Wild ramification of schemes and sheaves

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Wild ramification of schemes and sheaves

Partly joint work with

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Two major problems in ramification theory

- Compute global cohomological invariants

  (e.g. Euler number, conductor)

in terms of invariants of ramification

  (e.g. Swan class, characteristic class)

- Describe these invariants of ramification

  using ramification groups

  and characteristic cycle

New inputs ($\sim$ 80's)

- Define invariants of ramification as 0-cycle classes (Bloch)

- Analogy with irregular singularity of $\mathcal{D}$-modules. (Deligne, Kato, ...)

New inputs (00's)

- Lefschetz trace formulas for open varieties or over a log point, dvr etc.

- Kill ramification by blow-up, instead of ramified covering

PLAN

1. Euler numbers and conductors
2. Characteristic class of an ℓ-adic sheaf
3. Ramification groups of a local field
4. Blow-up to kill ramification
5. Graded pieces of ramification groups
6. Bounding wild ramification
7. Characteristic cycle of an ℓ-adic sheaf

1. EULER NUMBERS AND CONDUCTORS

(a) Generalization of the Grothendieck-Ogg-Shafarevich formula (SGA5)

U/k : separated of finite type, $k = \bar{k}$,

$\mathcal{F}/U$: $\mathbb{Q}_\ell$-sheaf, $\ell \neq \text{char } k$,

$$\chi_c(U, \mathcal{F}) := \sum_{q=0}^{2\dim U} (-1)^q \dim H_c^q(U, \mathcal{F}).$$

THEOREM 1 (Kato-S.)

$k = \bar{k}$, $\ell \neq \text{char } k > 0$, U/k smooth, $\mathcal{F}/U$ smooth,

Then :

$$\chi_c(U, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U, \mathbb{Q}_\ell) - \deg \mathrm{Sw}_U \mathcal{F}.$$

$\mathrm{Sw}_U \mathcal{F} \in CH_0(X \setminus U)_{\mathbb{Q}}$: Swan class of $\mathcal{F}$,

X : compactification of U.

(b) Conductor formula

K : complete discrete valuation field (=cdvf),

$k = \bar{k}$: residue field,

U/K : separated of finite type,

$\mathcal{F}/U$: $\mathbb{Q}_\ell$-sheaf, $\ell \neq \text{char } k$,

$$\mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F}) := \sum_{q=0}^{2\dim U} (-1)^q \mathrm{Sw}_K H_c^q(U_{\bar{K}}, \mathcal{F})$$

Sw_K: Swan conductor.

THEOREM 2 (Kato-S.)

K cdvf, $k = \bar{k}$, $\ell \neq \text{char } k > \text{char } K = 0$,

U/K smooth, $\mathcal{F}/U$ smooth.

Then :

$$\mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F}) = \mathrm{rank}\, \mathcal{F} \cdot \mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathbb{Q}_\ell) - \deg \mathrm{Sw}_U \mathcal{F}.$$

$\mathrm{Sw}_U \mathcal{F} \in F_0 G(X_k)_{\mathbb{Q}}$: Swan class of $\mathcal{F}$,

X : compactification of U over $\mathcal{O}_K$.

- Formula for $\mathsf{Sw}_K H^*_c(U_{\bar{K}}, \mathbb{Q}_\ell)$ (Bloch, Kato-S.)
- Relative version (Kato-S.) :

ℓ-adic Grothendieck-Riemann-Roch

$$\overline{\mathsf{Sw}}_V Rf_!\mathcal{F} = f_!\overline{\mathsf{Sw}}_U\mathcal{F}.$$

$f : U \to V/K$: separated of finite type,

$\mathcal{F}/U$: $\mathbb{Q}_\ell$-sheaf.

2. CHARACTERISTIC CLASS OF AN ℓ-ADIC SHEAF

X/k : separated of finite type,

$\ell \neq \text{char } k$,

$\mathcal{F}/X$: $\mathbb{Q}_\ell$-sheaf

DEFINITION Characteristic class $C(\mathcal{F})$:

$$\begin{array}{ccccc} 1 & \in & \mathrm{End}(\mathcal{F}) & = & H^0_X(X \times_k X, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})) \\ \downarrow & & & & \downarrow \delta^* \\ C(\mathcal{F}) & \in & H^0(X, K_X) & \xleftarrow{\mathrm{Tr}} & H^0(X, R\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes K_X)) \end{array}$$

$\delta : X \to X \times X$: diagonal, $\mathrm{pr}_i : X \times X \to X$ projections,

$K_X = Ra^!\mathbb{Q}_\ell$, $a : X \to \mathrm{Spec}\ k$,

$(= \mathbb{Q}_\ell(d)[2d]$ if X smooth of dim $d)$,

Tr : Trace map.

