k alg closed char $p > 0$. X/k smooth of dim d

Λ finite field char $\neq p$. K const cx of Λ-mod $/X$.

Char K. linear combination of irred conic closed subset of T^*X of dim d.

Deligne Notes sur Euler–Poincaré. 8/2/2011 August 8 Illu 4/11/1976

Propositions & Conjectures. assuming $\exists$ — ccot. conject

~~Aim~~ produce Today + Tomorrow What & How I can prove ~~under~~ thm assuming the existence of Sing supp characterised by local acyclicity of families of morphisms to curves / surfaces.

Prove in particular. Milnor formula. dim tot of can cycle. E-P formula.

Ramification theory $\Rightarrow$ the assumption is satisfied if $\dim X \leq 2$. outside codim ≥ 2

$\Rightarrow$ unconditional result for surfaces.

~~Method~~ Tool - (semi) continuity of Swan conductor

Ingredients univ. family of hyperplane sections - geom.

vanishing cycles over a general base scheme - - abs.

Plan. 1. Sing supp & Char cycles.

2. Construction of Char cycle.

3. Properties of Char cycle. Sing supp loc. acyclicity of family of mor. to curves; Char cycle Milnor formula at iso. sing.

1.1 Singular supp & loc. acy

X smooth dim d/k. $S = (S_i)$ finite family of closed conic subset of T^*X (of dim. d)

Def 1.1. 1. $f: X \to C$ flat morphism C. smooth curve $/k$

f. $\boxed{\text{non char}}$ w.r.t S if ~~the fiber~~ inv. image of S_i ~~$\subset T^*X$~~ by the can map $X \times_C T^*C \to T^*X \supset S_i$ is a subset of ~~contained in~~ the 0-section

$T_i = S_i \cap T_X^*X \subset X$ flat $/C$

2. $\boxed{\text{relative}}$

$$\begin{array}{ccc} W & \xrightarrow[\text{flat}]{f} & C \\ \text{étale} \downarrow & & \downarrow \text{smooth rel dim } 1 \\ X \times B & \longrightarrow & B \ni \text{smooth} \end{array}$$

f. non char w.r.t S if $\forall b \in B$

$f_b : W_b \to C_b$ is n. char w.r.t the pull-back of S to T^*W_b.

$$T^*W_b = W_b \times_X T^*X$$

Def 1.2. K const cx of Λ-mod. $S = (S_i)$ is a $\boxed{\text{sing supp}}$ if (SS1) For a CD as above,

$f : W \to C$ is loc. acyclic rel to the pull-back of K if it is univ. non char w.r.t. S.

SSK sing supp of K. not unique

· smooth pull-back. finite unram. push forward.

K smooth except at $Z \subset X$ finite.

$$S = T^*_X X \cup T^*_Z X$$

0-sect fiber

Ram th $\Rightarrow$ $X - Z$ Z codim 2. SSK exists

$\Rightarrow$ X dim 2 =

1.2. Char cycle & Milnor formula

Def 1.3 $f : X \to C$ C smooth curve. $u \in X$ closed pt

u is an $\boxed{\text{iso char pt}}$ if $\exists U$ nbd of u s.t $f|_{U - \{u\}} : U - \{u\} \to C$ is non char with r.t the res'n of $SSK = \bigcup S_i$

ω basis of T^*C at $f(u)$ $f^*\omega$ section of T^*X

$\{f^*\omega\} \cap S_C$ isolated $\Leftarrow$ $(\sum a_i [S_i], \underset{df}{f^*\omega})_{T^*X, u}$ well-defined

Thm 1.4* Assm SSK exists. Then there exist a unique $\mathbb{Z}[\frac{1}{p}]$-lin combination $ChK = \sum a_i S_i$ s.t

(M) $\quad -\mathrm{dimtot}\,\phi_u(K, f) = (ChK, df)_{T^*X, u}$

for $f : V \to C$ s.t u is an isolated char pt of f.

