Wild Ramification and the Cotangent Bundle

Introduction

Goal: Define the characteristic cycle of a smooth ℓ-adic sheaf on a smooth variety in positive characteristic ramified along boundary as a cycle in the cotangent bundle of the variety and derive several consequences of the construction.

(Non-logarithmic version of an earlier result.)

Motivations:

• Analogy between the irregularity of $\mathcal{D}$-modules on a variety in characteristic 0 and the wild ramification of ℓ-adic sheaves on a variety in characteristic $p > 0$.

• Filtration by ramification groups of the absolute Galois group of a local field with not necessarily perfect residue field.

• (Mysterious) Appearance of differential forms in ramification theory.

Consequences:

• Compatibility with the pull-back by non-characteristic morphism.

• Characterization of the support of the characteristic cycle by cutting by curves.

• Local acyclicity for non-characteristic morphism.

• Description of the graded pieces of the filtration by ramification groups in terms of differential forms.

• Computing the characteristic class and the Euler number of an ℓ-adic sheaf.

Known results:

• Approach by Deligne using jet bundles.

• Rank 1 case by Kato.

• Logarithmic version by S. and Abbes-S.

• Characteristic class by Illusie and Abbes-S.

New aspects:

• Tangent bundle suffices.

• Localization to allow denominators in slope.

Main machinary to link ramification and the cotangent bundle:

• An additive structure on the boundary of a dilatation of the self product induced by an obvious groupoid structure.

1. Characteristic cycle

Notation:

k: a perfect field of characteristic $p > 0$.

X: a smooth scheme of dimension d over k.

D: a divisor of X with simple normal crossings. $D_1, \ldots, D_h$: irreducible components.

$U = X - D$: the complement.

T^*X: the cotangent bundle.

Λ: a local ring over $\mathbf{Z}[\frac{1}{p}, \zeta_p]$.

$\mathcal{F}$: a locally constant constructible sheaf of free Λ-modules on U.

Assumptions:

• (simplifying) The ramification of $\mathcal{F}$ along D is *isoclinic* of slope $R = r_1D_1 + \cdots + r_hD_h$ where $r_i > 1$ are rational numbers (in general $r_i \geq 1$).

(*isoclinic*: there is a *unique* jump on the representation of the Galois group of a local field associated to the sheaf.)

($R = D$: $\mathcal{F}$ is tamely ramified along D.)

• (serious) The ramification of $\mathcal{F}$ along D is *non-degenerate* at multiplicity R.

Satisfied on a dense open subscheme such that the complmement has codimension ≥ 2.

Main construction: We define

$$\mathrm{Char}\ \mathcal{F} \in Z_d(T^*X)_{\mathbf{Z}[\frac{1}{p}]}.$$

It is a linear combination of positive rational coefficients of the classes of sub line bundles defined over finite étale coverings of finite radicial coverings of D + rank $\mathcal{F}$-times the class of the 0-section $T^*_X X$.

If X is a curve and $D = \{x\}$, $\mathrm{Char}\ \mathcal{F} = \mathrm{rank}\ \mathcal{F} \cdot [T^*_X X] + \dim \mathrm{tot}_x \mathcal{F} \cdot [T^*_D X]$.

$\dim \mathrm{tot}_x \mathcal{F} = \mathrm{rank}\ \mathcal{F} + \mathrm{Sw}_x \mathcal{F}$.

Example: Artin-Schreier sheaf on $U = \mathrm{Spec}\ k[x^{\pm 1}, y] \subset X = \mathrm{Spec}\ k[x, y]$.

(1) $t^p - t = 1/x^n$ ($p \nmid n$). $\mathrm{Char}\ \mathcal{F} = [T^*_X X] + (n+1) \cdot [T^*_D X]$.

(2) $t^p - t = y/x^n$ ($p \mid n$). $\mathrm{Char}\ \mathcal{F} = [T^*_X X] + n \cdot$ [line bundle generated by dy over D].

Pull-back:

$f\colon X' \to X$: morphism of smooth schemes such that $U' = f^{-1}(U)$ is the complement of a divisor $D' \subset X'$ with simple normal crossings.

$f\colon X' \to X$ is *non-characteristic* with respect to the ramification of $\mathcal{F}$ along D: The intersection of the inverse image of Char $\mathcal{F}$ by $T^*X \times_X X' \to T^*X$ and $\mathrm{Ker}(T^*X \times_X X' \to T^*X')$ is contained in the zero-section.

(*non-characteristic*: generic with respect to the intersection with the characteristic cycle. A standard definition in the theory of $\mathcal{D}$-modules.)