THEOREM 3 (Abbes-S.)

U/k smooth,

$\mathcal{F}/U$ smooth, $\ell \neq$ char k,

$j\colon U \to X/k$ open immersion.

Then, if $\mathcal{F}$ is "potentially of Kummer type", :

$$C(j_!\mathcal{F}) = \text{rank } \mathcal{F} \cdot C(j_!\mathbb{Q}_\ell) - \text{cl}(\text{Sw}_U\mathcal{F}).$$

cl: $CH_0(X \setminus U) \to H^0(X, K_X)$, cycle class map.

- Theorem 3 $\Rightarrow$ Theorem 1.
- If X proper, Lefschetz trace formula (SGA5)

$\Rightarrow$

$$\begin{array}{ccc} C(\mathcal{F}) & \in H^0(X, K_X) \\ \downarrow & \big\downarrow \mathrm{Tr}_X \\ \chi_c(U, \mathcal{F}) \in & \mathbb{Q}_\ell \end{array}$$

GOAL:

(1) Define the characteristic cycle $CC(\mathcal{F})$ as a cycle on the (log) cotangent bundle.

(2) Prove

$$C(j_!\mathcal{F}) = \mathrm{cl}(CC(\mathcal{F})).$$

METHOD:

Blow-up the ramification locus in the diagonal.

3. RAMIFICATION GROUPS OF A LOCAL FIELD

K : complete discrete valuation field,

F : residue field (may be non-perfect),

L/K : finite Galois extension,

$G = \mathrm{Gal}(L/K)$,

(a) Lower numbering filtration (classical)

$$G = G_0 \supset G_1 \supset \cdots$$
$$\supset G_i = \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i)) \supset \cdots \supset 1.$$

compatible with subgroup

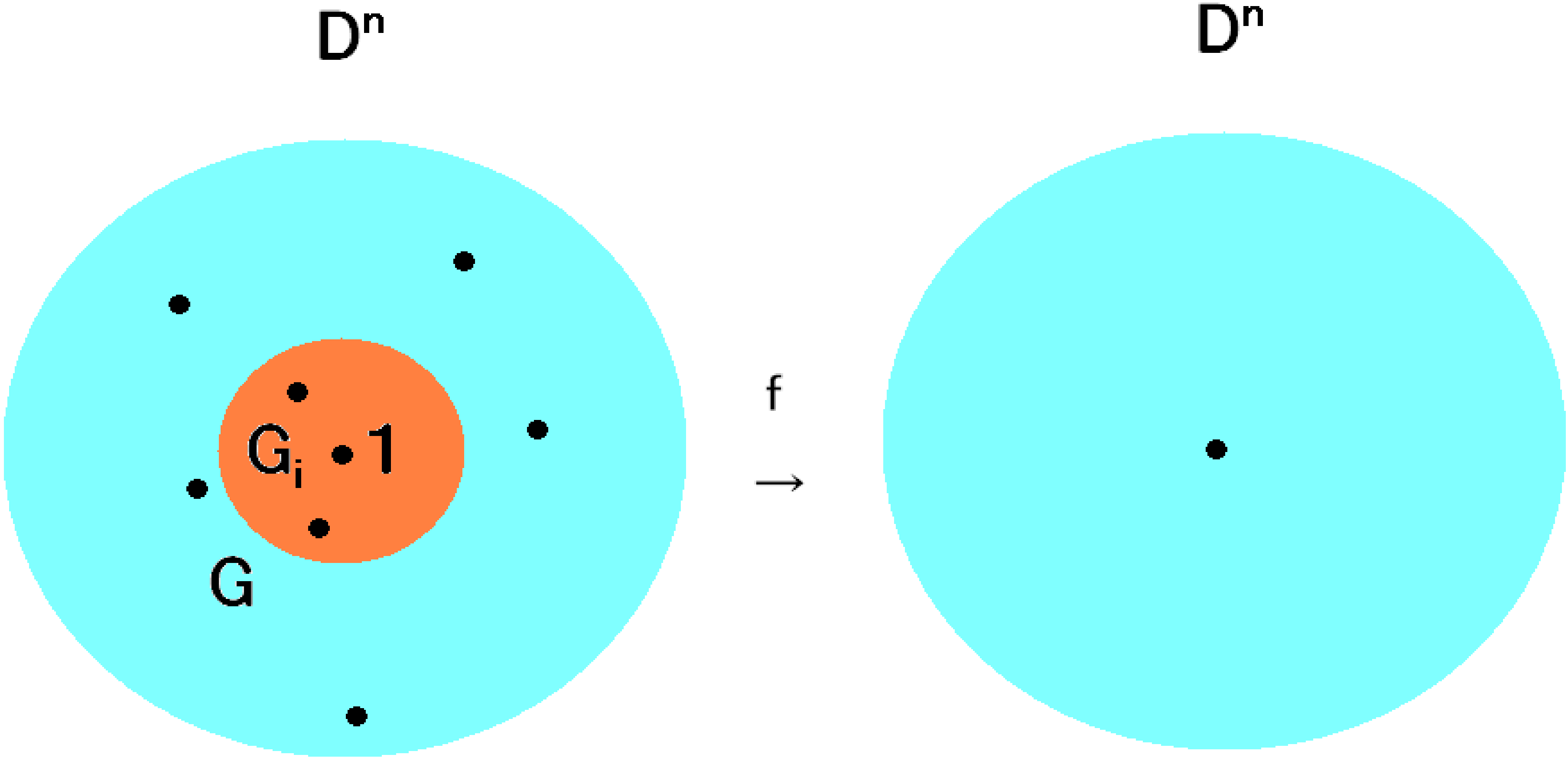
Geometric interpretation.