étale nbd of u $\quad$ closed pt $\in X$

$\phi_u(K,f)$ vanishing cycles at u. Gal rep of loc field K_u u = fcn 3

dim Tot = dim + Sw

Milnor formula in SGA 7 XVI $K = \Lambda$. $\mathrm{Ch}\, K = (-1)^d [T^*_X X]$ 0-sect.

Example 1 $U = X - D$ ~~compl~~ $D = \bigcup D_i$ div w/ SNC C

$K = j_! \mathcal{F}$ $j : U \to X$ open $\mathcal{F}$ loc. const$/U$ tame along D

$\Rightarrow$ $\mathrm{Ch}\, K = (-1)^d \sum \mathrm{rk}\,\mathcal{F} \cdot [T^*_{X_I} X]$ conormal b'dle

$X_I = \bigcap_{i \in I} D_i \subset X$

(M) E. Yang.

2. $d = 1$ $\mathrm{Ch}\, K = -(\mathrm{rk}\, K [T^*_X X] + \sum a_x(K) \cdot [T^*_x X])$ fiber

$a_x(K) = \dim_\eta K - \dim_x K_x + \mathrm{Sw}_x K$

(M) induction formula.

2. Construction of char cycle

- Def. $\mathrm{Char}_E K$ dep a priori on an embedding to a proj space
- Prove (M) for morphism defined by a pencil
- Prove (M) general & independence of embedding.

2.1. univ. family.

X quasi-proj. $\mathcal{L}$ ample $E \subset \Gamma(X, \mathcal{L})$.

$X \hookrightarrow \mathbb{P} = \mathbb{P}(E^\vee) = \mathrm{Proj}\, S^\bullet E$. contra Grothendieck.

(E) $\forall u \neq v \in X(k)$ $E \twoheadrightarrow \mathcal{L}_u / m_u^2 \mathcal{L}_u \oplus \mathcal{L}_v / m_v^2 \mathcal{L}_v$

Satisfied for $E^{(n)} = \mathrm{Im}(E^{\otimes n} \to \Gamma(X, \mathcal{L}^{\otimes n}))$ for $n \geq 3$.

$\mathbb{P}^\vee = \mathbb{P}(E)$ hyp planes in $\mathbb{P}$

$L \subset \mathbb{P}^\vee$ $A_L \subset \mathbb{P}$ intersections codim 2.

$X_L \to X$ blow up at $X \cap A_L$ isom on $X_L^\circ = X - (X \cap A_L)$

$p_L : X_L \to L$. $p_L^\circ : X_L^\circ \to L$.

universal family $H = \{(x, H) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in H\}$

$(X \times \mathbb{P}^\vee) \cap H = X \times_{\mathbb{P}} H \longrightarrow \mathbb{P}^\vee$

$$0 \to \Omega^1_{\mathbb P}(1) \to E \otimes \mathcal O_{\mathbb P} \xrightarrow{\text{can}} \mathcal O_{\mathbb P}(1) \to 0$$

$$\hookrightarrow \quad \mathbb H \cong \mathbb P(T^*\mathbb P) \subset \mathbb P \times \mathbb P^\vee = \mathbb P(\mathbb P \times E)$$

$\mathbb G = G(2,E)$ lines in $\mathbb P^\vee$. $X \times_{\mathbb P} \mathbb H = \mathbb P(X \times_{\mathbb P} T^*\mathbb P)$

$\mathbb A \subset \mathbb P \times \mathbb G$ univ. subspace of codim 2.

