

Singular support and Characteristic cycle

k perfect field of char $p(>0)$

X smooth/ k $n = \dim X$

Λ/\mathbb{F}_e finite local ring \mathbb{F}_e res char $\neq p$

\mathcal{F} constructible complex ($\text{tg } \mathcal{H}^i(\mathcal{F})$ const, $\text{at } \mathcal{H}^i = 0$)

$$SS\mathcal{F} = \cup C_a \subset T^b X \quad \text{cotangent bundle} \quad \dim \geq n$$

irred closed conical subset of $\dim n$
stable under multiplication

$$\text{Char } \mathcal{F} = \sum m_a [C_a] \quad m_a \in \mathbb{Z}[\frac{1}{p}]$$

controls behavior of \mathcal{F} cf. micro local analysis.

classical

Example $\dim X = 1$ $j: U \hookrightarrow X$ dense open $\dim = 1$

\mathcal{F} locally constant, $\neq 0$

$$SS j_! \mathcal{F} = \underbrace{T_x^b X}_{0\text{-sect}} \cup \bigcup_{x \notin U} T_x^b X \text{ fibers.}$$

$$\text{Char } j_! \mathcal{F} = (-1) (\text{rk } \mathcal{F} \cdot [T_x^b X]) + \sum_{x \notin U} \text{dim } \text{tot}_x \mathcal{F} \cdot [T_x^b X] \in \mathbb{N}$$

rk + Swan conductor
measures wild ramification

X projective, $k = \bar{k}$

$$\chi(X, j_! \mathcal{F}) = \sum (\text{Char } j_! \mathcal{F}, T_x^b X) \cdot \text{vol } T_x^b X$$

Grothendieck - Ogus - Shafarevich.

1. Singula support (Beilinson)

$C \subset T^*X$ closed conical.

$f: X \rightarrow Y$ C -transversal f

$df: X \times_{\mathbb{C}} T^*Y \rightarrow T^*X$ $df^{-1}(C) \subset O$ -section.

Theorem (Beilinson)

~~The smallest closed conical subset $C = \cup C_i \subset T^*X$~~

Let $U = \cup C_i \subset T^*X$ be the smallest closed conical subset s.t for every $X \xrightarrow{j} U \xrightarrow{f} A'$, f is j^*C -transversal

f is locally acyclic relatively to F^* .

Then $dim C_i = dim X$ for every C_i .

locally acyclic -- no non-zero vanishing cycle

2. Char. cycle

$X \xrightarrow{j} U \xrightarrow{f} A'$
étale u

We say u is an isolated char pt if $U - \{u\} \rightarrow A'$ is j^*C -transversal.

$C = \cup C_i$ $df: U \rightarrow T^*U$. $C_i \cap df(U)$ isolated, $(\sum m_i C_i, df) \cap T^*u$.

Theorem 1 There exists a unique $\mathbb{Z}[\frac{1}{p}]$ -linear combination $Ch \gamma = \sum m_i [C_i]$ such that for every $X \xrightarrow{j} U \xrightarrow{f} A'$ with isolated char pt u ,

$-dim \text{tot } \phi_u(\gamma, df) = (Ch \gamma, df) \cap T^*u$

Milnor formula

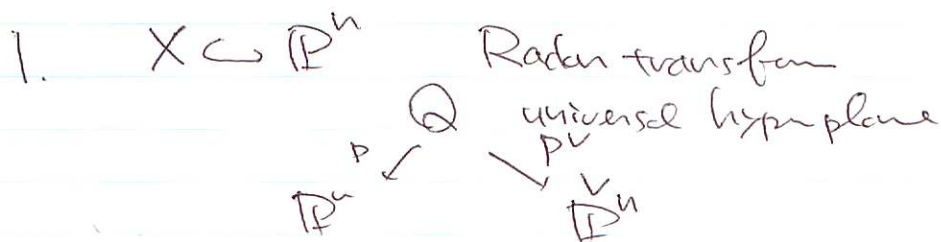
$\gamma = \Lambda$, $Ch \gamma = [T^*_x X]$.

Deligne. SGA7.

Theorem 2 X projective

$$\chi(X, \mathcal{F}) = C(\text{Char } \mathcal{F}, T_x^* X) T_x^* X.$$

Outline of Proof



\leadsto coefficient ma.

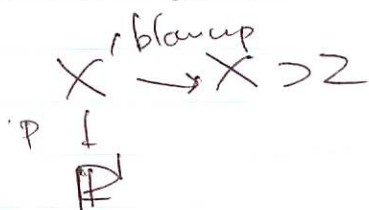
• indep.

• Milnor formula

Stability of dim tot.

\Leftarrow continuity of Swan conductors
Deligne/Lauerman

2 Induction on dim reduce to GOS formula



$$\chi(X, \mathcal{F}) = 2 \cdot \chi(\mathcal{F}|_Y) - \sum \text{dim tot} + \chi(\mathcal{F}|_Z)$$

Milnor formula

$\chi(\mathcal{F}|_Y), \chi(\mathcal{F}|_Z)$ induction

$$\text{Char } \mathbb{R}^c \mathcal{F} = \mathbb{R}^c \text{Char } \mathcal{F}$$

for strongly \mathbb{C} -transversal morphism $f: W \rightarrow X$

$$W \times_X T_x^* X \rightarrow T_x^* W \quad \mathbb{R}^c \text{C} \cap \mathbb{K} \subset \mathcal{O}\text{-section}$$

$$\mathbb{R}^c \text{C} \quad \dim W.$$

Compatibility with pull-back.

Reduction to divisor, $\dim X = 2$.

Ramification theory + global argument