$\mathcal{O}_L = \mathcal{O}_K[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$.

D^n : rigid analytic disk of dimension n over K.

$$\begin{array}{rccc} f: & D^n & \longrightarrow & D^n \\ & (x_1, \ldots, x_n) & \mapsto & (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \end{array}$$

$$\begin{aligned} G &= \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_L, \overline{K}) = f^{-1}(0) \\ &\supset G_i = G \cap D^n\left(\frac{i}{e_{L/K}}\right) = \{x \in G \mid d(x, 1) \le |\pi_L|^i\}. \end{aligned}$$



D^n
G_i
1
G
f
→
D^n

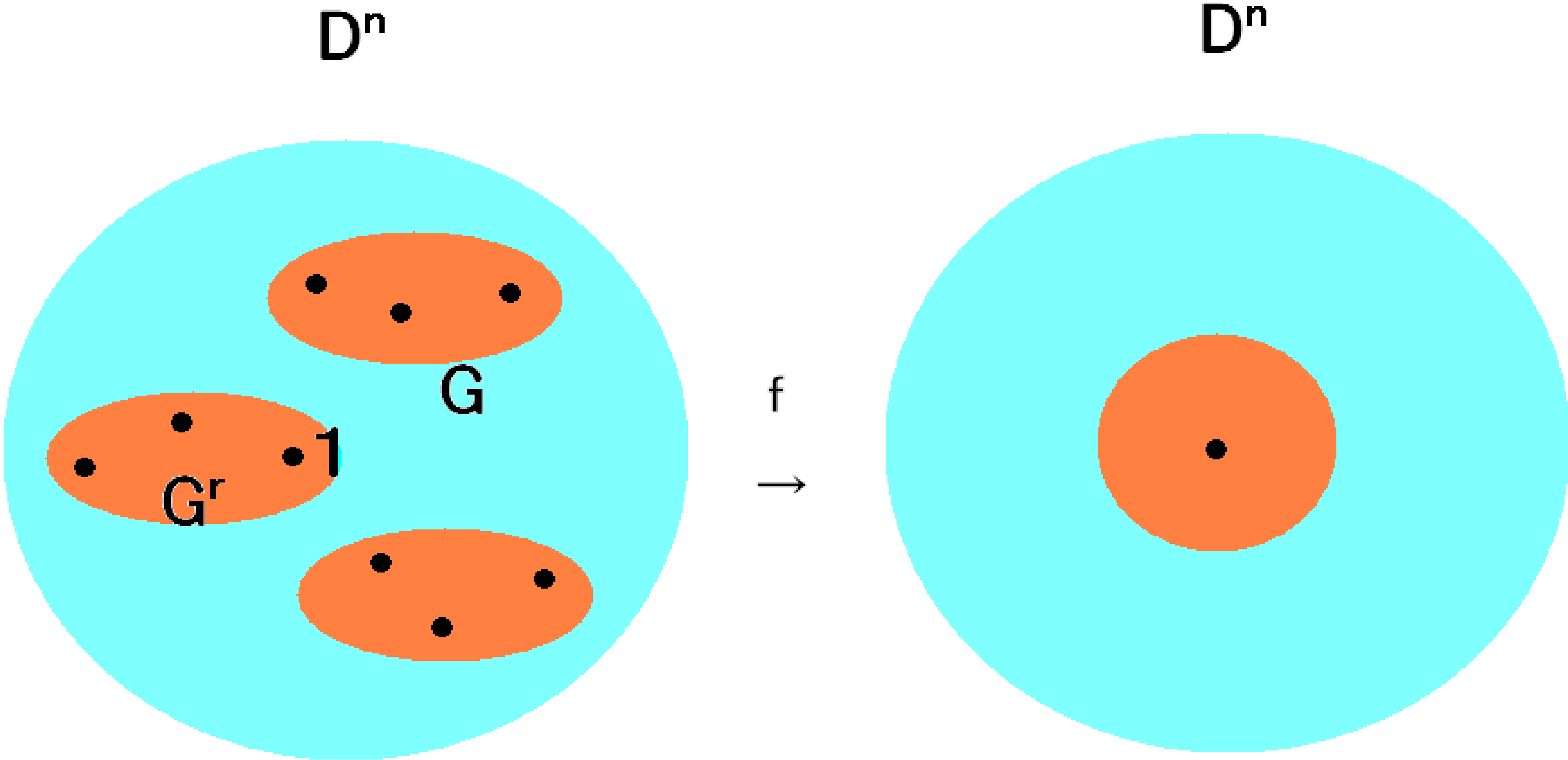
(b) Upper numbering filtration

DEFINITION $r \in \mathbb{Q}, r > 0$,

$$G^r := G \cap \begin{pmatrix} \text{connected cpt of } f^{-1}(D^n(r)) \\ \text{containing } 1 \in G \end{pmatrix}.$$

$$f\colon D^n \to D^n \supset D^n(r) = \{x \in D^n \mid d(x, 0) \leq |\pi_K|^r\}.$$

- compatible with quotient
- classical if F perfect


D^n
G
1
G^r
f
→
D^n

FIRST STEPS :

(1a) Relate characters of graded pieces of ramification groups to differential forms.

(1b) Define the characteristic cycle of an ℓ-adic sheaf using these differential forms.

- shrinking the radius (rigid geometry)

  $=$ blowing-up (algebraic geometry),
- logarithmic variant $G^r_{\log}$,
- globalizing,
- analogy with $\mathcal{D}$-modules ...

$$\Downarrow$$

- Graded pieces of ramification groups
- Definition of the characteristic cycle

4. BLOW-UP TO KILL RAMIFICATION

$k = \bar{k}$, char $k = p > 0$,

X/k smooth,

$U = X \setminus D$ complement of a divisor

$D = \cup_i D_i$ simple normal crossings (=sncd).

