k char p mostly perfect or even alg closed

X smooth $/k$ $n = \dim X$

Λ $/\mathbb{F}_\ell$ finite $\ell \neq p$

T^*X cotangent b'dle

$C \subset T^*X$ conical (stable under mult'n) closed subsch

locally def'd by a graded ideal

$\mathcal{F}$ constructible complex of Λ-modules on X

Singular support -- conical closed subset today

$$SS\mathcal{F} = C = \bigcup C_a \subset T^*X.$$

$\dim C_a = n$. $\forall a$

Characteristic cycle -- $\mathbb{Z}[\frac{1}{p}]$-linear combination later

$$\mathrm{Char}\,\mathcal{F} = \sum m_a [C_a]. \quad m_a \in \mathbb{Z}[\tfrac{1}{p}].$$

Properties of constructible sheaf on X are controlled by a cycle on T^*X.

Example X curve k perfect $D \subset X$ divisor

$j: U = X - D \hookrightarrow X$. $\mathcal{F} = j_! \mathcal{G}$. $\mathcal{G}$ loc. const $/U$, $\neq 0$

$$SS\mathcal{F} = \underset{0\text{-section}}{T^*_X X} \cup \underset{x \in D}{\bigcup} \underset{\text{fibers}}{T^*_x X}$$

$$\mathrm{Char}\,\mathcal{F} = (-1)\Big(\mathrm{rk}\,\mathcal{G} \cdot [T^*_X X] + \sum_{x \in D} \mathrm{dimtot}_x \mathcal{G} \cdot [T^*_x X]\Big)$$

dimtot = dim + Sw$_x$ — Swan conductor $\in \mathbb{N}$
measure of wild ramification

$\mathcal{F}$ perverse $\Rightarrow$ $m_a \geq 0$.

1. Singular support

1.1 C-transversality

$f : X \to Y$, $h : U \to X$ morphisms of smooth schemes$/k$

Definition 1. $f : X \to Y$ C-transversal if

$$\begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{df} & T^*X \\ \cup & & \cup \\ df^{-1}(C) & \longrightarrow & C \end{array}$$

for $df^{-1}(C) \subset X \times_Y T^*_Y Y$ 0-section

Example • If $C = T^*_X X$.

f C-transversal $\Leftrightarrow$ df injection $\Leftrightarrow$ f smooth

• If $Y = pt$. every f is C-transversal to every C

2. $h : U \to X$ C-transversal if

for
$$\begin{array}{ccc} h^*C = W \times_X C & & \\ \cap & & \\ W \times_X T^*X & \xrightarrow{dh} & T^*W \end{array}$$
$dh^{-1}(T^*_W W) \cap h^*C \subset W \times_X T^*_X X$

$h^\circ C = dh(h^*C) \subset T^*W$. $h^*C \to h^\circ C$ finite.

Example If h is smooth.

$h : W \to X$ is transversal to any C.

Remark. open condition.

3. $X \xleftarrow{h} W \xrightarrow{f} Y$ is C-transversal

if h is C-transversal & f is $h^\circ C$-transversal

Exercise. 1. $h \Leftrightarrow (\cdot, h)$. $f \Leftrightarrow (f, id)$

$X \leftarrow W \to \cdot$ $\qquad$ $X \leftarrow X \to Y$

2. $$\begin{array}{ccc} h^*C \times (W \times_Y T^*Y) & & T^*_W W \\ \cap & & \cap \\ (W \times_X T^*X) \times_W (W \times_Y T^*Y) & \longrightarrow & T^*W \end{array}$$ int. with the image $\subset 0$

2' If $f : W \to Y$ smooth
$$\begin{array}{ccc} h^\circ C & & W \times_Y T^*Y \\ \cap & & \cap \\ W \times_X T^*X & \longrightarrow & T^*W \end{array}$$
int with the image $\subset 0$

1.2 local acyclicity -- no vanishing cycles

$f: X \to Y$

Milnor fibers $x \mapsto y \leftarrow z$ generalization

$X_x \times_{Y_y} z$

Defn

- f is loc acyclic rel to $\mathcal{F}$ if

$\forall x \mapsto y \leftarrow z$ $\mathcal{F}_x \to R\Gamma(X_x \times_{Y_y} z, \mathcal{F})$ is an isom

- univ. loc acyclic $\forall$ base change

($\Leftarrow$ $\forall$ smooth base change)

Facts: 1. (loc. acyclicity of smooth morphism)

$f: X \to Y$, ~~[illegible]~~ smooth. $\mathcal{F}$, loc. const ($= \forall q\ \mathcal{H}^q(\mathcal{F})$ loc. const)

$\Rightarrow f$. univ. loc acyclic rel to $\mathcal{F}$.