$\mathbb D = Fl(1,2,E) \subset \mathbb P^\vee \times \mathbb G$ flag univ. line in $\mathbb P^\vee$

$$\begin{tikzcd} X \times_{\mathbb P} \mathbb H \arrow[d] & (X\times\mathbb G)' \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\square"] & X \times \mathbb G \arrow[d] \\ \mathbb P^\vee & \mathbb D \arrow[l] \arrow[r] & \mathbb G \end{tikzcd}$$

def

blow up at $(X\times\mathbb G) \cap \mathbb A = X \times_{\mathbb P} \mathbb A$.

ison on $(X \times \mathbb G)^\circ = (X\times\mathbb G) - X \times_{\mathbb P} \mathbb A$.

$$\begin{tikzcd} X \times_{\mathbb P} \mathbb H \arrow[d] & X_L \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\square"] & X \arrow[d] \\ \mathbb P^\vee & L \arrow[l] \arrow[r] & [L] \end{tikzcd}$$

$\tilde S \subset X \times_{\mathbb P} T^*\mathbb P$ inv image of $S \subset T^*X$

$\mathbb P(\tilde S) \subset \mathbb P(X \times_{\mathbb P} T^*\mathbb P) = X \times_{\mathbb P} \mathbb H$ projectivisation

$\mathbb P(S) \subset X \times_{\mathbb P} \mathbb H \subset X \times \mathbb P^\vee$

$\| $

$\bigcup T_i \times \mathbb P^\vee_i \qquad T_i = S_i \cap T^*_X X \subset X.$ $\{\dim T_i > 0\}$

$\mathbb P^\vee_i = \mathbb P(E_i) \subset \mathbb P^\vee = \mathbb P(E) \quad E_i = \mathrm{Ker}(E \to \Gamma(T_i, \mathcal L \otimes \mathcal O_{T_i}))$

$(X\times\mathbb G)^{\circ\circ} \subset (X \times \mathbb G)^\circ$ — inv. i. of $\mathbb P_{>0}(S)$

largest open s.t. $Z \subset (X\times\mathbb G)^{\circ\circ}$ is quasi-finite $/\mathbb G$.

$\downarrow$ $\mathbb G$

inv image of $\mathbb P(\tilde S)$

2. *

Lem 3.1. 1. For $(u, L) \in (X \times \mathbb G)^\circ$ the following are equiv

(1) $(u, L) \in (X \times \mathbb G)^{\circ\circ}$

(2) u iso. char pt of $p_L^\circ : X_L^\circ \to L$.

2. On the complemt $X \times_{\mathbb P} \mathbb H - (\mathbb P(\tilde S) \cup \mathbb P(S))$

$\downarrow$

$\mathbb P^\vee$

is univ. loc. acyclic rel. to pull-back of K.

(SS1) $\Rightarrow$ 2.

Def 2.2

1. $i: Y \to X$ unram. reg of codim 1 (= étale locally on Y closed imm of codim 1 reg.) $T^*_Y X$

is non char if $(S \times_X Y) \cap \ker(Y \times_X T^*X \to T^*X) \subset 0$-section.
w.r.t. S & $T_i \times_X Y \to T_i$ reg of codim 1. divisor

2. $Y \xrightarrow{f} B$ flat

smooth $\downarrow$ smooth , $Y \to X \times B$ unram. reg of codim 1
$X \to \mathrm{Spec}\, k$

is non char if $\forall b \in B$ cl. pt $Y_b \to X$ non char
w.r.t. S.

(CSS 1b)($\Leftarrow$ (SS 1)) non char $\Rightarrow$ $f: Y \to B$ loc. acy. rel. to the pull back of K

2.2 Continuity of Swan conductor.

Def 2.3 $f: Z \to S$ quasi-finite morphism of noetherian sch.
$\varphi: Z \to \mathbb{Z}$ is flat over S if $\forall x \to s = f(x) \leftarrow t$.

$$\varphi(x) = \sum_{z \in Z_{(x)} \times_{S_{(s)}} t} \varphi(z) \qquad \text{flat} \Rightarrow \text{const'ble}$$

~~φ flat~~ $\Rightarrow$ f étale $\Rightarrow$ φ flat $\Leftrightarrow$ loc. constant
(φ constructible)

Prop. 2.4 (a partial gen'n of (semi) continuity of Swan by Deligne-Laumon

$$\underset{\text{closed}}{Z} \subset X \xrightarrow{f} Y \qquad \text{noetherian}$$
$p \searrow \quad \swarrow$ smooth curve
q-finite $\to$ S

K on X $\quad$ $p: X \to S$, $f: X - Z \to Y$ loc. acy. rel to K

$\Rightarrow$ $\varphi(z) = \dim \mathrm{tot}_z (K|_{X_s}, f_s)$ $\quad s = p(z)$
is flat over S.