(a) Blow-up to kill tame ramification

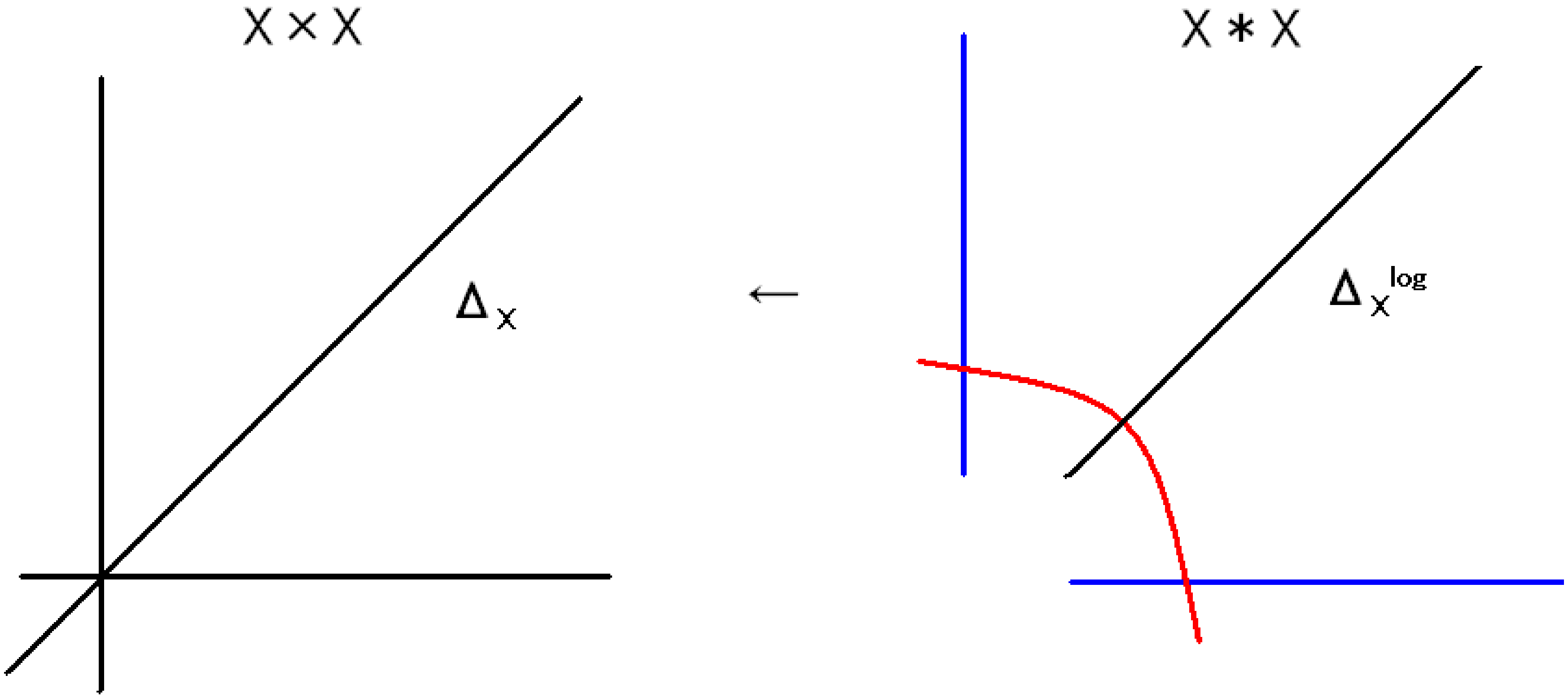
$$X * X \to X \times X$$

Blow up at all $D_i \times D_i$,

Remove proper transform of $(D \times X) \cup (X \times D)$.

log diagonal $X \to X * X$.

conormal sheaf $N_{X/X*X} = \Omega^1_X(\log D)$.


$X \times X$
Δ_X
$\longleftarrow$
$X * X$
$\Delta_X^{\log}$

EXAMPLE

$X = \mathbf{A}^1_k =$ Spec $k[x] \supset U =$ Spec $k[x^{\pm 1}]$