2 (generic local acyclicity)

$f: X \to Y$ $\mathcal{F}$ constructible

$\Rightarrow \exists V \subset Y$ dense open st $f_V: X_V \to V$ is u.l.a rel to $\mathcal{F}|_{X_V}$.

3. $f: X \to Y$. $g: Y \to Z$, $\mathcal{F}$ const on X.

f. (univ). l.a rel to $\mathcal{F}$ + g smooth $\Rightarrow$ $g \circ f$ (univ). l.a rel to $\mathcal{F}$

4 f. g. $\mathcal{F}$.

f proper $g \circ f$ (univ) l.a. rel to $\mathcal{F} \Rightarrow g$ (univ). l.a rel to $Rf_*\mathcal{F}$.

5 ~~If~~ For $\mathcal{F}$ on X

$\mathcal{F}$ is locally const $\Leftrightarrow$ $id: X \to X$. is loc. acyclic rel to $\mathcal{F}$.

1.3 micro support

$C \subset T^*X$ conical closed. $\mathcal{F}$ const cx on X

Defn 1. $\mathcal{F}$ is micro supported on C if for every $X \xleftarrow{h} U \xrightarrow{f} Y$. C-transversal, $f: U \to Y$ is univ. l.a. rel to $h^*\mathcal{F}$

2 weakly micro supported

for every $X \xleftarrow{h} U \xrightarrow{f} Y$ C-trans. h open imm, Y. curve

Example $\mathcal{F}$ locally constant $\Leftrightarrow$ $C = T^*_X X$ $\Rightarrow$ Fact 1 + Example; $\Leftarrow$ ~~char. of l.const~~ Fact 5

Lemma $\mathcal{F}$ w.m.s on C & C' $\Rightarrow$ $\mathcal{F}$ w.m.s on $C \cap C'$

for m.s not a priori so but true.

C minimal among m.s. $\rightarrow$ tightly $\quad C$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$

C ″ w.m.s $\rightarrow$ smallest. $SS\mathcal{F}$

singular support

Thm Every irred comp of $SS\mathcal{F}$ is of dim n & $\mathcal{F}$ is micro supported on $C = SS\mathcal{F}$.

Thm 1 (1.2) $\exists C$ s.t. $\mathcal{F}$ is micro supp on C & $\dim C \leqq n$

Thm 2 (1.3) Assume $\mathcal{F}$ is tightly micro supp on C. Then every irred cpt of C is of dim n & $C = SS\mathcal{F}$

Thm 1 + 2 $\Rightarrow$ Thm

Reduce to $X = \mathbb{P}^n$ & use Radon transform

1.4 Reduction to $X = \mathbb{P}^n$.

Lemma 1. 1. $f: U \to X$ étale, Th1 for $(X, \mathcal{F})$ implies = for $(U, f^*\mathcal{F})$

2 Th1 for $\mathcal{F} \oplus \mathcal{G} \Rightarrow$ Th1 for $\mathcal{F}$ & $\mathcal{G}$

Cor $f: X \to Y$ étale Th1 for $(Y, f_*\mathcal{F})$ implies = for $(X, \mathcal{F})$

Lemma $\Rightarrow$ Cor $\quad (Y, f_*\mathcal{F}) \rightsquigarrow (X, f^*f_*\mathcal{F})$

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$f^*f_*\mathcal{F} = p_{2*}p_1^*\mathcal{F}$

$\delta: X \to X \times_Y X$ open & closed. $\mathcal{F}$ direct summand of $p_{2*}p_1^*\mathcal{F}$

Th1 is reduced to $X = \mathbb{P}^n$. Zariski local

$f: X \to \mathbb{P}^n$ étale

Lemma 2 $\quad i: X \to Y$ closed imm

1. $\mathcal{F}$ (weakly) micro supported on C $\Rightarrow$ $i_*\mathcal{F}$ —— on $i_\circ C$

2. $\mathcal{F}$ tightly —— on C $\Rightarrow$ $i_*\mathcal{F}$ —— on $i_\circ C$

or $C = SS\mathcal{F}$ $\qquad$ or $i_\circ C = SS i_*\mathcal{F}$

Lemma 3 $\quad j: U \to X$ open imm

1. $\Rightarrow$ $j^*\mathcal{F}$ —— on $C|_U$

2

2. trans open condition.