Proof Apply Deligne-Laumon to $R\Phi_f K$ on $X \overleftarrow{\times}_Y Y$.

Cor 2.5 Notation be as in Lem 2.1. Then

$$\varphi(u, L) = -\operatorname{dimtot}\phi_u(K, p_L)$$

on Z is flat over $\mathbb{G}$.

Proof Apply Prop 2.4 to

$$\begin{tikzcd} Z \subset (X \times \mathbb{G})^{oo} \arrow[r, "f"] \arrow[dr, "p"'] & \mathbb{D} \arrow[d] \\ & \mathbb{G} \end{tikzcd}$$

universal line

loc. acyclicity of $f : (X\times \mathbb{G})^{oo} - Z \to \mathbb{D}$. Lem 2.1 (SS$^{\prime}$ 1)

$p : (X\times\mathbb{G})^{oo} \to \mathbb{G}$ gen. loc. acy. $\Rightarrow$ SGA $4\frac{1}{2}$.

Cor 2.5 $\Rightarrow \varphi$ const on $Z_i^{\circ} \hookrightarrow Z$
a_i a dense open of inv image of $\mathbb{P}(\widetilde{S}_i)$

$$\begin{array}{ccc} (E) \ \xi_i \in \mathbb{P}(\widetilde{S}_i) & \subset & X \times_{\mathbb{P}} \mathbb{H} \\ \downarrow & & \downarrow p \\ \eta_i \in p(\quad : \quad) & \subset & \mathbb{P}^{\vee} \end{array}$$

gen. finite

Define

$$\operatorname{Char}_E K = -\sum \frac{a_i}{[\xi_i : \eta_i]_{\mathrm{insep}}} [S_i]$$

Def $\Rightarrow$

$$-\operatorname{dimtot}\phi_u(K, p_L) = (\operatorname{Char}_E K, dp_L)_{T^*X, u}$$

if $z = (u, L) \in Z$ is in $\coprod_i Z_i^{\circ}$.

general z. ~~apply~~ consider Cor 2.5 ~~to the conic case~~.
derive from

$\mathbb{P}(\widetilde{\mathrm{Ch}_E K})$

$$\begin{array}{ccccc} X \times_{\mathbb{P}} \mathbb{H} & \leftarrow & (X\times \mathbb{G})^{oo} & \hookleftarrow & X_L^{o} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^{\vee} & \leftarrow & \mathbb{D} & \longleftarrow & L \\ & & \downarrow & & \downarrow \\ & & \mathbb{G} & \leftarrow & [L] \end{array}$$

$(\operatorname{Ch}_E K, dp_L)_{T^*X, u}$
$\|$
$(\mathbb{P}(\operatorname{Ch}_E K), X_L^{o})_u$
$\|$
degree at u of the fiber over $[L]$ of the pull-back of $\mathbb{P}(\operatorname{Ch}_E K)$ to $(X\times\mathbb{G})^{oo}$

the right hand side is also ~~the~~ value of another flat funct on Z

= of flat fns on a dense open

$\Rightarrow$ = everywhere

2.3 Stability of dim tot of van. cycles.

$f: X \to Y$, $g: X \to Y$ $Z \subset X$ $f \equiv g \mod Z \Leftrightarrow f|_Z = g|_Z$

Prop 2.6* $f: X \to C$. morphism to smooth curve C.
$u \in X$ iso. char pt w.r.t K. Then $\exists N \geq 1$ st
$g: V \to C$ on an étale nbd V of u, $g \equiv f \mod m_u^N$
u iso ch pt of g w.r.t K & $\dim \mathrm{tot}\, \phi_u(K, g) = \dim \mathrm{tot}\, \phi_u(K, f)$

Proof. $\exists N$ for the first part is easy.
Assume $g \equiv f \mod m_u^N$, $V = X$. & $C = \mathbb{A}^1$.
Define $h: X \times \mathbb{A}^1 \to C \times \mathbb{A}^1$ by $h = (1-t)f + tg$.