$$\begin{array}{lcl} X \times_k X & \leftarrow & X * X \\ \quad = \text{Spec } k[x,y] & & \quad = \text{Spec } k[x,y,u^{\pm 1}]/(x-uy). \end{array}$$

$\mathcal{F}/U \,:\, T^n = x,\ p \nmid n,$

$\Rightarrow \mathcal{H} = \mathcal{H}om(\text{pr}_2^*\mathcal{F}, \text{pr}_1^*\mathcal{F})/U \times U \,:\, T^n = x/y = u$

smooth extension on $X * X \supset U \times U$.

(b) Blow-up to kill wild ramification

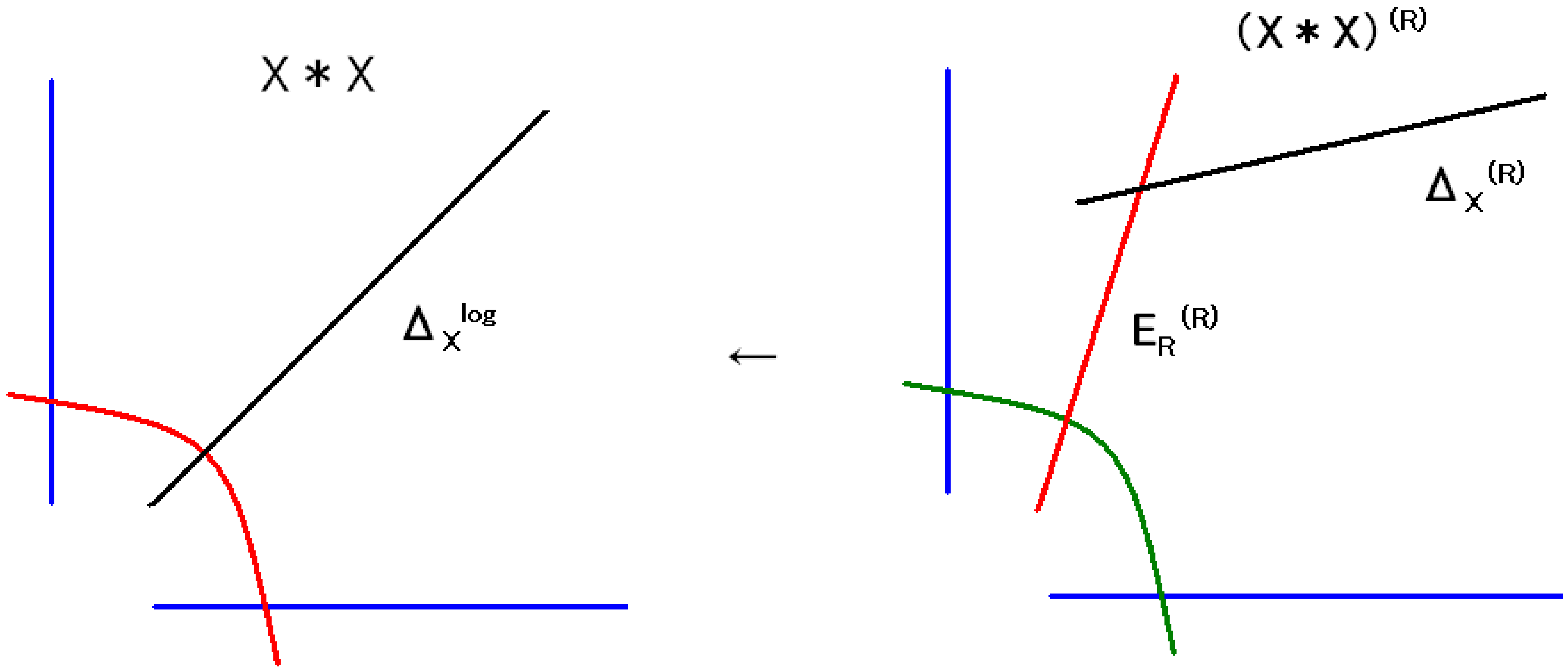
$R = \sum_i r_i D_i,\ r_i \in \mathbb{Q},\ r_i \geq 0$.

