Euler-Poincaré characteristic of an ℓ-adic sheaf (with Kazuya Kato)

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Notation:
F algebraically closed field of characteritic $p > 0$.
U smooth varity over F, d $\dim U$.
$\mathcal{F}$ smooth ℓ-adic sheaf (or $\mathbb{F}_\ell$-sheaf) on U, $\ell \neq p$
$\chi_c(U, \mathcal{F}) = \sum_{q=0}^{2d} \dim H_c^q(U, \mathcal{F})$
Aim: Define a 0-cycle class $\mathrm{Sw}\mathcal{F} \in CH_0(X - U) \otimes \mathbb{Q}$ and prove

$$\chi_c(U, \mathcal{F}) = \chi_c(U)\mathrm{rank}\ \mathcal{F} - \deg \mathrm{Sw}\mathcal{F}$$

where X is a compactification of U.

Note: There is another way to compute $\chi_c(U, \mathcal{F})$ using the zeta function. This approach is more geometric and relies on the ramification along the boundary, while the zeta function has nothing to do with the boundary.

Plan:
1. Classical case.
2. Definition.
3. Results.
4. Lefschetz trace formula for open varieties.

1.
U: a smooth curve
$X \supset U$: smooth compactification.
$CH_0(X - U) = \bigoplus_{x \in X-U} \mathbb{Z}$.

For simplicity, assume $\mathcal{F}$ is trivialized by a finite Galois covering $V \to U$. In general, we consider $\mathcal{F}$ mod ℓ and use the Brauer trace.

Y: the normalization of X in V.

$$\begin{array}{ccc} Y & \overset{\supset}{\longleftarrow} & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \overset{\supset}{\longleftarrow} & U \end{array}$$

$\sigma \in G = \mathrm{Gal}(U/V), \sigma \neq 1$.

Figure 1.
Γ_σ : graph of σ.
$(Y \times Y)' \to Y \times Y$: Blow up at (y, y) for each $y \in Y \setminus V$.
Figure 2.
$\overline{\Gamma_\sigma}$: closure of $\Gamma_\sigma \subset V \times_U V$ in $(Y \times Y)'$.
tame ramification : no intersection.
wild ramification : non-empty intersection.
Define

$$s_{V/U}(\sigma) = -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(Y - V) = \bigoplus_{y \in Y - V} \mathbb{Z},$$

$s_{V/U}(1) = -\sum_{\sigma \neq 1} s_{V/U}(\sigma)$ and

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_{V/U}(\sigma) \mathrm{Tr}(\sigma : M) \in CH_0(X - U) \otimes \mathbb{Q}. \tag{1}$$

M : representation of G corresponding to $\mathcal{F}$.

Theorem 1 (Hasse-Arf)

$$\mathrm{Sw}(\mathcal{F}) \in CH_0(X - U). \tag{2}$$

Theorem 2 (Grothendieck-Ogg-Shafarevich)

$$\chi_c(U, \mathcal{F}) = \chi_c(U) \cdot \mathrm{rank}\ \mathcal{F} - \deg\ \mathrm{Sw}(\mathcal{F}).$$

2.
We do not know if we have a smooth compactification. However, we may use an alteration.

$$\begin{array}{ccc} Z & \overset{\supset}{\longleftarrow} & W \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \overset{\supset}{\longleftarrow} & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \overset{\supset}{\longleftarrow} & U \end{array}$$

X, Y: compactifications.
Z proper smooth, W complement of a divisor with simple normal crossings.
$g : Z \to Y$: alteration=proper, surjective and generically finite.

$(Z \times Z)'$: Blow-up of $Z \times Z$ at $D_1 \times D_1, \dots, D_m \times D_m$ where $D_1, \dots, D_m$ are the irreducible components of D.
$\Delta_Z : Z \to (Z \times Z)'$: the log diagonal map.
$(Z \times Z)'$ is smooth and the immersion $Z \to (Z \times Z)'$ a regular immersion of codimension d.

