Characteristic cycle and restriction to curves

k perfect $\quad X$ smooth $/k$

Λ finite field $\quad \ell = \operatorname{char}\Lambda \neq p = \operatorname{char} k \geqq 0$

$K(X,\Lambda)$ Grothendieck group of constructible sheaves of Λ-mod. on X.

Char 0 $\quad K(X,\Lambda) \longrightarrow L(T^*X)$ free group of Lagrangian cycles

closed, conical, $\omega|_C = 0$.

$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d, "\text{rank}"'] & L(T^*X) \arrow[dd] \\ \mathrm{Const}(X,\mathbb{Z}) \arrow[ur, "\text{unique isom.}"] \arrow[dr] & \\ & CH_0(X) \end{tikzcd}$$

$[\mathcal{F}] \longmapsto [CC\mathcal{F}]$

$\mathbb{Z}$-valued constructible f'ns

$$\begin{array}{ccc} A & \leftarrow & \bar{A} \\ \cap & & \cap \\ T^*X & \hookrightarrow & \mathbb{P}(T^*X \oplus \mathbb{A}^1_X) \end{array}$$

composition MacPherson's Chern class

proper push forward. $\quad f : X \to Y$ proper morphism

$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d, "f_*"] & \mathrm{Const}(X,\mathbb{Z}) \arrow[r] \arrow[d, "f_*"] & L(T^*X) \arrow[r] \arrow[d, "f_*"] & CH_*(X) \arrow[d, "f_*"] \\ K(Y,\Lambda) \arrow[r] & \mathrm{Const}(Y,\mathbb{Z}) \arrow[r] & L(T^*Y) \arrow[r] & CH_*(Y) \end{tikzcd}$$

$f_*\chi_Z(y) = \chi(Z \cap f^{-1}(y)) \qquad T^*X \xleftarrow{p} X \times_Y T^*Y \xrightarrow{g} T^*Y$

$\qquad\qquad A \qquad\qquad p^!A \qquad\qquad g_*p^!A$

Char p $\quad K(X,\Lambda) \longrightarrow ZC_n(T^*X)$

closed conical dim $n = \dim X$

No CD $\quad K(X,\Lambda) \longrightarrow ZC_n(T^*X)$

$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d] & ZC_n(T^*X) \\ \mathrm{Const}(X,\mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

$\dim X = 1$

$$CC\mathcal{F} = -\Big(\operatorname{rk}\mathcal{F}|_U \cdot [T^*_X X] + \sum_{x \notin U} a_x(\mathcal{F})[T^*_x X]\Big)$$

Artin conductor

$a_x\mathcal{F} = \operatorname{rk}\mathcal{F}_{\bar\eta} - \operatorname{rk}\mathcal{F}_{\bar x} + \mathrm{Sw}_x\mathcal{F} \leftarrow$ wild ramification

$C_1(X)$ suggested by Beilinson
group of $\mathbb{Z}$-valued f'ns on the set of $(v \in \tilde{C} \to C \hookrightarrow X)$

$C \hookrightarrow X$ closed integral curve
$\tilde{C} \to C$ normalization v. closed pt

$$\begin{array}{ccc} K(X,\Lambda) & \to & \mathrm{Const}(X) \oplus C_1(X) \\ \mathcal{F} & \mapsto & (\mathrm{rank}\,\mathcal{F},\ ((v,C) \mapsto \mathrm{Sw}_v \mathcal{F}|_{\tilde{C}})) \end{array}$$

$\searrow \ \ \cup$
$\tilde{C}(X)$ image

Theorem. 1. There exists a unique morphism $\tilde{C}(X) \to ZC_n(T^*X)$
s.t the diagram
$$\begin{tikzcd} K(X,\Lambda) \arrow[rr, "CC"] \arrow[dr] & & ZC_n(T^*X) \\ & \tilde{C}(X) \arrow[ur] & \end{tikzcd}$$
is commutative.

Variant for $\Lambda \neq \Lambda'$. $(\ell \neq \ell')$
Preceeding results S. Yatagawa. H. Kato.
Larger qts of $K(X,\Lambda)$

Proper pushforward $f: X \to Y$ proper $n = \dim X$, $m = \dim Y$

$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d, "f_*"] & \tilde{C}(X) \arrow[r] \arrow[d] & ZC_n(T^*X) \arrow[r] & CH_\cdot(X) \\ K(Y,\Lambda) \arrow[r] & \tilde{C}(Y) \arrow[r] & ZC_m(T^*X) \arrow[r] & CH_\cdot(Y) \end{tikzcd}$$

Theorem 2. There exists a unique morphism $f_*: \tilde{C}(X) \to \tilde{C}(Y)$
s.t the diagram is commutative

Preceeding results. Vidal, Yatagawa, H. Kato.
Larger qts of $K(X,\Lambda)$

No CD
$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d, "f_*"] & ZC_n(T^*X) \arrow[d] \\ K(Y,\Lambda) \arrow[r] & ZC_n(T^*Y) \end{tikzcd}$$

Example $X = Y = \mathbb{A}^2$ $\quad Y = \mathrm{Sp}\, k[x,y] \supset D$ $\ x=0$

$U = Y - D$.

