrm{Gal}(L/K)$ lower numbering ram. gp

$G_i := \mathrm{Ker}\,(G \to \mathrm{Aut}(L^\times/(1+m_L^i)))$

$G_1 = P = p$-Sylow of $I = \mathrm{Ker}(G \to \mathrm{Aut}(E/F))$
$\bigcap G_i = 1$
$\sigma \in G, \neq 1, \; i_G(\sigma) = \max\{i \mid \sigma \in G_i\}$

L/K cyclic.
cond $= 0 \Leftrightarrow$ tame

$$\text{conductor} = \frac{1}{e_{L/K}}\left(\sum_{\sigma \in P, \neq 1} i_G(\sigma) + \max\{i_G(\sigma) \mid \sigma \in G, \neq 1\}\right)$$

Thm (H-A) L/K cyclic.
version 2 Thm. $\mathrm{Sw}\,\chi \leq \mathrm{cond}$
& if E/F sep $\Rightarrow$ $=$ holds "
Proof of $\leq$: $L^\times \xrightarrow[\text{compute}]{N} K^\times \xrightarrow{\{\chi, \cdot\}} \mathrm{Br}\,K \to \mathrm{Br}\,L$ exact

Question: Condition for $=$

Wild ramification

$P \subset \mathrm{Gal}(L/K)$ not nec. cyclic

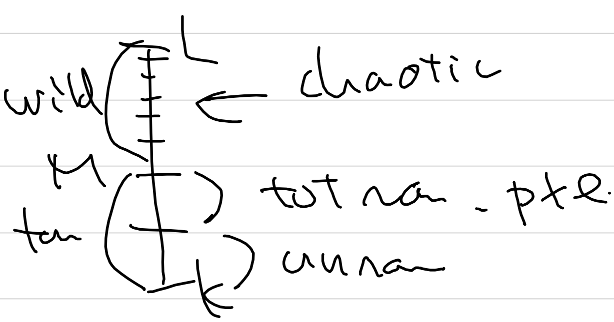
P p-gp nilpotent

$K \subset M = L_0 \subset L_1 \subset \cdots \subset L_n = L$

L_i/L_{i-1} either tot. ram.

or $e=1$. res. purely insep

there is no order in general



Def We say L is a <u>layered</u> ext'n of K

if there exist a unif t of M &

s of L

s.t. $u = s^{e_{L/M}}/t$ is a generator of

a purely insep ext E of F_M. //

E/F sep $\Rightarrow$ L/K layered.

L/K layered ext

Prop Let $M \subset L' \subset L$.

max tame

If $[L':M] \leq [E:F_M]$
$\Rightarrow e_{L'/M} = 1$

If $[L':M] \geq [E:F_M]$
$\Rightarrow E' = E$.

tot.
M — $e=1$
pte
unram

Thm (H-A) version 3 L/K cyclic

Then $\mathrm{Sw}\,\chi \leq \mathrm{cond}$ &

$=$ $\iff$ L/K layered.

classical = $\leq$ in gen'l + $=$ $\Leftarrow$ Layered.

Conductor-discriminant formula.

$D_{L/K}^{-1} = \max\{I \mid$ frac. ideal of L. s.t. $\mathrm{Tr}_{L/K} I \subset O_K\}$

$D_{L/K}^{\log\,-1} = \min\{J \mid$ " s.t. $\mathrm{Tr}_{L/K} J \supset O_K\}$

and $D_{L/K} \overset{\mathrm{ord}}{=} D_{L/K}^{\log} \cdot + e_{L/K} - 1$

Prop $\operatorname{ord} D^{\text{ly}}_{L/K} \leq \sum_{\sigma \in P_1, \neq 1} i_G(\sigma)$

& $=$ $\iff$ L/K layered.

Trace of diff fms

$\mathrm{Tr} : \Omega^1_{\mathcal{O}_L} \to \Omega^1_{\mathcal{O}_K}$.

$\underbrace{\mathcal{O}_L}_{B} \simeq \underbrace{\overset{\cup A}{\mathcal{O}_K}[x_1, \dots, x_n]}_{C} / (f_1, \dots f_n)$

$N_{B/C} \to \Omega^1_C \otimes_C B \to \Omega^1_B \to 0$ exact

$0 \to \Omega^1_A \otimes_A C \to \Omega^1_C \to \Omega^1_{C/A} \to 0$ split exact

$\Omega^1_B \otimes_B \Lambda^n N_{B/C} \to \Omega^{n+1}_C \otimes_C B$

$\Omega^{n+1}_C \to \Omega^1_A \otimes_A \Omega^n_{C/A}$

$\Omega^1_B \to \underbrace{\mathrm{Hom}_B(\Lambda^n N_{B/C}, \Omega^n_{C/A} \otimes_C B)}_{\displaystyle \| \atop \displaystyle \mathrm{Hom}_A(B, A)} \otimes_A \Omega^1_A$

$\| \atop \mathrm{Hom}_A(B, \Omega^1_A)$

$\mathrm{Tr}_{B/A} \, \Omega^1_B \to \Omega^1_A$

$\mathrm{Tr}: \Omega^1_{O_L} \to \Omega^1_{O_K}$
variant
$\Omega^1_{O_L}(\log) \to \Omega^1_{O_K}(\log)$

$\mathrm{Tr}_{L/K}: \Omega^1_E(\log) \to \Omega^1_F(\log)$

$0 \to \Omega^1_F \to \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \to 0$

$\quad a\, d\log t \longmapsto 1$

$\quad \Omega^1_{O_K}(\log) \otimes_{O_K} F$

Thm (H-A) version 4
L/K cyclic. Then $\mathrm{Sw}\,X \leq \mathrm{cond}\,X$
& TFAE
(1) = holds
(2) L/K layered
(3) $\mathrm{Tr}_{L/K}: \Omega^1_E(\log) \to \Omega^1_F(\log)$ is non-zero

Application L/K Layered Gal $G = \mathrm{Gal}(L/K)$

V rep'n of G $n = \mathrm{Sw}\,V \in \mathbb{N}$ ism of F-v.sp of
rsw V : $m_K^n / m_K^{n+1} \to Z_{F\backslash E}(\log)^{\otimes d}$ dim 1
$d = \dim(V/V^P)$
$Z_{F\backslash E}(\log) = \mathrm{Ker}\,(\Omega^1_F(\log) \to \Omega^1_E(\log))$