Characteristic cycle

X smooth / k perfect. $\dim X = n$.

Λ finite field char ℓ inv. in k

$\mathcal{F}$ constructible complex of Λ-mod on X.

$\operatorname{supp} \mathcal{F} \subset X$ closed subset

Beilinson $SS\,\mathcal{F} \subset T^*X$ singular support $\dim SS\mathcal{F} = n$

closed conical subset of the cotangent b'dle . $\dim T^*X = n$

stable under $\mathbb{G}_m$-action

$SS\mathcal{F} = \bigcup C_a$. C_a irred dim n. $SS\mathcal{F} \cap T^*_X X = \operatorname{supp}\mathcal{F}$

$CC\mathcal{F} = \sum m_a C_a$. $m_a \in \mathbb{Z}$. characteristic cycle.

$\mathcal{F}$ perverse $\Rightarrow$ $m_a > 0$ $\forall a$.

1. Classical example.
2. Characteristic class.
3. Direct image.

Example. $\dim X = 1$. $D \subset X$ smallest $\mathcal{F}|_{X-D}$ locally const $\neq 0$

$$SS\mathcal{F} = \underset{\text{0-section}}{T^*_X X} \cup \underset{\text{fiber}}{\bigcup_{x\in D} T^*_x X}$$

$$CC\mathcal{F} = -\left(\operatorname{rk}\mathcal{F}|_{X-D} \cdot T^*_X X + \sum_{x\in D} a_x\mathcal{F} \cdot T^*_x X\right)$$

$a_x\mathcal{F}$ Artin conductor

$= \operatorname{rk}\mathcal{F} - \operatorname{rk}\mathcal{F}_{\bar{x}} + Sw_x\mathcal{F}$ Swan conductor.

not Lagrangean in higher dim.

2 Characteristic class

X possibly singular $i: X \to M$ closed imm M smooth
$N = \dim M$

$CC\, i_* \mathcal{F} = \sum m_a C_a \quad C_a \subset X \times_M T^*M$

$\overline{CC\, i_* \mathcal{F}} = \sum m_a \overline{C_a} \quad \overline{C_a} \subset \mathbb{P}(X \times_M T^*M \oplus \mathbb{A}^1_X)$

$CH_N(\quad\text{——}\quad) = \bigoplus_q CH_q(X)$
/rational eq.

$cc_X \mathcal{F} \in CH_\bullet(X)$ char class. $CH_\bullet(X)$
indep of $i: X \to M$.

$K(X, \Lambda)$ Grothendieck gp of category of constructible cx of Λ-mod on X

$cc_X : K(X, \Lambda) \to CH_\bullet(X)$

char $k = 0$ $\searrow \quad \nearrow (-1)^? c_M$ MacPherson Chern class
$F(X)$ $\mathbb{Z}$-valued constructible fns.

char $k > 0$ generalization of fns.

$\mathcal{V}_X = ($Sep. of f. type$/k\ /X) \quad K_0(\mathcal{V}_X)$.

$\widetilde{F}(X) = \mathrm{Hom}(K_0(\mathcal{V}_X), \mathbb{Z})$

$K(X, \Lambda) \times K_0(\mathcal{V}_X) \to \mathbb{Z} \quad (\mathcal{F}, g: Z \to X) \mapsto \chi_c(Z_{\bar{k}}, g^*\mathcal{F})$

$K(X, \Lambda) \to \widetilde{F}(X)$

Prop. $cc_X : K(X, \Lambda) \to CH_\bullet(X)$
$\searrow \ \mathbb{Q} \ \nearrow \exists 1$
$\mathrm{Im}(K(X, \Lambda) \to \widetilde{F}(X))$

Conj 1.
$f: X \to Y$ proper

$$\begin{array}{ccc} K(X, \Lambda) & \xrightarrow{cc_X} & CH_\bullet(X) \\ f_* \downarrow & & \downarrow \\ K(Y, \Lambda) & \xrightarrow{cc_Y} & CH_\bullet(Y) \end{array}$$

OK. If k finite & X, Y proj smooth$/k$ U-Y-Z.

3. Direct image.

$f: X \to Y$ proper morphism of smooth schemes (or more generally proper on supp $\mathcal{F}$).

$\dim X = n$. $\dim Y = m$.

$f_* CC\mathcal{F}$. $\quad T^*X \leftarrow X \times_Y T^*Y \to T^*Y$

$f_\circ SS\mathcal{F}$.

$$f_* CC\mathcal{F} \in CH_m(f_\circ SS\mathcal{F})$$

Conj 2 $\quad CCRf_*\mathcal{F} = f_* CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$

Conj 2 $\Rightarrow$ Conj 1.

Theorem 3. If $f: X \to Y$ is proj & $\dim f_\circ SS\mathcal{F} = m$,
$CCRf_*\mathcal{F} = f_* CC\mathcal{F}$ in $Z_m(f_\circ SS\mathcal{F})$.

Example 1. $Y = \operatorname{Spec} k$. $\dim f_\circ SS\mathcal{F} = 0$ always.

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X} \quad \text{index formula}$$

In particular, if $\dim X = 1$.

$$\chi(X_{\bar{k}}, \mathcal{F}) = \operatorname{rk}\mathcal{F} \cdot \chi(X_{\bar{k}}) - \sum_{x \in D} a_x \mathcal{F} \cdot \deg x$$

Grothendieck-Ogg-Shafarevich

2 $\dim Y = 1$. $\dim f_\circ SS\mathcal{F} = 1$ is satisfied after b.c by iteration of Frob.

$$-\underset{\shortparallel}{a_y} Rf_* \mathcal{F} = (CC\mathcal{F}, df)_{T^*X, y} + 1.$$

$$\chi(X_{\bar{\eta}}, \mathcal{F}) - \chi(X_{\bar{y}}, \mathcal{F}) + Sw_y H^*(X_{\bar{\eta}}, \mathcal{F})$$

Pf. G-O-S $\Rightarrow$ index formula $\Rightarrow$ $\overset{\text{new.}}{2}$ $\overset{\text{easy}}{\Rightarrow}$ Thm.

reduced to Y. $f: X \to Y$ proj smooth.

$1 \Rightarrow \sum_{y'} — = \sum_{y'} —$.

fix y $\exists Y' \to Y$ étale at y. killing other terms

analog of St- ned them