K cd.v.f F res fd ch $p>0$

L/K finite Galois ext. $G = \mathrm{Gal}(L/K)$

$G_i = \mathrm{Ker}(G \to \mathrm{Aut}\, L^\times/1+m_L^i)$ $i \geq 1, \in \mathbb{N}$

lower ram. gp $G_1 = P \subset I$.

Not stable under qt.

even after renumbering if

$\mathcal{O}_L$ is not gen'd by a single elt $/\mathcal{O}_K$

G^r upper ram gp $r>0, \in \mathbb{Q}$ closed disk

Stable under qt. $r+, r \geq 0, \in \mathbb{Q}$ open disk

1. Definition.

p-Sylow

$G^1 = I = \mathrm{Ker}(G \to \mathrm{Aut}\, \mathcal{O}_L/m_L)$, $G^{1+} = P \subset I$

2. Graded qt $\mathrm{Gr}^r G = G^r/G^{r+}$ is an $\mathbb{F}_p$-v. sp for $r>1$.

3. Characteristic fn

inj $\mathrm{Gr}^r G^\vee = \mathrm{Hom}(\mathrm{Gr}^r G, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\bar{F}}(m_K^r/m_K^{r+}, H_1(L_{\bar{F}}/\mathcal{O}_K))$

$H_1 = \Omega^1_{\mathcal{O}_K/m_K^2} \otimes_{\mathcal{O}_K/m_K^2} \bar{F}$ if p is not a unif

$0 \to m_K/m_K^2 \otimes_F \bar{F} \to H_1 \to \Omega_F^1 \otimes_F \bar{F} \to 0$

Heuristic observation using terminology of rigid geometry

$\mathcal{O}_L = \mathcal{O}_K[\underline{x}]/(\underline{f})$ $\qquad \underline{x} = x_1, \dots, x_n$

$\qquad\qquad\qquad\qquad\qquad \underline{f} = f_1, \dots, f_n$

$\underline{f} : D^n \longrightarrow D^n$

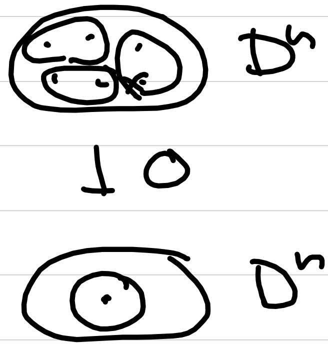
$G = \underline{f}^{-1}(0) = \mathrm{Mor}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_L, \mathcal{O}_{\bar{K}})$

$r > 0 \qquad D^{n,r} = \{ \underline{y} \in D^n \mid \mathrm{ord}_K(y_i) \geq r \quad i = 1, \dots, n \}$

$f^{-1}(0) \subset f^{-1}(D^{n,r}) \subset D^n$ rational subdomain

htp equiv't $\qquad \leftarrow$ fiber disks

$\mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{\bar{K}}/m_{\bar{K}}^r)$



$\boxed{G \twoheadrightarrow \pi_0(f^{-1}(D^{n,r}))} = G/G^r$

$G^r = \{ z \in G \mid$ contained in the same conn cpt of $f^{-1}(D^{n,r})$ as to conn. to $1 \in G \}$

(rigid geometry $\xleftrightarrow{\text{equiv.}}$ formal geom)

invert blowup

rational subdomain $\longleftrightarrow$ partial blowup

dilatation

$\uparrow$

language of schemes

forget

Construction K hens. d.v.f. L/K fin. sep. ext'n

$S = \mathrm{Sp}\,\mathcal{O}_K$. $T = \mathrm{Sp}\,\mathcal{O}_L$ Q smooth sch $/S$

$T \to Q$ imm over S

E.g 1. $\mathcal{O}_L = \mathcal{O}_K[\underline{x}]/(\underline{f})$, $Q = \mathrm{Sp}\,\mathcal{O}_K[x] = \mathbb{A}^n_S$
T

2. X sm$/k$ $D \subset X$ irred div. sm$/k$

$$\begin{array}{c} D \subset X \\ \uparrow \text{ finite} \\ Y \text{ smooth}/k \end{array}$$

$K = \mathcal{O}^h_{X,\xi}$ ξ gen pt of D

$$\begin{array}{ccc} Y \to X \times Y & \rightsquigarrow & T \to Q \\ \downarrow & & \downarrow \\ X & \times_X S & S \end{array}$$

$r > 0$ rational. K'/K $r' = e_{K'/K} \cdot r \in \mathbb{N}$

$S' = \mathrm{Spec}\,\mathcal{O}_{K'}$

$Q^{[r]}_{S'}$ dilatation

$Q = \mathrm{Sp}\,A$. $\mathcal{O}_L = A/I$.

$$Q^{[r]}_{S'} = \mathrm{Sp}\, A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}\left[\frac{I}{\pi'^{r'}}\right] \quad \pi' \text{ unif}$$

$Q^{(r)}_{S'}$ normalization.