$f^*\mathrm{Char}\ \mathcal{F}$: If $f\colon X' \to X$ is *non-characteristic* with respect to the ramification of $\mathcal{F}$ along D, define $f^*\mathrm{Char}\ \mathcal{F}$ to be the push-forward of the pull-back by $T^*X \leftarrow T^*X \times_X X' \to T^*X'$. A cycle on T^*X' of dimension $\dim X'$

Proposition 1 *If $f\colon X' \to X$ is non-characteristic with respect to the ramification of $\mathcal{F}$ along D, we have*

$$\mathrm{Char}\ f^*\mathcal{F} = f^*\mathrm{Char}\ \mathcal{F}.$$

Cutting by curves:

$DT(\mathcal{F}) := \mathrm{rank}\ \mathcal{F} \cdot R$.

C: curve on X meeting components of D transversally at x.

Proposition 2

$$\dim \mathrm{tot}_x \mathcal{F}|_C \leq (C, DT(\mathcal{F}))_x.$$

$=$ *is equivalent to that the immersion $C \to X$ is non-characteristic.*

$\Sigma \subset TX$: union of hyperplane bundles annihilated by non-vanishing sections of Char $\mathcal{F}$.

$=$ *is further equivalent to that $T_x C \subset T_x X$ is not in Σ.*

Local acyclicity:

smooth morphism $f\colon X \to Y$ is *non-characteristic* with respect to the ramification of $\mathcal{F}$ along D: D has simple normal crossings relatively to Y and for every closed point $y \in Y$, the immersion $X_y \to X$ of the fiber is non–characteristic with respect to the ramification of $\mathcal{F}$ along D:

Proposition 3 *If $f\colon X \to Y$ is non-characteristic with respect to the ramification of $\mathcal{F}$ along D and if f is of* relative dimension 1*, $j_!\mathcal{F}$ is universally locally acyclic.*

$j\colon U = X - D \to X$: open immersion.

locally acyclic: $H^*(X_{\bar{x}}, j_!\mathcal{F}) \to H^*(X_{\bar{x}} \times_{Y_{f(\bar{x})}} \bar{t}, j_!\mathcal{F})$ is an isomorphism for every geometric point $\bar{x} \to X$ and every generalization $\bar{t} \to Y$ of the composition $\bar{x} \to X \to Y$.

Ramification groups: D: irreducible. ξ: generic point of D. $K = \mathrm{Frac}\hat{\mathcal{O}}_{X,\xi}$: local field at ξ. complete dvf with residue field $\kappa(\xi) =$ function field F of D.

$G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$: absolute Galois group of K. G_K^r: filtration by ramification groups defined by Abbes-S. V: representation of G_K defined by $\mathcal{F}$. $R = rD$.

$G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$ acts trivially on V. Induced action of $\mathrm{Gr}^r G_K = G_K^r / G_K^{r+}$.

Proposition 4 *$\mathrm{Gr}^r G_K$ is a pro-finite abelian group killed by p. There is a canonical injection*

$$\mathrm{Hom}_{\mathbf{F}_p}(\mathrm{Gr}^r G_K, \mathbf{F}_p) \to \mathrm{Hom}_{\bar{F}}(\mathfrak{m}^r_{K^{\mathrm{sep}}}/\mathfrak{m}^{r+}_{K^{\mathrm{sep}}}, \Omega^1_{X/k,\xi} \otimes \bar{F}).$$

$\mathfrak{m}^r_{K^{\mathrm{sep}}} = \{a \in K^{\mathrm{sep}} \mid \mathrm{ord}_K a \geq r\}, \mathfrak{m}^{r+}_{K^{\mathrm{sep}}} = \{a \in K^{\mathrm{sep}} \mid \mathrm{ord}_K a > r\}$.

Characteristic class and Euler number:

Characteristic class $C(j_!\mathcal{F}) \in H^{2d}(X, j_!\Lambda(d))$. If X is proper, Tr $C(j_!\mathcal{F}) = \chi_c(U_{\bar{k}}, \mathcal{F}) = \sum_{i=0}^{2d}(-1)^i \mathrm{rank}\ H^i_c(U_{\bar{k}}, \mathcal{F})$.

Proposition 5

$$[\mathrm{Char}\ \mathcal{F}] = C(j_!\mathcal{F}) \in H^{2d}(X, j_!\Lambda(d)).$$

$[\mathrm{Char}\ \mathcal{F}]$: the cohomological cycle class.

2. Additive structure (Definition of the characteristic cycle)

Assumption • (simplifying) The coefficients $r_i > 1$ are integers (in general rational numbers).

$P_n = X^{n+1}$. $P_n^{(R)}$: Blow up $P_n = X^{n+1}$ at $R \subset X \subset X^{n+1}$ embedded by the diagonal. Then, remove the proper transforms of the inverse images of $D \subset X$ by the $n+1$ projections $X^{n+1} \to X$.