$f: X \to Y$ Artin-Schreier covering $\quad t^p - t = \dfrac{y}{x^{pn}}$ $\quad w = x^n t$

$w^p - x^{(p-1)n} w = y \qquad X = \mathrm{Sp}\, k[x,w] \supset D'$ $\ x=0$

h, m integers satisfying $\ ph > m > ph-(p-1)n$, $p n$, $(h>n)$

$p \nmid m$. $\qquad (>h)$

$\chi \quad t^p - t = \dfrac{y}{x^{ph}}$, $\quad \varphi \quad t^p - t = \dfrac{1}{x^m}$. $\quad$ char on V.

$CC\chi = CC\chi\circ\varphi = [T^*_Y Y] + ph\langle dy\rangle_D$.

$\langle dx\rangle_D$

$\neq CC\varphi = [T^*_Y Y] + m\,[T^*_D Y]$

$f^*\chi \quad t^p - t = \dfrac{w^p - x^{(p-1)n} w}{x^{ph}} \sim \dfrac{w}{x^h} - \dfrac{w}{x^{ph-(p-1)n}} \qquad m > h$

$ph-(p-1)n$

$CC(f^*\chi\circ\varphi) = CC(f^*\varphi) = [T^*_X X] + m[T^*_{D'} X]$.

$CC(f_* f^*\chi\circ\varphi) = p\,CC(\chi\circ\varphi) \neq p\,CC\varphi = CC(f_* f^*\varphi)$.

No CD
$$\begin{array}{ccc} K(X.\Lambda) & \longrightarrow & CH_\cdot(X) \\ f_* \downarrow & & \downarrow \\ K(Y.\Lambda) & \longrightarrow & CH_\cdot(Y) \end{array}$$

Counterexample Frobenius.
However
Conjecture
$$\begin{array}{ccc} K(X.\Lambda) & \longrightarrow & CH_0(X) \\ f_* \downarrow & \circlearrowleft & \downarrow f_* \\ K(Y.\Lambda) & \longrightarrow & CH_0(Y) \end{array}$$

If X, Y proj / k finite Umezaki-Yang-Zhao.

Outline of Proof of Theorems 1 & 2.

- Crucial case $\dim X = 2$. $\dim Y = 1$
- Induction on dim.

Thm 2 By proper b.c. $\dim Y = 1$.
devissage red. to $\dim X = 2$ $\dim Y = 1$

Thm 1. CC is characterized by the Milnor formula
for the total dimension of vanishing cycles.
~~To replace~~ p.b.c. use nearby cycles over gen'l basis.

$$\begin{array}{ccc} Z \subset X & \xrightarrow{g} & \mathbb{A}^1_Y \\ & {\scriptstyle f}\searrow & \downarrow p \\ & & Y \end{array} \qquad R\Psi_f = Rp_* \circ R\Psi_g$$

g is transversal ($\Rightarrow$ locally acyclic)
away from Z quasi-finite over Y
$\Downarrow$
$R\Psi_g$ commute with b.c.

X normal $U \subset X$ dense open $j: U \hookrightarrow X$ open imm
$\mathcal{F}, \mathcal{F}'$ loc. const const'ble on U

$\mathcal{F}$ & $\mathcal{F}'$ have the same wild ram'n (cf. Deligne-Illusie, Gabber i.e. Vidal)

if $\exists$ prop $X' \to X$ isom on U. X' normal. s.t.
$\forall x' \in X' - U$ $\forall \sigma \in I_{x'}$ inertia of order power of p
we have
$$\dim \mathcal{F}_{\bar{\eta}_x}^{\langle\sigma\rangle} = \dim \mathcal{F}'^{\langle\sigma\rangle}_{\bar{\eta}_x}$$

$\mathcal{F}$ & $\mathcal{F}'$ have the same rk at a closed pt x.

if $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{F}'$ & $\forall$ closed integral curve $C \hookrightarrow X$ $\forall v \in \tilde{C} \to C$ normalization above x, we have
$$\mathrm{Sw}_v\, \mathcal{F}|_{\tilde{C}} = \mathrm{Sw}_v\, \mathcal{F}'|_{C}.$$

Key Proposition Assume $\dim X = 2$

Same rk + cond at x $\Rightarrow$ Same wild ram'n on a nbd of x.

Key Prop + Preceding results $\Rightarrow$ $\dim X = 2$. $\dim Y = 1$ case of Thm 1 & 2.

Proof of Key Prop.

- First prove the case $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{F}' = 1$, char of order p
  Kato's cleaning theorem

- This case implies
  - univ. Same rk + cond at x $\Rightarrow$ Same wild ram on a nbd of x
    replace $C \hookrightarrow X$ by $C \to X$
    $\Uparrow$ not necessarily imm
  - Same rk + cond at x
    deformation of curve. Stability of the Swan conductor
    use Eℓkik.