Reduced fiber theorem. For suff large K' the geom closed fiber $Q^{(r)}_{\bar F} = Q^{(r)}_{S'} \times_{S'} \mathrm{Sp}\,\bar F$ is reduced

$$F^r(L) = \pi_0(Q^{(r)}_{\bar F}) \quad \text{indep of } T \to Q$$

functorial in L.

A variant

functoriality of dilatation & normalization.

$$\bar T_{S'} = (\text{normalization of } T \times_S S') \to Q^{(r)}_{S'}$$

$$F^{r+}(L) = \mathrm{Im}\,(\bar T_{\bar F} \to Q^{(r)}_{\bar F})$$

$$F(L) = \mathrm{Mor}_k(L, k_s) \to F^{r+}(L) \to F^r(L)$$

$$\mathbb{Q}+ = \{r \in \mathbb{Q} \mid r > 0\} \amalg \{r+ \mid r \in \mathbb{Q}, r \geq 0\}$$

Theorem 1. L/k fin. Galois ext'n $G = \mathrm{Gal}(L/k)$

1. There exists a decreasing fil. (G^r) indexed by $r \in \mathbb{Q}+$ by normal subgps st. $F(M)/G^r \to F^r(M)$ are bij for $r \in \mathbb{Q}+$ and $M \subset L$

2. There exist $0 = r_0 < r_1 \cdots < r_n$ rat'l numbers s.t G^r are const' on $[r_i+, r_{i+1}]$ for $i = 0, \ldots, n-1$ & $G^r = 1$ on $[r_n, \infty)$

Cor G^r compatible w qt.

Idea of proof

1. $$\begin{tikzcd} F(L) \arrow[r, "\subseteq G"] \arrow[d] & F^r(L) = G/G^r \arrow[d] \\ F(M) \arrow[r] & F^r(M) \end{tikzcd}$$ cocartesian $\Leftarrow$ $F(L) \to F(M) \times_{F^r(M)} F^r(L)$ surjection.

$\Uparrow$

Going down thm.

2. • Construction in family $0 \leq r \leq m$.
parametrized by a curve $/S$.
Reduced fiber thm + Stable reduction thm for
stable under b.c. curves
and doublept $\leftrightarrow$ interval.

• $F(L) \to F^r(L)$ bij for $r >> \infty$
construct idempotents

$\mathcal{O}_L$ gen by a single elt $T \to Q = \mathbb{A}^1_S$

Explicit computation of $Q^{(r)}_T$
$\rightsquigarrow$ Herbrand fn.

$G^1 = I$. $G^{H} = P$ $\supset$ easy

$\subset$ reduction to monogenic case
$[L:K] = p \Rightarrow$ monogenic.
+ tameness criterion by different.

2. Graded qts $Gr^r G = G^r/G^{r+}$ $r>1$.

By compatibility with qt. we may assume $G^{r+}=1$.

Thm Assume $r>1$ and $F(L)\to F^{r+}(L)$ bij

$$\Rightarrow \quad Q^{(r)}_{\bar F} \longrightarrow Q^{[r]}_{\bar F,\mathrm{red}} \text{ is finite étale.}$$

$$Q^{[r]}_{F,\mathrm{red}} = \mathrm{Hom}_F(m_{K'}^{r}/m_{K'}^{r+}, N_{T/Q}\otimes E)^{\vee} \quad F'\otimes_F E\text{-module}$$

Pf.
$$\begin{array}{ccc} T & \to & Q \\ \downarrow & \square & \downarrow \\ S & \to & P \end{array} \qquad \begin{array}{ccc} Q^{(r)}_{S'} & \to & Q^{[r]}_{S'} \\ \downarrow & & \downarrow \text{finite} \\ P^{(r)}_{S'} & = & P^{[r]}_{S'} \text{ smooth}/S' \end{array} \qquad \begin{array}{c} Q^{[r]}_{F',\mathrm{red}} \\ \downarrow \text{ étale} \\ P^{[r]}_{F'} \end{array}$$

Assumption $F(L)\to F^{r+}(L)$ bij

$\Rightarrow$ étale on a nbd of image of $\bar T_{S'} \to \bar Q^{(r)}_{S'}$.