P_n have a natural multiplicative structure: $P_n \times_X P_m \to P_{n+m}$. $P_n^{(R)}$ inherit it and have $P_n^{(R)} \times_X P_m^{(R)} \to P_{n+m}^{(R)}$. (groupoid)

$T_n^{(R)} = P_n^{(R)} \times_X D$ have an additive structure. $T_n^{(R)} \times_D T_m^{(R)} \to T_{n+m}^{(R)}$. (commutative group) Canonical isomorphism $T^{(R)} = T_1^{(R)} = TX(-R) \times_X D$.

$V \to U = X - D$: finite étale G-torsor.

$$\begin{array}{ccc} W_n^{(R)} \subset Q_n^{(R)} & \xleftarrow{\supset} & V^{n+1}/\Delta G \\ \downarrow & & \downarrow \\ P_n^{(R)} & \xleftarrow{\supset} & U^{n+1} \end{array}$$

$Q_n^{(R)}$: normalization.

$W_n^{(R)} \subset Q_n^{(R)}$: the largest open subschemes étale over $P_n^{(R)}$.

$V^{n+1}/\Delta G$ have a multiplicative structure.

Definition 6 *The ramification of V over U along D is* bounded by $R+$ *if the image of the canonical map $X = Q_0^{(R)} \to Q_1^{(R)}$ is in $W_1^{(R)}$.*

If bounded by $R+$, then $W_n^{(R)}$ inherit a multiplicative structure.

Definition 6 is an obvious necessary condition (the existence of unit).

It is in fact a sufficient condition.

Cartesian diagram

$$\begin{array}{ccc} E_n^{(R)} & \xrightarrow{\subset} & W_n^{(R)} \\ \downarrow & & \downarrow \\ T_n^{(R)} & \xrightarrow{\subset} & P_n^{(R)}. \end{array}$$

$E^{(R)} = E_1^{(R)}$ is a smooth group scheme over D and $E^{(R)} \to T^{(R)} = TX(-R) \times_X D$ is an étale morphism of smooth group schemes over D.

$E^{(R)0} \subset E^{(R)}$: open subgroup scheme such that for every point x of D, the fiber $E_x^{(R)0}$ is connected.

Definition 7 *The ramification of V over U along D is* non-degenerate *at multiplicity R if $E^{(R)0} \to T^{(R)}$ is* finite *(and étale).*

Finite: No splitting $E_x^{(R)} \to T_x^{(R)}$ for $x \in D$.

Extension

$$0 \longrightarrow G^{(R)} \longrightarrow E^{(R)0} \longrightarrow T^{(R)} \longrightarrow 0$$

by a finite étale group scheme $G^{(R)} = \mathrm{Ker}(E^{(R)0} \to T^{(R)})$ of $\mathbf{F}_p$-vector spaces over D.

Classification of extensions of a vector bundle by a finite étale group scheme of $\mathbf{F}_p$-vector spaces:

(Characteristic form) $$G^{(R)\vee} \longrightarrow T^{(R)\vee} = T^*X(R) \times_X D$$

injection defined over a radicial covering of D.

Abelian and logarithmic setting: refined Swan conductor defined by Kato.

Character of $G^{(R)}$ defines a sub line bundle of T^*X defined over the pull-back of a finite étale scheme $G^{(R)\vee} - D$ to a radicial covering of D.

Definition 8 *Characteristic cycle.*

$$\mathrm{Char}\ \mathcal{F} = \mathrm{rank}\ \mathcal{F} \cdot [T_X^* X] + \sum_i r_i \cdot \mathrm{rank} \cdot [L(-R)|_{G^{(R)\vee} \times_D D_i - D_i}]$$

where the pull-back to $G^{(R)\vee} \times_D D_i - D_i$ of the line bundle $L(-R)$ is regarded as a cycle of the cotangent bundle T^*X by the map induced by the characteristic form.

(rank is a locally constant function on $G^{(R)\vee} \times_D D_i - D_i$.)

Rational coefficients $M = m_1 D_1 + \cdots + m_h D_h$: $m_i \geq 1$ and $m_i r_i$ integers.

$P_n^{(D,M)} \to P_n(D)$: log smooth defined by $t_i = u_i s_i^{m_i}$, u_i invertible.

$P_n^{(D,M)}$ is the log product $(P_n(D) \times \mathbf{A}^h) \times_{[\mathbf{N}^h + \mathbf{N}^h]} [\mathbf{N}^h]$ with respect to the surjection $\mathrm{id} + (m_i) \colon \mathbf{N}^h + \mathbf{N}^h \to \mathbf{N}^h$.

$P_n^{(R,M)} \to P_n^{(D,M)}$: Blow-up $P_n^{(D,M)}$ at the inverse image of $R - D$ by $P_0^{(D,M)} \to X = P_0$ embedded by the map $P_0^{(D,M)} \to P_n^{(D,M)}$ induced by the diagonal map $X = P_0 \to X^{n+1} = P_n$. Then, remove the proper transform of the inverse image of D by $P_n^{(D,M)} \to P_n^{(D)} \to X$.