$\mathbb{A}^1 = \mathrm{Spec}\, k[t]$

and apply Prop 2.4 to

$$\begin{tikzcd} \{u\} \times \mathbb{A}^1 \arrow[r, hook] & X \times \mathbb{A}^1 \arrow[r, "h"] \arrow[dr, "pr_2"'] & C \times \mathbb{A}^1 \arrow[d, "pr"] \\ & & \mathbb{A}^1 \end{tikzcd}$$

$pr_2: X \times \mathbb{A}^1 \to \mathbb{A}^1$ gen loc acy.
$h: X \times \mathbb{A}^1 - \{u\} \times \mathbb{A}^1 \to C \times \mathbb{A}^1$ (SS1).
Prop 2.6 $\Rightarrow$ Indep of E & Milnor formula. i.e Thm 1.4

3 Properties of char cycles.

Condition on family of morphisms to surfaces. of rel.dim m

Def 3.1. 1. $f: X \to P$ smooth morphism to smooth surface.
non char. if $S_i \cap I(df: X \times_P T^*P \to T^*X) \subset$ 0-sect
& $T_i = S_i \cap T^*_X X \subset X$ flat $/P$.

2. $$\begin{tikzcd} W \arrow[r, "smooth", "f"'] \arrow[d, "étale"'] & P \arrow[d, "smooth \\ rel. dim 2. m"] \\ X \times B \arrow[r] & B \end{tikzcd}$$ f non ch $\Leftrightarrow$ $\forall b \in B$ f_b non char

(SS2) (SSm) non char $\Rightarrow$ loc. acyclic 5

Def 3.2 $i: Y \to X$ imm. of smooth divisor, non char
is strictly non char. if $T_i \cap Y \subset T_i$ divisor
is integral (= irred + red)

Theorem 3.3.** Assume SSK satisfies (SS1) & (SS2)

$i: Y \to X$ strictly non char. imm of smooth div.

Then $i^{!} SSK \cup T^{*}_{Y}Y$ is a sing supp of $i^{*}K$ &

$$Ch\, i^{*}K = i^{!}\, Ch\, K$$

$$i^{!}: \quad T^{*}X \leftarrow Y \times_X T^{*}X \to T^{*}Y$$

$$\cup$$

$$Y \times_X SSK \nearrow \text{finite} \Leftarrow \text{n. char}$$

Theorem 3.4 $j: U \hookrightarrow X$ open im of compl of div D w/ SNC, $U = X - D$

$\exists$ on U s.l.c.s. ram along D is ~~sm.~~ nondeg

$\Rightarrow$ $Ch\,K$ equals that def'd by ram. th.

I.e. $Ch\,K$ def'd by ram th satisfies the Milnor fml

Pf of Th's. 3.4 in dim $2 \Rightarrow 3.3 \Rightarrow 3.4$ dim >2.

$\Uparrow$ global argument

Cor 3.5** $f: Y \to X$ smooth

$$\Rightarrow \quad Ch\, f^{*}K = f^{*}\, Ch\, K$$

$$T^{*}X \leftarrow Y \times_X T^{*}X \hookrightarrow T^{*}Y$$

Thm 3.6** X proj smooth, SSK satisfies (SSm) for all $m \leq n$

$$\chi(X, K) = (Ch\,K, T^{*}_{X}X)_{T^{*}X}.$$

Pf. Ind'n on $\dim X$. Clear if dim 0.

Lem 3.7. $\exists$ pencil L s.t.

- $p_L: X_L \to L$ at most fin. many char pt,
- $Y = X \cap H \to X$. strict. n. char
- $Z = X \cap A_L \to Y$ =
- char pt $\notin$ inv. image of $X \cap A_L$.