(Assume $r_i \in \mathbb{N}$ for simplicity.)

$$(X * X)^{(R)} \to X * X$$

Blow up at $R \subset X$ in the log diagonal $X \to X * X$,

Remove proper transform of $(X * X) \times_X R$.



$X * X$
$\Delta_X^{\log}$
$\leftarrow$
$(X * X)^{(R)}$
$\Delta_X^{(R)}$
$E_R^{(R)}$

EXAMPLE

$X = \mathbf{A}^1_k = \text{Spec } k[x] \supset U = \text{Spec } k[x^{\pm 1}]$,

$$\mathcal{F}/U: \;\; T^p - T = \frac{1}{x^r}, \quad p \nmid r, \; r \in \mathbb{N},$$

$$\mathcal{H} = \mathcal{H}om(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F})/U \times U$$

$R = r(0)$,

$$\begin{array}{ccl} (X * X)^{(R)} & = & \text{Spec } k[x, y, t, (1 + ty^r)^{\pm 1}]/(x - (1 + ty^r)y) \\ \downarrow & & \\ X * X & = & \text{Spec } k[x, y, u^{\pm 1}]/(x - uy). \end{array}$$

$$\begin{aligned} \mathcal{H}: \quad T^p - T &= \frac{1}{x^r} - \frac{1}{y^r} \\ &= -(1 + ty^r)^{-r} \left(rt + \binom{r}{2} t^2 y^r + \cdots \right) \end{aligned}$$

smooth extension on $(X * X)^{(R)} \supset U \times U$.

lift of the diagonal $\delta^{(R)}\colon X \to (X * X)^{(R)}$.

conormal sheaf $N_{X/(X*X)^{(R)}} = \Omega^1_X(\log D)(R)$.

$$
\begin{array}{ccccc}
E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
\downarrow & & \downarrow \mathrm{pr}_2 & & \downarrow \mathrm{pr}_2 \\
R & \xrightarrow{i_R} & X & \xleftarrow{j} & U
\end{array}
$$

$E_R^{(R)} = V(\Omega^1_X(\log D)(R)) \times_X R$: twisted tangent b'dle

$(V(\mathcal{E}) = \mathrm{Spec}\ S^\bullet \mathcal{E})$

$\mathcal{F}/U$ smooth $\mathbb{Q}_\ell$-sheaf. $\ell \neq p = \text{char } k > 0$.

$\mathcal{H} = \mathcal{H}om(\text{pr}_2^*\mathcal{F}, \text{pr}_1^*\mathcal{F})$ on $U \times U$.

- $i_R^{(R)*} j_*^{(R)} \mathcal{H}$ on $E_R^{(R)} = V(\Omega^1_X(\log D)(R)) \times_X R$.

$$\begin{array}{ccccc} E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\ & & \delta^{(R)} \uparrow & & \uparrow \delta_U \\ & & X & \xleftarrow{j} & U \end{array}$$

- $\delta^{(R)*} j_*^{(R)} \mathcal{H} \to j_* \delta_U^* \mathcal{H} = j_* \mathcal{E}nd(\mathcal{F})$ base change map

5. GRADED PIECES OF RAMIFICATION GROUPS

$U = X \setminus D$, D sncd,

$K = \text{Frac}(\widehat{\mathcal{O}}_{X,\xi})$: local field at the generic point ξ

of an irreducible component of D.

$F = F(\xi)$: residue fd = function fd of the cpt.

non perfect if $\dim X > 1$

$G_K = \mathrm{Gal}(\bar{K}/K)$, log upper numbering filtration

DEFINITION $r \in \mathbb{Q},\ r > 0$,

$$\mathrm{Gr}^r_{\log} G_K = G^r_{K,\log} / G^{r+}_{K,\log},$$

$$G^{r+}_{K,\log} = \overline{\bigcup_{s>r} G^s_{K,\log}}$$

PROPOSITION 4 (Abbes-S., S.) $r \in \mathbb{Q},\ r > 0$,

$\mathrm{Gr}^r_{\log} G_K$ is abelian, killed by p.

