Analogy

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1. Analysis (Sato. Kashiwara. Schapira, ...

D-modules on complex manifolds

~~Linear~~ PDE

2. Algebraic (~~Grothendieck~~. Beilinson. ...

ℓ-adic sheaves on smooth varieties over a perfect field / e.g $\mathbb{F}_p$

3 Arithmetic

ℓ-adic sheaves on regular schemes / e.g $\mathbb{Z}_p$

Microlocal analysis.

Singular supports
& characteristic cycles
on the cotangent b'dle
T^*X

Irregular singularities . 1.

⇕

Wild ramification 2.3

Algebraic.

Notation. $\underline{k}$: perfect field e.g. $\mathbb{F}_p$

X smooth variety$/k$ $\underline{n = \dim X}$.

$\underline{\Lambda: \text{ finite field of char } \ell}$ $\neq$ char k

(cf. $\underline{\ell\text{-adic}}$ $\to$ model.. $\mathbb{F}_\ell$)

$\mathcal{F}$ $\underline{\text{constructible sheaf}}$ or complex of Λ-modules on $X_{\text{ét}}$.

$\boxed{T^*X}$ $\underline{\text{cotangent b'dle}}$ X

$\uparrow$ $\underline{\text{vector b'dle}}$ on loc. free rank n $\dim n$

$2n = n + n$ $\underline{\Omega^1_{X/k}}$ $\leftarrow$ $\underline{\text{rank } n}$

$C \subset T^*X$ closed conical subset

stable under $\mathbb{G}_m$-action.

Singular support (Beilinson)

$C = SS\mathcal{F} \subset T^*X$

closed conical subset

$= \bigcup_a C_a$ irreducible cpt

$\dim C_a = \dim X = n.$

holonomic.

Characteristic cycle

$CC\mathcal{F} = \sum_a m_a C_a \quad m_a \in \mathbb{Z}$

$\mathcal{F}$ perverse. $\Rightarrow m_a > 0.$

Examples

1. $\mathcal{F} = \Lambda$ constant.

$SS\mathcal{F} = T^*_X X$ 0-section $\cong X$

$CC\mathcal{F} = (-1)^{\dim X} T^*_X X$

2. $\dim X = 1$.

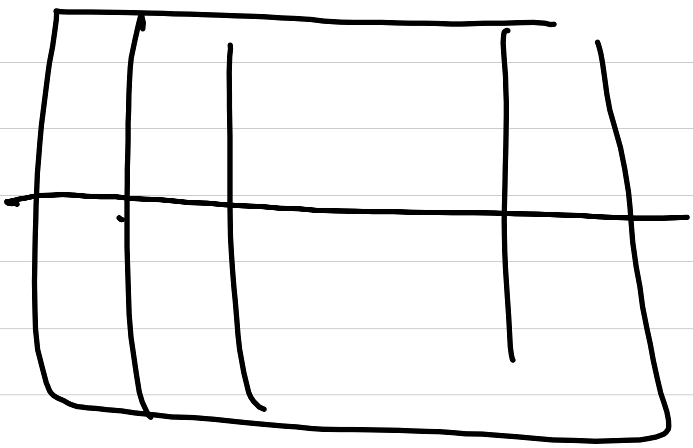
$U \subset X$ dense open s.t.

$\mathcal{F}|_U$ is loc. const.

$$\boxed{CC\mathcal{F} = (-1)\left(\mathrm{rank}\,\mathcal{F}|_U \cdot T^*_X X + \sum_{x \in X-U} a_x\mathcal{F} \cdot T^*_x X\right)}$$

Artin conductor fiber at x

T^*X = line bdle on curve



$$a_x \mathcal{F} = \operatorname{rank} \mathcal{F}|_U - \operatorname{rk} \underline{\mathcal{F}_x} + \operatorname{Sw}_x \mathcal{F}$$

Swan conductor : measure of wild ramification

Index formula.

Assume X projective (and smooth).

$\sum m_a C_a$, dim X

$$\chi(X_{\bar{k}}, \mathcal{F}) = \big(CC(\mathcal{F}), T^*_X X \big)_{T^*X}$$

$$\| $$

$$\sum_q (-1)^q \dim_\Lambda H^q(X_{\bar{k}}, \mathcal{F})$$

Intersection number

0-section.