$X \xrightarrow{i} U \overset{j}{\hookrightarrow} \mathbb{P}^n$

1.5 Radon transform, Legendre transform

$P = P(V) = \{\text{lines in } V\} = \{\text{points in } P\}$

$P^\vee = P(V^\vee) = \{\text{hyperplanes in } V\} = \{\text{hyperplanes in } P\}$

$Q = \{(x, x^\vee) \in P \times P^\vee \mid x \in x^\vee\} \subset P \times P^\vee$

$$\begin{tikzcd} & Q \arrow[dl, "p"'] \arrow[dr, "p^\vee"] & \\ P & & P^\vee \end{tikzcd}$$

$\mathcal{F} \longmapsto R(\mathcal{F}) = Rp^\vee_* p^* \mathcal{F}[n-1]$

$\mathcal{G} \longmapsto \check{R}(\mathcal{G}) = Rp_* p^{\vee *} \mathcal{G}[n-1]$

almost inverse to each other except up to geom. const sheaves

$$\begin{tikzcd} & & Q \times_{P^\vee} Q \arrow[dl] \arrow[dr] & & \\ & Q \arrow[dl] \arrow[dr] & & Q \arrow[dl] \arrow[dr] & \\ P & & P^\vee & & P \end{tikzcd}$$

$$\begin{tikzcd} & Q \times_{P^\vee} Q \subset P^\vee \times P \times P \arrow[d, "\pi"] & \\ P & P \times P \arrow[l, "p_1"'] \arrow[r, "p_2"] & P \end{tikzcd}$$

$\mathbb{P}^{n-2}$-bdle outside P

$Q = \mathbb{P}^{n-1}$ —— P

$\check{R} \circ R(\mathcal{F}) = Rp_{2*}(p_1^* \mathcal{F} \otimes R\pi_* \Lambda)$

$\to R\Gamma(\mathbb{P}^{n-2}, \Lambda) \otimes \Lambda_{P \times P} \to R\pi_* \Lambda \to \Lambda_P(-(n-1))[-2(n-1)] \to$ dist tri

$Q = P(T^*P) \subset P \times P^\vee$

$= P(T^*P^\vee)$

$0 \to \Omega^1_P \to V^\vee \otimes \mathcal{O}(-1) \to \mathcal{O}_P \to 0$

~~$N_{Q/P\times}$~~ $T^*_Q(P \times P^\vee) \hookrightarrow T^*(P \times P^\vee) \times_{P \times P^\vee} Q$

$\downarrow$

univ. subline bdle $\;\; T^*P \times_P Q$

Similarly for $P^\vee$

1.6 Reformulation of Thms.

Def $f: X \to Y$. $\mathcal{F}$ on X.

$E_f(\mathcal{F}) \subset X$ closed. the complent of the largest open $U \subset X$

s.t. $f|_U : U \to Y$ is univ. l.a. rel to $\mathcal{F}|_U$

Thm 1' (1.4) For $\mathcal{G}$ on P, $E_{p^\vee}(p^* \mathcal{G})$ is of dim $\leq n-1$.

$d \geq 1$ $\quad i_d : P \hookrightarrow \tilde{P}$ $\quad$ d-th Veronese embedding

$\mathcal{O}(d)$

Thm 2' (1.6) $\mathcal{G}$ on P Assume $d \geq 3$. and let $D \subset \tilde{\mathbb{P}}$ be the complnt of the largest open when $\tilde{R}(i_{d*}\mathcal{G})$ is loc. const

(1) D is a divisor

(2) For each irred cpt D_a of $D = \bigcup D_a$, there exists a unique irreducible subset $C_a \subset T^*\mathbb{P}$ of dim n. s.t $D_a = \tilde{p}^\vee P(i_{0} \circ C_a)$. More over $P(i_0 C_a) \to D_a$ is radicial

(3) $C = \bigcup C_a$ is $SS\mathcal{G}$

Rmk

1. If $\mathcal{G}$ is m.s on C, the $E_{p^\vee}(p^* \mathcal{G}) \subset P(C)$ in $\mathcal{Q}$

$1' \Rightarrow 1$, $2'$, $1'$.