$+\ r>1 \Rightarrow$ étale on gen fiber

$\Rightarrow$ étale

———— Zariski-Nagata.

$\dim Q$ minimal $\Rightarrow N_{T/Q}\otimes_Q E \xleftarrow{\sim} \mathrm{Tor}_1^{\mathcal O_L}(\Omega^1_{T/S}, E)$

$$Q^{[r]}_{\bar F,\mathrm{red}} = \Theta^{(r)}_{L/K,\bar F} \qquad \boxed{\text{Functorial construction indep of } Q}$$

$\uparrow$ finite étale morphism

$$Q^{(r)}_{\bar F} = \Phi_{L/K,\bar F} \quad / S_{F'\otimes_F E}$$

with $\mathrm{Gal}(K'/K)\times \mathrm{Gal}(L/K)$-action arithmetic $\quad$ geometric

$$\begin{array}{ccc} & G' & G \\ & \| & \| \\ \Phi_{L/K,F'} \longrightarrow \Theta_{L/K,F'} & & Gal(K'/K) \times Gal(L/K) \\ \cup & \| & \\ Mor_K(L, K') & \mathrm{Hom}_F(m_{K'}^{r'}/m_{K'}^{r'+1}, \mathrm{Tor}_1^{O_L}(\Omega_{O_L/O_K}, E))^\vee & \end{array}$$

Lemma. Actions of $G'^{r+} \times G'$ on Θ is trivial.

$$G'^{r+} \curvearrowright m_{K'}^{r'}/m_{K'}^{r'+1} \qquad G' \curvearrowright \mathrm{Tor}_1^{O_L}(\Omega_{O_L/O_K}, E)$$

later

Fix $L \to K'$

conn cpt

$$\begin{array}{ccc} \Phi^0_{L/K, F'} & \longrightarrow & \Theta_{L/K,F'} = \mathrm{Hom}(m_{K'}^{r'}/m_{K'}^{r'+1}, \mathrm{Tor}_1^{O_L}(\ , F'))^\vee \\ \cup & & \Downarrow \\ \mathrm{Fiber} & \longrightarrow & 0 \end{array}$$

G^r-torsor $\qquad G^r \xrightarrow{\sim} \mathrm{Aut}(\Phi^0/\Theta^0)$

$$\uparrow ?$$

$$G'^r$$

$\mathrm{Im}\, G^r$, $\mathrm{Im}\, G'^r$ commute to each other

<u>**Prop**</u> $\quad G^r$ is abelian. p-gp

$$[\Phi^0] \in H^1(\Theta^0, G^r)$$

$$\| \qquad \qquad \|$$

$$[\ ] \in \mathrm{Hom}(G^{r\vee}, H^1(\Theta^0, \mathbb{Q}_p/\mathbb{Z}_p))$$

Cotangent cx. $X \to S$ $\quad L_{X/S}$ complex of $\mathcal{O}_X$-mod

$H_q = 0$ unless $q \geq 0$

$H_0(L_{X/S}) = \Omega^1_{X/S}$

$L_{X/S} \simeq \Omega^1_{X/S}[0]$ if $X \to S$ smooth

$H_1(L_{X/S}) \simeq N_{X/S}$ if $X \to S$ imm

$L_{X/S} \simeq N_{X/S}[1]$ conormal if $X \to S$ reg. imm

$X \xrightarrow{f} Y \to S$

$Lf^* L_{Y/S} \to L_{X/S} \to L_{X/Y} \to$.