If $\dim X = 1$,

we recover

the Grothendieck-Ogg-Shafarevich formula.

Arithmetic case

Notation. K complete discrete valuation field with perfect residue field k of ch $p > 0$

e.g $K = \mathbb{Q}_p$ $\quad n = \dim X$

X regular flat scheme of finite type over $\mathcal{O}_K$ e.g $\mathbb{Z}_p$

Λ finite field ch $\neq p$

$\mathcal{F}$ const. sheaf Λ cx on X.

Problem: What is T^*X?

$\Omega^1_{X/\mathcal{O}_K}$ not locally free of rk n

Solution: Modify $\Omega^1_{X/\mathcal{O}_K}$ differentials

so that $\boxed{\text{"}dp\text{"} \neq 0}$

e.g. $X = \operatorname{Spec} A$ $\quad A \xrightarrow{d} \Omega^1_{A/\mathcal{O}_K}$

$\boxed{\text{Frobenius}}$ - $\boxed{\text{Witt}}$ derivation
• additive
• Leibniz
localization

p prime number.

A ring flat / $\mathbb{Z}_{(p)}$ at p

Definition

1. A mapping $w: A \to M$ to an A-module M is an FW-derivation if

(cf Dupuy, Katz, Rabinoff, Zuveick-Brown).

(cf. δ-str Bhatt-Scholze)

$$w(a+b) = w(a) + w(b) + \underbrace{\frac{a^p + b^p - (a+b)^p}{p}}_{\text{polynomial in } a,b \text{ with } \mathbb{Z}\text{-coeff}} \cdot \underline{\underline{w(p)}}$$

polynomial in a, b with $\mathbb{Z}$-coeff

appearing in the definition of the addition in the ring of <u>Witt-vectors</u>

(e.g. $W(\mathbb{F}_p) = \mathbb{Z}_p$)

* $w(a \cdot b) = b^p w(a) + a^p w(b)$

2. $\underline{F\Omega^1_A}$ $\underline{w}: A \to \underline{F\Omega^1 A}$

universal pair of A-module & FW-derivation

<u>Proposition</u> Suppose X is regular flat and of finite type over $\mathcal{O}_K$ (e.g. $\mathbb{Z}_p$)

Then $\underline{F\Omega^1_X}$ is a locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$ ($X_{\mathbb{F}_p} = X \times_{\mathbb{Z}_p} \mathbb{F}_p$) reduction modulo p.