Example $i: X \hookrightarrow P \subset \tilde{P}$ $\quad \mathcal{G} = i_* \Lambda$

$i_\circ C_a = T^*_X \tilde{P}$ $\quad D_a = X^\vee$

1.7 Thm 1' ⇒ Thm 1.

$\mathcal{F}$ on $\mathbb{P}^\vee$. wma $\mathcal{F} = R(\mathcal{G})$ since loc. const - Ob.

$$\begin{tikzcd} \mathbb{P} & Q \arrow[l, "p"'] & Q_U \arrow[l] \arrow[ll, bend right, "p'"'] \arrow[d, "p'^\vee"] \\ & P^\vee \arrow[u, phantom, "\square"] & U \arrow[l, "h"'] \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

$E = P(C^\vee)$

$C^{\vee +} = C^\vee \cup T^*_{P^\vee} P^\vee$

$\dim E \leq n-1 \Rightarrow \dim C^{\vee +} \leq n$

(A) $P^\vee \leftarrow U \rightarrow Y$ $C^{\vee +}$ transversal $\overset{\text{Thm 1}}{\Rightarrow}$ f univ. l-a rel to $h^*\mathcal{F}$

⇓ ⇗ Fact 4. + $p^\vee$ proper

(C) $f \circ p'^\vee$ univ. l-a rel to $p'^*\mathcal{F}$

Fact 3 ⇒ (C) holds on the complement of the inv. image of E
+
$f : U \to Y$ smooth. ($\Leftarrow C^{\vee +} \supset T^*_{P^\vee} P^\vee$) $f \circ p'^\vee$ u.l-a

Define $Q_U^f \subset Q_U$ largest open where $Q_U \to P \times Y$ is smooth

Fact 2 ⇒ (C) holds on Q_U^f = u.l-a

Suffices to show that inv. image
$P^\vee \leftarrow U \rightarrow Y$ $C^{\vee +}$-trans ⇒ of $E \subset Q_U^f$.
(A) i.e $Q_U \to P \times Y$ is smooth on a nbd of inv. image of E.

$\subset Q$

- E is the locus where $T^*_Q(P \times P^\vee) \subset T^*(P \times P^\vee) \times_{P \times P^\vee} Q$ is inside $(T^*P \times C^\vee) \times_{P \times P^\vee} Q$

- Exercise 2' ⇒ $\begin{matrix} C^\vee & & U \times T^*Y \\ \cap & & \cap \\ T^*P^\vee & \to & T^*U \end{matrix}$ intersection with inv. image $\subset$ 0-sect

⇒ on the inv. image $T^*_Q(P \times P^\vee) \to (T^*P \times T^*U) \times_{P \times U} Q_U$ inv. image $\subset$ 0-sect
$\uparrow$
$(T^*P \times T^*Y)$:

1.8 Thm 2'

$C \subset T^*\mathbb{P}$ $\quad C^+ = C \cup T^*_{\mathbb{P}}\mathbb{P}$ $\quad C^\vee$ Legendre transf

$C^{\vee +} = C^\vee \cup T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee$.

Lemma 1. $\mathcal{G}$ micro supported on $C^+ \Leftrightarrow \mathcal{F} = R(\mathcal{G})$ m.s. on $C^{\vee +}$

$\mathcal{G}$ tightly micro supp on C.

Cor $D = p^\vee(P(C)) \subset \mathbb{P}^\vee$ is the complement of the largest U s.t. $\mathcal{F}|_U$ is loc.const.

Pf. Complement of D is the largest op. where ~~$\mathcal{G}$~~ $\mathcal{F}|_U$ is micro supp on the 0-sect.. Example in 1.3.

Lemma 2. $d \geq 3$.