$\mathcal{F}/U$: smooth $\mathbb{Q}_\ell$-sheaf,

V : ℓ-adic representation of G_K defined by $\mathcal{F}$

$$V = \bigoplus_{r \geq 0} V^{(r)} : \text{slope decomposition}$$

$$V^{(r)}|_{G^r_{K,\log}} = \bigoplus_{\chi} V_\chi \otimes \chi$$

decomposition by characters of $\mathrm{Gr}^r_{\log} G_K$.

Assume $r > 0$, $V = V^{(r)} = V_\chi \otimes \chi$, $R = rD$, D irred.

THEOREM 5 $\mathcal{H} = \mathcal{H}om(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F}) / U \times U$

Then : $\quad j_*^{(R)} \mathcal{H}$ is smooth. (=ramification killed)

$\exists !$ linear form $f_\chi \neq 0$ on $E_{\bar{\xi}}^{(R)}$ s.t.

$$(j_*^{(R)} \mathcal{H})|_{E_{\bar{\xi}}^{(R)}} \simeq \text{End}(V_\chi) \otimes \mathcal{L}_\chi$$

$\mathcal{L}_\chi$: sm rk 1 def'd by Artin-Schreier eqn $T^p - T = f_\chi$

$$\begin{array}{ccccccc}
E_{\bar{\xi}}^{(R)} & \longrightarrow & E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
\downarrow & & \downarrow & & \downarrow & & \\
\bar{\xi} & \longrightarrow & R = rD & \longrightarrow & X & &
\end{array}$$

χ : char. of $\mathrm{Gr}^r_{\log} G_K$, $V = V_\chi \otimes \chi$, $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$

$$(j_*^{(R)} \mathcal{H})|_{E_{\bar{\xi}}^{(R)}} = \mathrm{End}(V_\chi) \otimes \mathcal{L}_\chi, \quad \mathcal{L}_\chi : T^p - T = f_\chi$$

DEFINITION Refined Swan conductor:

linear form $f_\chi \neq 0$ on $E_{\bar{\xi}}^{(R)} = V(\Omega^1_X(\log D)(R)) \times_X \bar{\xi}$

regarded as a differential form

$$\mathrm{rsw}\chi \in \Omega^1_X(\log D)(R) \otimes F(\bar{\xi}).$$

(Kato : rank 1 case.)

6. BOUNDING WILD RAMIFICATION

$U = X \setminus D, D = \cup_i D_i$ sncd.

$\mathcal{F}/U$ smooth,

$R = \sum_i r_i D_i,\ r_i \in \mathbb{Q},\ r_i \geq 0$,

$(r_i \in \mathbb{N},\ r_i > 0$ for simplicity).

DEFINITION $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})/U \times U$,

Ramification of $\mathcal{F}$ along D is bounded by $R+$:

base change map

$$\delta^{(R)*} j_*^{(R)} \mathcal{H} \longrightarrow j_* \delta_U^* \mathcal{H}$$

is an isomorphism.

$$\begin{array}{ccc}
(X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
{\scriptstyle \delta^{(R)}} \uparrow & & \uparrow {\scriptstyle \delta_U} \\
X & \xleftarrow{j} & U
\end{array}$$

ramification of $\mathcal{F}$ along D is bounded by $R+$

$\Rightarrow G^{r_i+}_{K_i,\log}$ acts on V trivially $\forall\ i$. $\qquad (R = \sum_i r_i D_i)$

Assume further $V = V^{(r_i)}$ $\forall i$. $\qquad (R = \sum_i r_i D_i)$

$V = \oplus_\chi V_\chi \otimes \chi$, χ non-trivial char. of $\mathrm{Gr}^{r_i}_{K_i,\log}$.

rswχ: linear form on $E^{(R)}_{\bar{\xi}_i}$ = a point of the dual $\overset{\vee}{E}{}^{(R)}_{\bar{\xi}_i}$.

$$\mathrm{rsw}\chi \in \overset{\vee}{E}{}^{(R)}_{\bar{\xi}_i} \to \overset{\vee}{E}{}^{(R)}_{D_i}, \quad \bar{\xi}_i \to \xi_i \in D_i.$$

PROPOSITION 6 (Abbes-S.)

$A_\chi = \overline{\mathrm{Image}(\mathrm{rsw}\chi)} \subset \overset{\vee}{E}{}^{(R)}_{D_i}$ is finite over D_i.