~~$X_L \to X$~~ ~~$\chi(X, K)$~~ Bertini

$\chi(X, K) = \chi(X_L, K) - \chi(Z, K)$ blow-up

$\chi(X_L, K) = 2\chi(Y, K) - \sum \mathrm{dim\,tot}\, \phi_u(K, p_L)$ GOS

Ind hyp + Th 3.3

$\chi(Y, K) = -(i^! \,\mathrm{Ch}\, K, T^*_Y Y)$

$\chi(Z, K) = (i'^! \,\mathrm{Ch}\, K, T^*_Z Z)$

Milnor fml

$-\mathrm{dim\,tot}\, \phi_u = (\mathrm{Ch}\, K, dp_L)$

Substituting them + computing chern classes, we obtain Th 3.6

4. Sketch of Pf of 3.3 ($\Leftarrow$ 3.4 in dim 2)

$p_{L'} : Y^\circ_{L'} \to L'$ $\quad (\mathrm{Ch}_a\, i^* K, dp_{L'})_u = -(i^! \,\mathrm{Ch}_a K, dp_{L'})_u$

LHS $= -\mathrm{dim\,tot}\, \phi_u(i^* K, p_{L'})$ Milnor formula

$= \overline{\mathrm{Ch}_a}\, R\Phi_{p_{L'}} i^* K$ $\quad \overline{\mathrm{Ch}_a} = \mathrm{Ch}_a -$ compt of 0-section.

$= \overline{\mathrm{Ch}_a}\, h^* R\Phi_f K$ $\quad$ isol. sing $\Rightarrow$ $R\Phi$ commute with b.c.

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle g = p_{L'}}\downarrow & \square & \downarrow f \\ L' & \xrightarrow{h} & P \end{array}$$

RHS $= -dg^* i^! \overline{\mathrm{Ch}_a K} = -h^! df^* \overline{\mathrm{Ch}_a K} = -h^! \overline{\mathrm{Ch}_a R\Phi_f K}$.

Reduced to constant C.D. satisfying

$\overline{\mathrm{Ch}_a}\, h^* R\Phi_f K = -h^! \overline{\mathrm{Ch}_a R\Phi_f K}$. P°_L

Same method as the proof of Milnor formula for the morphisms defined by a pencil

- univ. family.
- flatness of the Swan conductor
- $=$ in generic case ($\Leftarrow$ ramification theory

univ ~~case~~ family.

$$\begin{tikzcd}
 & \mathbb{D} \arrow[r] \arrow[dl] & \mathbb{G} & \\
\mathbb{D}^\vee & & \mathbb{C} \arrow[ul] \arrow[r] \arrow[dl] & \mathbb{F} \arrow[ul] \arrow[dl] \\
 & \mathbb{B} \arrow[ul] \arrow[r] & \mathbb{E} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Fl}(1,2,E) \arrow[r, "\mathbb{P}^1"] \arrow[dl] & G(2,E) & \\
\mathbb{P}(E) = G(1,E) & & \mathrm{Fl}(1,2,3,E) \arrow[ul] \arrow[r, "\mathbb{P}^1"] \arrow[dl, "\mathbb{P}^1"'] \arrow[ur, phantom, "\square"] & \mathrm{Fl}(2,3,E) \arrow[ul] \arrow[dl, "\mathbb{P}^2"] \\
 & \mathrm{Fl}(1,3,E) \arrow[ul] \arrow[r, "\mathbb{P}^2"'] & G(3,E) &
\end{tikzcd}$$

dual

$$\begin{tikzcd}
Y'_{\mathbb{B}} \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & (X \times \mathbb{B})' \arrow[r] \arrow[d, "\in"] \arrow[dr, phantom, "\square"] & X \times_{\mathbb{P}} A \arrow[d] \\
\mathbb{C} \arrow[r, "e"'] \arrow[dr, "\mathbb{P}^1"'] & \mathbb{B} \times_{\mathbb{E}} \mathbb{F} \arrow[r] \arrow[d, "\mathbb{P}^2"] & \mathbb{G} \\
 & \mathbb{B} &
\end{tikzcd}$$