$T \to Q \to S$

$$\mathcal{O}_T \underset{\mathcal{O}_Q}{\otimes} \Omega^1_{Q/S}[0] \to L_{T/S} \to N_{T/Q}[1] \to$$

$\Rightarrow L_{T/S} \simeq \Omega^1_{T/S}[0]$

$E \to T \to S$

$$L_{T/S} \overset{L}{\underset{\mathcal{O}_T}{\otimes}} E \to L_{E/S} \to N_{E/T}[1] \to$$

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_Q}(\Omega^1_{T/S}, E) \to H_1(L_{E/S}) \to N_{E/T}$$

$$\to \Omega^1_{T/S} \otimes_{\mathcal{O}} E \to 0$$

Lemma G' acts triv. on $H_1(L_{E/S})$ & Tor_1

$E \to F \to S$ $\quad E \overset{\otimes}{_F} N_{F/S}[1] \to L_{E/S} \to L_{E/F} \to$

$E \otimes_E \Omega^1_F \to \Omega^1_E \to L_{E/F} \to$

$\Rightarrow 0 \to E \overset{\otimes}{_F} N_{F/S} \to H_1(L_{E/S}) \to E \otimes_E \Omega^1_F \to \Omega^1_E \to \Omega^1_{E/F} \to 0$

Functoriality

$$\begin{array}{ccc} L & - & L_1 \\ | & & | \\ K & - & K_1 \\ G & \leftarrow & G_1 \\ \cup & & \cup \\ G^r & \leftarrow & G_1^r \end{array}$$

$e_{K_1/K}=1$ for simplicity

$$\begin{array}{l} [\Phi] \in \mathrm{Hom}(G^{r\vee}, H^1(\Theta^0, \mathbb{Q}_p/\mathbb{Z}_p)) \\ \quad \downarrow \qquad\qquad\qquad\qquad \downarrow \\ [\Phi_1] \in \mathrm{Hom}(G_1^{r\vee}, H^1(\Theta_1^0, \mathbb{Q}_p/\mathbb{Z}_p)) \end{array}$$

If. $H_1(L_{E/S}) \to H_1(L_{E_1/S_1})$ inj

$\Rightarrow G_1^n \to G^r$ surj & the vertical arrow is inj

Further if res field F_1 is perf

$\Rightarrow$ monogenic case $p\cdot[\Phi_1]=0$

Lemma $\exists K_1$ s.t. F_1 perf

& $H_1 \to H_1$ inj.

Theorem 2 $\quad p\cdot[\Phi]=0$. i.e.

G^n is an $\mathbb{F}_p$- v. sp.

3 Characteristic form

$$[\bar\Phi] : G^{r\vee} \longrightarrow H^1(\Theta^0_{\bar F}, \mathbb{F}_p) \quad \text{inj}$$

$$\cup$$

$$\mathrm{Ext}(\Theta^0_{\bar F}, \mathbb{F}_p)$$

$$\|$$

$$\Theta^{0\vee} = \mathrm{Hom}_{\bar F}(m_K^r/m_K^{r+}, \mathrm{Tor}_1^{\mathbb{Q}}(\Omega_{T/S}, \bar F))$$

$k = \bar F$

V k-vect. space of finite dim.

$V^\vee = \mathrm{Spec}\, S^\bullet V$ regarded as a sm. gp sch/k

$$\begin{tikzcd} H^1(V^\vee, \mathbb{F}_p) \arrow[r, phantom, "="] & S^\bullet V/(F-1) \\ \mathrm{Ext}(V^\vee, \mathbb{F}_p) \arrow[u] \arrow[r, phantom, "="] & V \ni f \arrow[u] \end{tikzcd}$$

Artin-Schreier

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F}_p \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & V^\vee \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \mathbb{F}_p \arrow[r] & \mathbb{G}_a \arrow[r, "F-1"] & \mathbb{G}_a \arrow[r] & 0 \end{tikzcd}$$

Theorem 3 Image of $\bar\Phi \subset \mathrm{Ext}$

i.e. $\bar\Phi$ defines an inj

$$\mathrm{char} : \mathrm{Hom}_{\mathbb{F}_p}(G^r, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\bar F}(m_K^r/m_K^{r+}, \mathrm{Tor}_1\ )$$

Need to show.

$\bar{\Phi}^0$ has a group sch str

s.t. the étale morph $\bar{\Phi}^0 \to \Theta^0$ is a morphism of gp schemes

Sketch of pf. in the case.

$\mathrm{ch}\, K = p > 0$, $r > 1$ is an integer.

$K = \mathrm{Frac}\, \mathcal{O}^h_{X,\xi}$ X/k smooth.

Where the group str comes from?