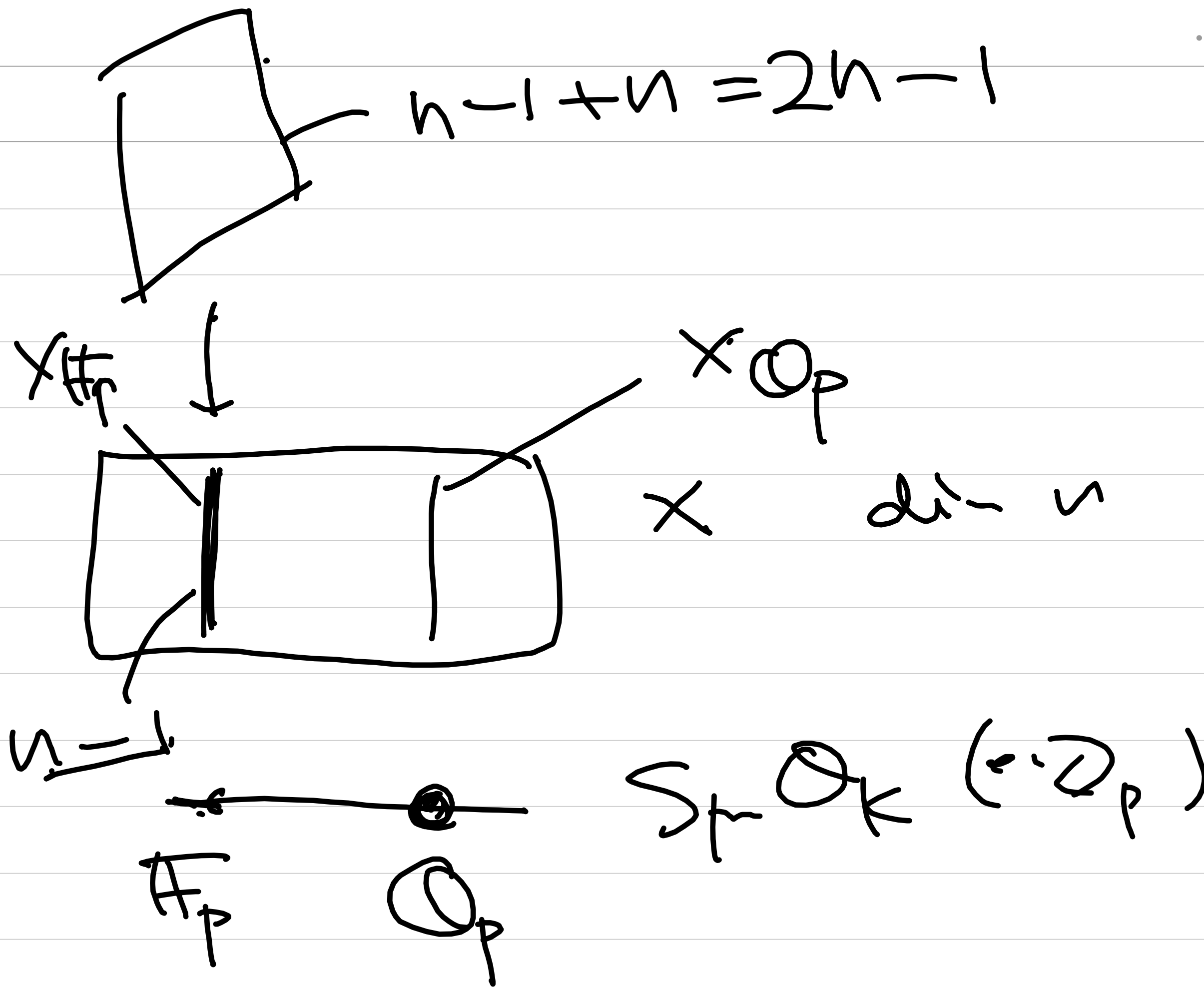
-module of rank $\underline{n} = \dim X$.

Definition.

$\boxed{F^*_p T^* X} / X_{\mathbb{F}_p}$: vector b'dle on $X_{\mathbb{F}_p}$ rk $\underline{n}$

Frobenius

associated to $F\Omega^1_X$.



microlocal analysis on $FT^*X|_{X_{\mathbb{F}_p}}$.

$\exists$: on X.

Singular support ?

(partial result)

Micro support on C

$C \subset T^*X|_{X_{\mathbb{F}_p}}$ closed conical subset.

$\boxed{SS\mathcal{F}}$: the smallest closed conical subset s.t. $\mathcal{F}$ is microsupported on C

For any morphism $h : W \to X$ of regular schemes of finite type over $\mathcal{O}_K$,

if h is C-transversal $\Rightarrow$ h is $\mathcal{F}$-transversal

$\mathcal{F}$-transversality

$$\underbrace{h^*\mathcal{F}}_{\text{pull-back}} \otimes Rh^!\Lambda \xrightarrow{\text{can mor}} Rh^!\mathcal{F}$$

isomorphism

Poincaré duality
smooth h is $\mathcal{F}$-transversal for any $\mathcal{F}$.

$\Rightarrow$ Any $\mathcal{F}$ is micro supported on $\boxed{FT^*X|_{X_{\mathbb{F}_p}}}$

Problem Does $SS\mathcal{F}$ exists?

Partial answer. $Z \xrightarrow{i} X$, $Z \to X_{\bar{\mathbb{F}}_p} \to X$ $\boxed{SS(i_*\mathcal{F}) = (i_*)SS\mathcal{F}}$

$$\begin{array}{ccc} W & \supset & V \\ \downarrow & & \downarrow \\ X & \overset{j}{\supset} & U = X_K \end{array}$$

finite Galois trivializing g

Frob pull + $F^*T^*Z \hookrightarrow F^*T^*X|_Z$

normalization

$\mathcal{F} = j_! g$

loc. const on U

Assume W regular

+ some other condition

$SS\mathcal{F}|_{X_t} \to SS\mathcal{F}$ exists.

X n.

$\downarrow$

$t \in C$ curve $/k$

dim n.

controls local acyclicity