$\{C \subset T^*\mathbb{P} \mid$ irred. conical closed subset of dim $\leq n\}$ $\quad C$

$\downarrow$ $\quad \mathbb{I}$

$\{D \subset \tilde{\mathbb{P}}^\vee \mid$ irreducible closed subset$\}$ $\quad D = \tilde{p}^\vee(P(i_0 C))$

(1) injection

(2) $P(i_0 C(C)) \longrightarrow D$ is generically radicial.

(2) $\Rightarrow$ $\quad n - \dim C = \operatorname{codim}(D.\tilde{\mathbb{P}}) - 1 \ (\geq 0.)$

$\Gamma(\mathbb{P}, \mathcal{O}(d)) \twoheadrightarrow \mathcal{O}(d)_x / m_x^2 \mathcal{O}(d)_x \times \mathcal{O}(d)_y / m_y^2 \mathcal{O}(d)_y$. for $x \neq y$

Cor ~~of~~ Lem 1 + Lem 2 $\Rightarrow$

Thm is reduced to

(1) D is a divisor

(2) $C = \bigcup C_a$ $(D = \bigcup D_a)$ is $SS(\mathcal{G})$

Perverse sheaves

$\mathrm{Perv}(X) \subset D(X)$ abelian subcategory.

every object is of finite length.

$\mathcal{G}$ on $\mathbb{P}$ simple perv not geom const $\Rightarrow \exists\, {}^p\mathcal{H}^0(R(\mathcal{G})) = \mathcal{F}_0$ s.t. $\mathcal{F}_0$ simple not geom const

$\mathcal{G}$ simple not geom const. Either

(a) $\mathcal{F}_0|_U = 0$

(b) $\mathcal{F}_0|_U \neq 0$

(a) $Y \subset \mathbb{P}$ supp of $\mathcal{G}$ $C \supset SS\mathcal{G} \supset N_Y$

$D \supset \quad \supset Z = Y^\vee \in$ irred cpt

$\mathcal{F}_0|_Z \neq 0$

simple $\quad D = Z \Rightarrow C = SS\mathcal{G}$
divisor $\quad = N_Y$

(b) $U \subset \mathbb{P}^\vee$ is the max open where $\mathcal{F}_0|_U$ is loc. const.

D div. Zariski–Nagata.

$C = SS\mathcal{G}$ $\quad L \cap D_a$ transversally

$p(i_0 C_a) \to D_a$ radicial at intersect.,

away from D_b $(b \neq a)$ $\quad$ by prop base change + isolated char pt.

$\Rightarrow \quad \to \quad \cong \quad \to$ van $\quad \to \quad$ dist.

$\mathcal{F}_0$

1.9 Thm 1'

Thm 1'' S smooth/k $P \to S$ proj space b'dle. $\mathcal{g}$ on P

$\Rightarrow \exists S^\circ$ dense open s.t. $E_{P^\vee}(p^* \mathcal{g})|_{Q \times_S S^\circ}$ is of dim $\leq \dim P - 1$

Induction on rel dim P/S

0. $P = P^\vee = S$ 1. $P = \cancel{P} \leftarrow Q \to P^\vee$ isom

$E\,\mathrm{id}(\mathcal{g})$ is of dim $< \dim P$ Exam in 1.4.

~~locally~~ $S^\circ = S$.

induction. $L \hookrightarrow \mathbb{P}$ sub $\mathbb{P}^1$-b'dle

$$\begin{tikzcd} L \arrow[r, hook] \arrow[dr] & \mathbb{P} \arrow[d] \\ & S \end{tikzcd}$$

$\bar{x}$ $L \in \bar{x} \subset P$

$$\begin{tikzcd} P \arrow[d] & \tilde{P}_L \arrow[l] \arrow[r] \arrow[d] & \bar{P}_L \arrow[dl] \\ S & L \arrow[l] & \end{tikzcd}$$

$\bar{P}_L = $ ~~$\mathbb{P}(\ldots)$~~ pull-back of univ. $\mathbb{P}^{n-1}$-b'dle by $L \to \mathbb{P}$