$X \times X$ has a canonical groupoid structure

Groupoid over k

$M \underset{t}{\overset{s}{\rightrightarrows}} X$, $m : M \times_{s \times t} M \to M$, $e : X \to M$, $i : M \to M$

$$\begin{tikzcd} M \times_X M \arrow[r, "m"] \arrow[d, "{t,\mathrm{com},s}"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "t \times s"] \\ X \times X \times X \arrow[r, "pr_{2\text{-}3}"] & X \times X \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "t \times s"] \\ X \arrow[r, "\delta"'] & X \times_k X \end{tikzcd} \quad \begin{tikzcd} M \arrow[r, "i"] \arrow[d, "t \times s"'] & M \arrow[d, "t \times s"] \\ X \times_k X \arrow[r, "\omega"] & X \times_k X \end{tikzcd}$$

$$\begin{array}{ccc}
\begin{tikzcd} X\times_S X\times_S X \arrow[r] \arrow[d] & X\times_S X \arrow[d] \\ X\times_S X \arrow[r] & X \end{tikzcd}
&
\begin{tikzcd} X \arrow[r, "1\times e_s"] \arrow[d, "e_t\times 1"'] \arrow[dr, "id"] & X\times_S X \arrow[d, "m"] \\ X\times_S X \arrow[r, "m"] & X \end{tikzcd}
&
\begin{tikzcd} X \arrow[r, "1\times i"] \arrow[d, "i\times 1"'] \arrow[dr, "e_t"] \arrow[dr, "e_s"'] & X\times_S X \arrow[d, "m"] \\ X\times_S X \arrow[r, "m"] & X \end{tikzcd}
\\
\text{ass} & \text{unit} & \text{inv}
\end{array}$$

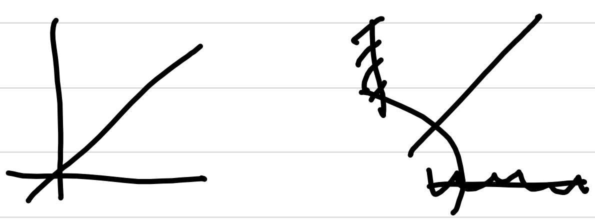
$s=t \Rightarrow$ group sch over X.

Example $M = X\times_S X$ $s,t = pr_2, pr_1$.

$m: X\times_S X\times_S X \to X\times_S X$ $i = w: X\times_S X \to X\times_S X$
$= pr_{13}$

$X \supset D$ $P^{(R)} \subset (X\times X)'$ blow-up at
$r \geqq 1$ integer $R = rD \subset X \subset X\times X$
diagonal

complement of prop transforms of
$D\times X$ & $X\times D$.



groupoid str on $X\times X$ induces one on $P^{(R)}$

$U = X - D$ $P^{(R)} \supset U\times U$
complement gp sch$/D$. $\mathbb{V}(\Omega^1_X(R)\otimes_{\mathcal{O}_X}\mathcal{O}_D)$

$(V\times V)/\Delta G$ $\qquad$ V G-tors $/U$, $\; V/G$

$$\begin{array}{ccc} (V\times V)/\Delta G \times_U (V\times V)/\Delta G & \longrightarrow & (V\times V)/\Delta G \\ \parallel & & \\ (V\times V\times V)/\underbrace{(\Delta G\times G)\cap(G\times\Delta G)}_{=\,\Delta G} & \longrightarrow & (V\times V)/\Delta G \end{array}$$

largest étale in the normalization

$$\begin{array}{ccc} W^{(R)} & \supset & (V\times V)/\Delta G \\ \downarrow & & \downarrow \text{ finite étale morphism of gpd.} \\ P^{(R)} & \supset & U\times U \end{array}$$

<u>Prop</u> TFAE

(1) $\delta^*: U = V/G \longrightarrow V\times V/\Delta G$ is lifted to $X \longrightarrow W^{(R)}$

(2) $W^{(R)} \rightrightarrows X$ is a gpd & $W^{(R)} \to P^{(R)}$ is a morphism of gpd.

<u>Thm 3</u> If $G^{int} = 1$, after shrinking X, the equiv. cond are satisfied & the structure in (2) induce a gp str on Φ^0 & a morphism $\Phi^0 \to \Theta^0$ of gp schemes.

general case

- r integer appropriate base ext.

- mixed char. imitate the geom.-constr.