CC

k char $p \geq 0$ perfect. X/k smooth. dim n

Λ finite f. ch $\ell \neq p$. $\mathcal{F}$ const sh on X_{et} ex of Λ-mod.

CC, SS. T^*X cotangent bdle.

v.b of rk n. / X dim $2n$

tot dim $= 2n$

Beilinson

$SS\mathcal{F}$ closed conical subset

stable under $\mathbb{G}_m$-action

$C = SS\mathcal{F} = \bigcup_a C_a$ irred cpt

$\dim C_a = \dim X = n$.

base of C

$B(C) = C \cap T^*_X X \subset X = T^*_X X$ 0-section

$= \operatorname{supp} \mathcal{F}$. cplt of largest U s.t. $\mathcal{F}|_U = 0$.

$CC\mathcal{F} = \sum_a m_a C_a \quad m_a \in \mathbb{Z}$

$\mathcal{F}$ perverse $\Rightarrow$ $m_a > 0 \quad \forall C_a$.

$\mathcal{F}$ l.c $\Leftrightarrow$ $SS\mathcal{F} \subset T^*_X X$.

$\Rightarrow$ $CC\mathcal{F} = (-1)^n \operatorname{rk} \mathcal{F} \cdot T^*_X X$.

Example 1'. $\dim X = 1$.

C_a either 0-section $T^*_X X$ or
fiber $T^*_x X$

$\operatorname{supp} \mathcal{F} = X \Rightarrow SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in D} T^*_x X$

$D = X - U$ U largest open st $\mathcal{F}|_U$ l.c.c.

$$C(\mathcal{F}) = -\left(\operatorname{rk} \mathcal{F}|_U \, T^*_X X + \sum_{x \in D} a_x \mathcal{F} \cdot T^*_x X\right)$$

$a_x \mathcal{F}$ Artin conductor

$= \operatorname{rk} \mathcal{F}|_U - \operatorname{rk} \mathcal{F}_{\bar{x}} + \operatorname{Sw}_x \mathcal{F}$

$\operatorname{Sw}_x \mathcal{F}$ Swan conductor measure of
<u>wild ramification</u>

higher dim. not Lagrangian in general.
exception dim 0.

~~Functoriality~~ S.S

$X \times_Y T^*Y \to T^*X$ 13

$(C(X \times (T^*X \times T^*Y) \to T^*X)$ $= C \times T^*Y$

$C \subset T^*X$ closed conical.

$h: W \to X$ C-trans. if

$h^*C (= W \times_X T^*X) \cap \ker(W \times_X T^*X \to T^*W)$

$\subset W \times_X T^*_X X$. 0-section

$X \xleftarrow{h} W \xrightarrow{f} Y$ C-trans. if

$(h, f): W \to X \times Y$ $C \times T^*Y$-trans

Def 1 We say $\mathcal{F}$ is micro supp on C

if $\forall$ C-trans (h, f),

f is ULA rel to $h^*\mathcal{F}$.

$f: X \to Y$ LA rel to $\mathcal{F}$

$\forall x \leftarrow y$ $\mathcal{F}_x \xrightarrow{\simeq} R\Gamma(X_{(x)} \times_{Y_{(f(x))}} y, \mathcal{F})$

~~smooth b.c.~~ l.a. of smooth. $\Rightarrow$ $\mathcal{F}$ l.c. $\Rightarrow$ micro supp on $T^*_X X$. ($\Leftarrow$ easier)

Def 2. $SS\mathcal{F}$. smallest C s.t. $\mathcal{F}$ is micro supp on C.

Th (Beilinson) 1. $SS\mathcal{F}$ exists.

2 $\forall C_a$ $SS\mathcal{F} = \bigcup C_a$. $\dim C_a = n$.

Radon trans. + Lefschetz pencil.

f is good with C $\Rightarrow$ f is good with $\mathcal{F}$

$C = SS\mathcal{F}$

Functoriality

$h: W \to X \quad C \subset T^*X$ closed conical dim n. $m = \dim W$

h C-trans. + $\dim h^\circ C = m$ ($\geq$ always)

properly C-transversal.

smooth $\Rightarrow$ $\forall C$ properly C-trans.

$$\begin{array}{ccccc} T^*X & \xleftarrow{p} & W \times_X T^*X & \xrightarrow{q} & T^*W \\ \cup & & \cup & & \cup \\ C & \leftarrow & h^*C & \xrightarrow{\text{fin}} & h^\circ C \end{array}$$

$A = \sum m_a C_a \mapsto p^! A \longmapsto (-1)^{m-n} q_! p^! A$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\quad h^! A \in Z_m(h^\circ C)$

$f: X \to Y$ proper

$$T^*X \xleftarrow{q} X \times_Y T^*Y \xrightarrow{p} T^*Y \qquad \dim (f_\circ C) \leq m$$

$A \qquad q^! A \longrightarrow p_! q^! A \in Z_m(f_\circ C)$

$\qquad\qquad\qquad\qquad \|$

Theorem. There exists $\exists$ way to associate $m_a \in \mathbb{Z}$

$CC\mathcal{F} = \sum m_a C_a$ ($SS\mathcal{F} = \bigcup C_a$) satisfying (1)-(5)

(1) (Normalization) $X = \mathrm{Spec}\, k$. $\mathcal{F} = \Lambda \Rightarrow CC\mathcal{F} = 1.$ [pt]

(2) (add.) $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$, $\Rightarrow CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''$.

(3) (pull-back) $h: W \to X$. properly $SS\mathcal{F}$-tran

$\Rightarrow CC h^*\mathcal{F} = h^! CC\mathcal{F}$.

(4) (closed im) $i: X \to P$ closed im

$\Rightarrow CC\, i_* \mathcal{F} = i_! CC\mathcal{F}$

(5) (Radon) $\mathbb{P}^n = P \xleftarrow{p_1} Q \xrightarrow{p_2} P^\vee$ = universal hyp plane $\mathcal{F}$ on $\mathbb{P}$

$CC(p_{2*} p_1^* \mathcal{F}) = p_{2!} p_1^! CC\mathcal{F}$

Cor (index formula) X proj smooth

$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{X}$ $\left(\begin{array}{c}\dim X = 1 \\ \text{G-O-S}\end{array}\right)$