$\mathbb{P}^1$-b'dle

blow up at $L \to P \times_S L$

$$\begin{tikzcd} x^\vee & x^\vee \arrow[l] \\ P^\vee \arrow[d] & P^\vee_L \arrow[l] \arrow[d, "\pi"] \\ S & L \arrow[l] \end{tikzcd}$$

blow up at $Y = \{x^\vee \mid L_s \subset x^\vee\}$

$P^\vee_{(L)} = \mathbb{P}^\vee - Y$ $x^\vee$

$$\begin{tikzcd} \hat{P}^\vee & P^\vee_L \arrow[l] \arrow[d] \\ & L \end{tikzcd}$$

$x^\vee \cap L_s = \pi(x^\vee)$

P $x \in \bar{x} \in \hat{x}^\vee$

blow up at the image of σ

$$\begin{tikzcd} \mathbb{P} & \tilde{P}_L \arrow[l] \arrow[d] & \tilde{Q}_L \arrow[l] \arrow[r] \arrow[d, "v"] \arrow[dl, phantom, "\square"] & Q_{(L)} \arrow[d, "p^\vee"] \\ & \bar{P}_L & \bar{Q}_{(L)} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}^\vee"] & P^\vee_{(L)} \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

x $x \in \bar{x}$ $x.x^\vee$ $(\pi(x^\vee), x^\vee)$

$\mathbb{P}^1$-b'dle $\bar{x}$ $\bar{x} \in \hat{x}^\vee$ $\hat{x}^\vee$ $x^\vee$

$$\left(\begin{matrix} \cap & & \cap \\ \bar{Q}_L & \longrightarrow & \hat{P}^\vee_{(L)} \end{matrix}\right)$$

$$
\begin{tikzcd}
g \arrow[d] \arrow[drr, bend left] \arrow[drrr, bend left=40, "K.\tilde{E}", "E"'] & & & \\
\bar{g} \arrow[rr, "\bar{E}"'] & & \bar{R}(\bar{g}) & R(g)
\end{tikzcd}
$$

If r is finite on $\tilde{E}$ $\Rightarrow$ $\bar{E} = r(\tilde{E})$

r: $\mathbb{P}$-bdle

& $\dim \bar{E} = \dim \tilde{E}$

$D \subset P \supset U$ g loc. const

$E \subset$ inv. image of D

If $L \not\subset D$, $E \cap \sigma(P^{\vee}_{(L^{\circ})}) = \phi$ &

$L^{\circ} = L \setminus (L \cap D)$

&

$\tilde{E} \to E$ isom.

$\dim \hat{E} = \dim E$.

2 Characteristic cycle $\quad k$ perfect/alg. closed.

$X/k \quad \mathcal{F}$ on $X \quad C = SS\mathcal{F} \subset T^*X$

$n = \dim X$ $\quad$ conic closed $\quad C = \bigcup C_a \quad C_a$ dim n

$X \xleftarrow{h} W \xrightarrow{f} Y$

h C-transversal. $\quad f$ $h^\circ C$-transversal

$\Rightarrow f: W \to Y$ univ. loc. acyclic rel to $h^*\mathcal{F}$

$\mathrm{Char}\,\mathcal{F} = \sum m_a [C_a] \quad m_a \in \mathbb{Z}[\frac{1}{p}]$.

2.1. Definition of $\mathrm{Char}\,\mathcal{F}$ — Milnor formula

2.2. Functoriality $\quad$ Index formula

3. Equivalent characterization of singular support.

2.1 $\quad X \xleftarrow{j} U \xrightarrow{f} Y \quad j$ étale. Y curve.

Def. We say a closed point $u \in U$ is an isolated characteristic point w.r.t C

if $X \leftarrow U - \{u\} \to Y$ is C-transversal.

Example $C = T^*_X X$.

Vanishing cycle $\quad$ $\mathcal{F}$-side $\quad$ dis. tri

$$\to \mathcal{F}_u \to R\Gamma(X_u \times_{Y_{(v)}} \bar{\eta}, \mathcal{F}) \to \phi_u(\mathcal{F}, f) \to$$

stalk at u of the complex of vanishing cycles

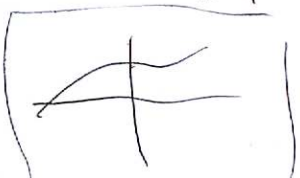
ϕ_u^q Λ-rep'n of finite dim of $\mathrm{Gal}(\bar{K}_v/K_v)$

$\dim\mathrm{tot}\,\phi_u = \sum (-1)^q \dim\mathrm{tot}\,\phi_u^q$

$\dim{} + \mathrm{Sw}$.

