SS & C.C

k field. (alg closed for simplicity) $\ell \neq$ char k. based on the work of Deligne, Beilinson.

X smooth / k. Λ finite field char ℓ.

$\mathcal{F}$ constructible sheaf (or complex) / X_{et}.

Kashiwara - Schapira real manifold $\quad \mathrm{supp}\,\mathcal{F} = SS\mathcal{F} \cap T^*_X X$.

Singular support $C = \bigcup C_a \subset T^*X$ closed conical subset
$= SS\mathcal{F}$ $\quad n = \dim C_a = \dim X \quad \dim T^*X = 2n$

Characteristic cycle $CC\mathcal{F} = \sum m_a C_a \quad m_a \in \mathbb{Z},\ \geq 0$ if $\mathcal{F}$ perverse & $SS\mathcal{F} = |CC\mathcal{F}|$

index formula. If X proj $\chi(X, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$
intersection number with 0-section

1 Example $\quad \dim X = 1$ Grothendieck - Ogg - Shafarevich

Example. 1. $\dim X = 1$. $\quad SS\mathcal{F} = T^*_X X \cup \bigcup_x T^*_x X$
0-section. $\quad$ fiber at x where $\mathcal{F}$ ramifies.

$$CC\mathcal{F} = -(\mathrm{rk}\,\mathcal{F} \cdot T^*_X X + \sum_{x \in X} a_x(\mathcal{F}) \cdot T^*_x X)$$

$a_x\mathcal{F}$ Artin conductor $= \mathrm{rk}\,\mathcal{F}_{\bar\eta} - \mathrm{rk}\,\mathcal{F}_x + \mathrm{Sw}_x \mathcal{F}$

2 $D \subset X$ div. w. SNC. $U = X - D$ $\mathcal{F} = j_! \mathcal{G}$. $\mathcal{G}$ loc. const on U tamely ramified along D

$SS\mathcal{F} = \bigcup_I T^*_{X_I} X \quad D = \bigcup D_i \quad X_I = \bigcap_{i \in I} D_i$
conormal b'dle.

$CC\mathcal{F} = (-1)^n\, \mathrm{rk}\,\mathcal{F} \sum_I T^*_{X_I} X \qquad n = \dim X$.

3 $X = \mathbb{A}^2$ $D = (x = 0)$ $p \neq 2$ $U = X - D$ $\mathcal{F} = j_! \mathcal{G}$ $\mathcal{G}$ $t^p - t = \frac{y}{x^p}$

$SS\mathcal{F} = T^*_X X \cup \langle dy \rangle_D \qquad \langle dy \rangle_D \simeq T^*D. \quad D \times_X T^*X = T^*D \oplus T^*_D X$

$CC\mathcal{F} = T^*_X X + p \langle dy \rangle_D$

2. S.S.

Def. 1. $f : X \to Y$ C-transversal. if $df^{-1}(C) \subset 0$-section
$df : X \times_Y T^*Y \to T^*X \supset C$.

Example $C = T^*_X X$. f C-trans $\Leftrightarrow$ f smooth.

2. $\mathcal{F}$ weakly micro-supp on C. if $\forall X \supset U \xrightarrow{f} Y$ curve.
f. C-trans $\Rightarrow$ f. loc acyclic rel to $\mathcal{F}$ (i.e. $\Phi(\mathcal{F}, f) = 0$)

Example $\mathcal{F}$ locally const $\Rightarrow$ $\mathcal{F}$ micro supp on $T^*_X X$

3 $SS\mathcal{F}$ smallest C. s.t $\mathcal{F}$ is weakly micro supp'd on C.
closure of $\{(u, df) \mid X \supset U \xrightarrow{f} Y \quad \Phi_u(\mathcal{F}, f) \neq 0\}$

Thm 1. (Beilinson). $SS\mathcal{F} = \bigcup C_a$. $\dim C_a = \dim X$.

3. C.C.

Def · $X \supset U \xrightarrow{f} Y$ $u \in U$ isol. char w.r.t $C \subset T^*X$
if $f|_{U - \{u\}}$ is C-transversal.
· If u isol. char $A = \sum m_a C_a$. $C = \bigcup C_a$. $\dim C_a = n$
$\Rightarrow$ $(A, df)_{T^*U, u}$ is defined.

Thm 2 $\exists 1$ $CC\mathcal{F} = \sum m_a C_a$ $(SS\mathcal{F} = \bigcup C_a)$
$\forall X \supset U \xrightarrow{f} Y$ u iso char pt.
$-\underset{\substack{\| \\ \dim + \mathrm{Sw}.}}{\dim \mathrm{tot}}\, \Phi_u(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*U, u}$.
Milnor formula
$m_a \in \mathbb{Z}[\frac{1}{p}]$. $, \in \mathbb{Z}$ (Deligne-Beilinson). generalization of Hasse-Arf.

$\mathcal{F} = \Lambda$ $CC\mathcal{F} = (-1)^n T^*_X X$. Milnor formula Deligne SGA 7 XVI

Thm 3 If X proj $\chi(X, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$.

Idea of Pf

Brylinski

1 Beilinson · Radon transform. $\mathbb{P} \leftarrow Q \rightarrow \mathbb{P}^\vee$ $\qquad \mathbb{P}(SS\mathcal{F}) = \mathbb{P}(SS R\mathcal{F})$

· perverse sheaves. structure of simple per.

2 $CC\mathcal{F}$ · $X \rightarrow \mathbb{P}$ $\qquad X \leftarrow X \times_{\mathbb{P}} Q \rightarrow \mathbb{P}^\vee$

$\cup$ $\mathbb{P}(\tilde{C_a})$ $\rightsquigarrow$ ma.

· Continuity of Swan conductor. Deligne - Laumon

(· $X \leftarrow U \rightarrow A^1$ s.t $C(a, df) \underset{u}{=} 1$ – integrality

formalism of ~~vanishing~~ vanishing cycles / general base Deligne – Orgogozo

3 index formula. $h: W \rightarrow X$ smooth hypersurface. properly C-transversal

$h^*C = W \times_X C$ $^{(1)}$ dim W. $\quad \subset W \times_X T^*X$

$\cap\ T^*_W X \overset{(2)}{\supset} C$ 0-section

$h^! CC\mathcal{F}$ $\qquad T^*X \leftarrow W \times_X T^*X \rightarrow T^*W$ $\qquad \times(-1)$.

$\Rightarrow$ $\quad CC h^*\mathcal{F} = h^! CC\mathcal{F}$.

induction on dim $\qquad$ dim $=1$ GOS

$X_L \rightarrow X$ $\qquad \chi(X_L, \mathcal{F}) = 2 \cdot \chi(W, \mathcal{F}) - \sum \mathrm{dimtot}_u(\mathcal{F}, p_L)$

$p_L \downarrow$ $\qquad \shortparallel \qquad$ ind. $\qquad$ Milnor formula

L $\qquad \chi(V, \mathcal{F}) + \chi(X, \mathcal{F})$

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