7. CHARACTERISTIC CYCLE OF AN ℓ-ADIC SHEAF

$U = X \setminus D, D = \bigcup_i D_i$ sncd.

$\mathcal{F}/U$ smooth,

$R = \sum_i r_i D_i, \; r_i \in \mathbb{Q}, \; r_i \geq 0,$

$(r_i \in \mathbb{N}, \; r_i > 0$ for simplicity).

Assume:

- Ramification of $\mathcal{F}$ along D is bdd by $R+$,
- $V = V^{(r_i)}$ $\forall$ i, $(R = \sum_i r_i D_i)$
- Cleanliness condition (Kato : rank 1 case):

$V = \oplus_\chi V_\chi \otimes \chi$, $A_\chi \cap$ (0-section) $= \emptyset$ $\forall$ i $\forall$ χ.

$$(A_\chi = \overline{\text{Image}(\text{rsw}\chi)} \subset \overset{\vee}{E}{}^{(R)}_{D_i})$$

(Wild ramification is controlled at codimension 1 points.)

$T^*X(\log D) = V(\Omega_X(\log D)^\vee)$: log cotangent bundle,

$L(-R) = V(\mathcal{O}_X(R))$: line bundle,

- $A_\chi \subset \overset{\vee}{E}{}^{(R)}_{D_i} = V(\Omega^1_X(\log D)(R)^\vee) \times_X D_i$

$\Rightarrow$ $a_\chi \colon L(-R) \times_X A_\chi \to T^*X(\log D) \times_X A_\chi$ linear

- $A_\chi \cap$ (0-section) $= \emptyset$ $\Leftrightarrow$ a_χ injection

$(R = \sum_i r_i D_i, V = \oplus_\chi V_\chi \otimes \chi)$,

$$\begin{array}{ccccc} L(-R) \times_X A_\chi & \xrightarrow{a_\chi} & T^*X(\log D) \times_X A_\chi & \longrightarrow & T^*X(\log D) \\ & \searrow & \downarrow & & \downarrow \\ & & A_\chi \longrightarrow D_i & \longrightarrow & X \end{array}$$

DEFINITION $d = \dim X$. Characteristic cycle:

$$CC(\mathcal{F}) = (-1)^d \Big(\text{rank} \mathcal{F} \cdot [\text{0-section}] + \sum_i r_i \sum_\chi \frac{\dim V_\chi}{[A_\chi : D_i]} \cdot a_{\chi *}[L(-R) \times_X A_\chi] \Big) \in Z^d(T^*X(\log D)) \otimes \mathbb{Z}\left[\frac{1}{p}\right]$$

THEOREM 7

Assume ramification of $\mathcal{F}$ along D is bdd by $R+$,

$V = V^{(r_i)}$ $\forall\ i$ and $A_\chi \cap$ (0-section) $= \emptyset$ $\forall\ i$ $\forall \chi$.

Then:

$$C(j_!\mathcal{F}) = \mathrm{cl}(CC(\mathcal{F}))$$

in $H^{2d}(X, \mathbb{Q}_\ell(d)) = H^{2d}(T^*X(\log D), \mathbb{Q}_\ell(d))$.

Key diagram in the proof

$$
\begin{array}{ccc}
1 & \in & \mathrm{End}(\mathcal{F}) = H^0_X(X \times_k X, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})) \\
\downarrow & & \big\downarrow \pi^{(R)*} \\
* & \in & H^{2d}_{\pi^{(R)-1}(X)}((X \times_k X)^{(R)}, j_*^{(R)}\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})(d)) \\
\uparrow & & \big\uparrow \\
[X] \otimes \mathrm{id} & \in & H^{2d}_X((X \times_k X)^{(R)}, \mathbb{Q}_\ell(d)) \otimes \\
 & & H^0(X, \delta^{(R)*} j_*^{(R)}\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F}))
\end{array}
$$

$C(\mathcal{F}) = \mathrm{Tr}\ \delta^*(1)$, $\delta = \pi^{(R)*} \circ \delta^{(R)*}$,

$$X \xrightarrow{\delta^{(R)}} (X \times_k X)^{(R)} \xrightarrow{\pi^{(R)}} X \times_k X.$$

COROLLARY

Further, if X proper,

$$\chi_c(U, \mathcal{F}) = (CC(\mathcal{F}), \text{0-section})_{T^*X(\log D)}.$$

(cf. Dubson-Kashiwara, Laumon, ...

for $\mathcal{D}$-modules in char. 0)

OPEN PROBLEMS

Mixed characterisitic case:

- Definition of characteristic class.
- Comparison with Kato's filtration.

Geometric case:

- Intrinsic definition of characteristic cycle.
- Existence of clean model.
- Characteristic cycle of a $\mathcal{D}$-module.

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