Intersection number. t. $f^\circ dt = df$ on ~~C~~. C-side

df $\quad j^*C$



$T^*U = U \times_X T^*X$

u $\quad U$ $\quad$ $df \cap j^*C$ isolated on a nbd of $T^*_u U$.

$\therefore \ (j^*\sum m_a C_a, df)_{T^*U,u}$.

Theorem 1. (Milnor formula) There exists a unique $\mathbb{Z}[\frac{1}{p}]$-linear combination $\mathrm{Char}\,\mathcal{F} = \sum m_a C_a$ of irred comp't C_a of $SS\mathcal{F} = C = \bigcup_a C_a$ such that for every $X \xleftarrow{j} U \xrightarrow{f} Y$ with isolated char pt $u \in U$, we have

$$-\dim\mathrm{tot}\,\phi_u(\mathcal{F}, f) = (j^*\mathrm{Char}\,\mathcal{F}, df)_{T^*U,u}$$

Example $t^q - t = \frac{y}{x^n}$ $\quad 2 \neq p | n$.

Example $\mathcal{F} = \Lambda$ $\quad$ r.h.s = length $\Omega^n_{U/Y,u}$

Milnor formula by Deligne SGA 7. Exp XVI.

Proof Idea. Modification of Deligne's pf.

- Tool. local version of Radon transform
  defined using the formalism of vanishing cycles over general base scheme (not nec. curve)
- Def. of coeff. generic pencil.
- Independence of Milnor formula.
  - Stability of dimtot.
  - Continuity of Swan conductor Deligne/Laumon

2.2 Functoriality

$h : W \to X$, $f : X \to Y$.

Def $h : W \to X$ is strongly C-transversal if $h^* C = W \times_X C$ is purely of dim W & h is C-transversal

$$\begin{array}{ccc} h^*C & \xrightarrow{\text{finite}} & h^{\circ}C \\ \cap & & \downarrow \\ W \times_X T^*X & \longrightarrow & T^*W \end{array}$$

$$h^!(\sum m_a [C_a]) = (-1)^{\dim X - \dim W} \sum m_a [h^{\circ} C_a].$$

Thm 2 If $h : W \to X$ is strongly C-transversal for $C = SS\mathcal{F}$, then

$$CC\, h^* \mathcal{F} = h^! \, CC\, \mathcal{F}$$

Pf. W divisor X. red to $\dim X = 2$.

global argument originally due to Deligne of ramification theory

Lemma $f : X \to Y$ proper + C-transversal for $C = SS\mathcal{F}$

Then $Rf_* \mathcal{F}$ is locally constant

Pf ~~proper~~ f proper + loc. acy rel to $\mathcal{F}$ $\Rightarrow$ $Rf_* \mathcal{F}$ l.c. (Index formula)

Theorem 3 If X is projective,

$$\chi(X, \mathcal{F}) = (CC\, \mathcal{F}, T^*_X X)_{T^*X}.$$

Pf Induction on dim. Take a good pencil.

$f : X' \to L$ ~~$\chi(X', \mathcal{F}') = (\ldots, 0)$~~

$X' \to X$

$$\chi(X', \mathcal{F}') = \chi(X, \mathcal{F}) + \chi(Z, \mathcal{F}|_Z)$$

ind. hypo + Thm 2

$\chi(X', \mathcal{F}') = \chi(L, Rf_* \mathcal{F}')$

$\underset{\uparrow}{=} \underset{\overset{\shortparallel}{2}}{\chi(L)} \cdot \underset{\uparrow}{\mathrm{rk}\, Rf_* \mathcal{F}'} - \sum_{v \in L} \underset{\uparrow}{\dim\mathrm{tot}_v \Phi_u(\mathcal{F}, f)}$

G-O-S   ind. hypo ~~th~~ Thm 2   Thm 1

	h	f	~~##~~ Deligne, Milnor
$\mathcal{F}$	?	loc acycl	int. product
C	trans	trans	

~~##~~ 3 Equiv. char

Def. $h: W \to X$ is $\mathcal{F}$-transversal if the canonical morphism

$$h^*\mathcal{F} \otimes Rh^! \Lambda \to Rh^! \mathcal{F}$$

is an isomorphism.

