Singular support and Characteristic cycle

k perfect field of char $p (>0)$

X smooth$/k$ $\quad n = \dim X$

$\Lambda / \mathbb{F}_\ell$ finite ~~local ring~~ ~~k~~ res char $\ell \neq p$

$\mathcal{F}$ constructible complex ($\forall q\ \mathcal{H}^q(\mathcal{F})$ const, a$\forall q\ \mathcal{H}^q = 0$) on X

$$SS\mathcal{F} = \bigcup C_a \subset T^*X \quad \text{cotangent bdle} \quad \dim 2n$$

irred closed conical subset of dim n
stable under multiplication

$$CC\mathcal{F} = \sum m_a [C_a] \qquad m_a \in \mathbb{Z}\left[\tfrac{1}{p}\right]$$

controls behavior of $\mathcal{F}$ cf. micro local analysis.

Classical

Example $\dim X = 1$ $\quad j : U \hookrightarrow X$ dense open imm.
$\mathcal{F}$ locally constant, $\neq 0$

$$SS j_!\mathcal{F} = \underbrace{T^*_X X}_{0\text{-sect}} \cup \bigcup_{x \notin U} \underbrace{T^*_x X}_{\text{fiber}}$$

$$CC j_!\mathcal{F} = (-1)\Big(\mathrm{rk}\,\mathcal{F}\cdot[T^*_X X] + \sum_{x \notin U} \overset{\in \mathbb{N}}{\mathrm{dimtot}_x \mathcal{F}} \cdot [T^*_x X]\Big)$$

rk + Swan conductor
measures wild ramification

X projective, $k = \bar{k}$.

$$\chi(X, j_!\mathcal{F}) = (CC j_!\mathcal{F}, T^*_X X)_{T^*X}$$

intersection number

Grothendieck–Ogg–Shafarevich.

1. Singular support (Beilinson)
$C \subset T^*X$ closed conical.
$f: X \to Y$ C-transversal if
$df: X \times_Y T^*Y \to T^*X$ $\supset C$ $\quad df^{-1}(C) \subset 0$-section.

Theorem (Beilinson)
~~The smallest closed conical subset $C = \bigcup C_a \subset T^*X$~~
s.t $C = SS\mathcal{F}$
Let $C = \bigcup C_a \subset T^*X$ be the smallest closed conical subset s.t for every $X \xleftarrow[\text{étale}]{j} U \xrightarrow{f} \mathbb{A}^1$, & f is $j^\circ C$-transversal

f is locally acyclic relatively to $j^*\mathcal{F}$.
Then. $\dim C_a = \dim X$ for every C_a.

locally acyclic -- no non-zero vanishing cycle

2. Char. cycle

$X \xleftarrow[\text{étale}]{j} U \xrightarrow{f} \mathbb{A}^1$, $u \in U$. We say u is an isolated char pt if $U - \{u\} \to \mathbb{A}^1$ is $j^\circ C$-transversal.

$C = \bigcup C_a$. $df: U \to T^*U$. $C_a \cap df(U)$ isolated,
$(\sum m_a C_a, df)_{T^*U, u}$

Theorem 1 There exists a unique $\mathbb{Z}[\frac{1}{p}]$-linear combination $CC\mathcal{F} = \sum m_a [C_a]$ such that for every $X \leftarrow U \xrightarrow{f} \mathbb{A}^1$ with isolated char pt u,

$$-\mathrm{dimtot}\, \phi_u(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*U, u}$$

Milnor formula
$\mathcal{F} = \Lambda$, $CC\mathcal{F} = [T^*_X X]$. Deligne. SGA 7.

Theorem 2 X projective

$$\chi(X, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$$

Outline of Proof

1. $X \hookrightarrow \mathbb{P}^n$ Radon transform

$$\begin{tikzcd} & Q \arrow[dl, "p"'] \arrow[dr, "p^\vee"] & \\ \mathbb{P}^n & & \mathbb{P}^{\vee n} \end{tikzcd}$$

Q universal hyperplane

$\rightsquigarrow$ coefficient mult. • indep.

• Milnor formula

Stability of dimtot $\Leftarrow$ continuity of Swan conductor

Deligne/Laumon

2 Induction on dim reduce to GOS formula

$$\begin{tikzcd} X' \arrow[r, "\text{blowup}"] \arrow[d, "p"'] & X \supset Z \\ \mathbb{P}^1 & \end{tikzcd}$$

Milnor formula

$$\chi(X, \mathcal{F}) = 2 \cdot \chi(Y, \mathcal{F}|_Y) - \sum \text{dimtot} + \chi(Z, \mathcal{F}|_Z)$$

$\chi(Y, \mathcal{F}|_Y)$, $\chi(Z, \mathcal{F}|_Z)$ induction

$+$ $\quad CC\, h^*\mathcal{F} = h^! CC\mathcal{F}$

for Strongly C-transversal morphism $h: W \to X$

$$W \times_X T^*X \to T^*W \qquad h^{*-1}C \cap K \subset 0\text{-section}$$

$$\cup$$

h^*C $\cdot$ dim W.

Compatibility with pull-back.

Reduction to divisor, dim $X = 2$.

Ramification theory + global argument