Theorem   For $C \subset T^*X$ and $\mathcal{F}$ on X, the following cond. are equiv.

(1) $\mathcal{F}$ is micro supported on C i.e.
for $X \xleftarrow{h} W \xrightarrow{f} Y$ C-transversal, f is univ. l.a. rel to $h^*\mathcal{F}$

(2) ~~##~~ For every $h: W \to X$, C-transversal $\Rightarrow$ $\mathcal{F}$-transversal

Pf. (1) $\Rightarrow$ (2) may assume h imm. by Example.

$$\begin{array}{ccc} W & \to & X \\ \downarrow & \square & \downarrow \\ y & \in & Y \end{array}$$

reduced to smooth base change thm

(2) $\Rightarrow$ (1) may assume f smooth. & $X = W$.

Induction on rel dim. $X \to Y$.
Reduction to Y curve.   reduced to loc. acyclicity of smooth morphism

2. If $\mathcal{F}$ is l.c. then any $h: W \to X$ is $\mathcal{F}$-transversal

Example 1 A smooth morph $h: W \to X$ is $\mathcal{F}$-transversal for any $\mathcal{F}$. Poincaré duality

2. ~~Properties~~ Functoriality

(1) $h: W \to X$ strongly C-transversal

pull-back $h^* C$ dim W & $h^\circ C = W \times_X C \to h^! C$ finite.

$$\begin{array}{ccccc} & & \cap & & \downarrow \\ T^*X & \leftarrow & W \times_X T^*X & \to & T^*W \end{array}$$

Theorem 2. If $h: W \to X$ st. C-transversal

$\Rightarrow$ $\mathrm{Cha}\, h^* K = h^! \,\mathrm{Cha}\, K$

$\uparrow$

$(-1)^{\dim W - \dim X}$

Pf. W div of X. red to $\dim X = 2$.

push forward.

(2) Theorem 3 ~~\#~~ If X is projective

$\chi(X, K) = (\mathrm{Cha}\, K, T^*_X X)_{T^*X}$

Pf Induction on dim. $\dim X = 1$ G.O.S

Th1 + Th2 + GOS

3. Equivalent characterization of singular support

Theorem 4. $C \subset T^*X$ conic closed

(1) For $X \xleftarrow{h} W$ h smooth. f $h^\circ C$-transversal

$\downarrow f$ $\Rightarrow$ f $h^* K$-acyclic

Y

(2) For $X \xleftarrow{h} W$ C-transversal

$\Rightarrow$ $h^* K \otimes^L Rh^! \Lambda \to Rh^! K$ isom

(1) $\Rightarrow$ (2). Conversely if $C \supset T^*_X X$ $\Rightarrow$, (2) $\Rightarrow$ (1)

$L_x \subset T^*X \times_X \bar{\xi}$

line spanned by the image of the char. for char x.

defined over a finite extension $F_x / F = k(\xi)$

$$\operatorname{Char} \mathcal{F}|_{S_p \mathcal{O}_{X,\xi}} = (-1)^n \Big(\operatorname{rk} \mathcal{F}|_U \cdot [T^*_X X] + \sum_{r \geq 1} r \sum_x m(x) \cdot \frac{[L_x]}{[F_x : F]} \Big)$$

$$SS\mathcal{F}| = T^*_X X \cup \bigcup_{r \geq 1} \bigcup_r L_x .$$

Example. $X = \mathbb{A}^2$ $U = \mathbb{A}^2 - D$ $D = (x = 0)$

$\mathcal{F} = j_! j^* \mathcal{F}$ $j^* \mathcal{F}$ dfd by $t^p - t = \frac{y}{x^d}$ $p \nmid d, \neq 2$ or $d > p$

$SS\mathcal{F} = T^*_X X \cup \langle dy / D \rangle$

$V = V^{(d)} = x$ $\operatorname{char} x : m^d_{\bar{K}} / m^{d+1}_{\bar{K}} \to \Omega^1 \otimes \bar{F}$.

$x^d \mapsto dy$

$\operatorname{Char} \mathcal{F} = [T^*_X X] + d \cdot [dy / D]$.

not Lagrangean.