Proposition 3 *Let $\sigma \in G, \neq 1$. Let $\Gamma' \in Z_d(\overline{W \times_U W \setminus W \times_V W})$ be an arbitrary extension of $[\Gamma'|_{W\times_U W \setminus W \times_V W}] = (g \times g)^! \Gamma_\sigma$. Then,*

$$\frac{1}{[W:V]} \bar{g}_*(\Gamma', \Delta_Z)_{(Z\times Z)'} =: (\Gamma_\sigma, \Delta_V)^{\log}$$

$\in CH_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ depends only on $U \leftarrow V \subset Y$ and σ.

Definition 4 *For $\sigma \in G, \neq 1$, put*

$$s_{V/U}(\sigma) = -(\Gamma_\sigma, \Delta_V)^{\log} \tag{3}$$

$s_{V/U}(1) = -\sum_{\sigma\in G,\neq 1} s_{V/U}(\sigma)$.

Define

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_{V/U}(\sigma) \mathrm{Tr}(\sigma : M) \in CH_0(X \setminus U) \otimes \mathbb{Q}_\ell.$$

In fact, $\mathrm{Sw}(\mathcal{F})$ is defined as an element of $CH_0(E)_{\mathbb{Q}}$ where $E \subset X - U$ is the wild ramification locus.

Variant: We may also define $\mathrm{Sw}(\mathcal{F})$ in a mixed characteristic situation.

3. Results.

Integrality.

Theorem 5 *Assume $d = \dim U \leq 2$.*

1. $(\Gamma_\sigma, \Delta_V)^{\log}$ is defined in $CH_0(Y - V)$.

2. $\mathrm{Sw}\mathcal{F} = 1$ is defined in $CH_0(X - U)$.

Proof. 1. Resolution.

2. By Brauer's induction, it is reduce to the case rank$\mathcal{F} = 1$. May assume X is smooth and U is the complement of a divisor $D = \sum_i D_i$ with simple normal crossings. Then, one can define a divisor $D_{\mathcal{F}} = \sum_i \mathrm{sw}_i(\mathcal{F}) D_i$. Further after blowing-up, we prove

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = (-1)^{d-1} \{c(\Omega_{X/F}(\log D)) \cap (1 - D_{\mathcal{F}})^{-1} \cap [D_{\mathcal{F}}]\}_{\dim 0}.$$

■

Theorem 6

$$\chi_c(U, \mathcal{F}) = \chi_c(U) \cdot \mathrm{rank}\ \mathcal{F} - \deg\ \mathrm{Sw}(\mathcal{F}).$$

Proof. Suffices to show the trace formula

$$\deg\ s_{V/U}(\sigma) = \begin{cases} -\mathrm{Tr}(\sigma^* : H_c^*(V, \mathbb{Q}_\ell)) & \text{if } \sigma \neq 1 \\ \chi_c(U)[V:U] - \chi_c(V) & \text{if } \sigma = 1. \end{cases}$$

for an open variety. $\mathrm{Tr}(\sigma^* : H^*_c(V,\mathbb{Q}_\ell)) = \sum_{q=0}^{2d}(-1)^q\mathrm{Tr}(\sigma^* : H^q_c(V,\mathbb{Q}_\ell))$.

$$\deg\ (\Gamma', \Delta_Z)_{(Z\times Z)'} = \mathrm{Tr}(\Gamma' : H^*_c(W,\mathbb{Q}_\ell)).$$

4.

Changing the notations.

X: proper smooth scheme over F of dimension d.

$U \subset X$: complement of a closed subscheme $D \subset X$.

$\Gamma \subset U \times U$: a closed subscheme of dimension d.

$p_i : \Gamma \to U$: the composition with the projections $pr_i : U \times U \to U$.

ℓ: a prime number different from the characterstic of F.

$\Gamma^* = pr_{1*} \circ pr_2^*$ on $H^q_c(U,\mathbb{Q}_\ell)$ is defined if $p_2 : \Gamma \to U$ is proper.

Write $\mathrm{Tr}(\Gamma^* : H^*_c(U,\mathbb{Q}_\ell)) = \sum_{q=0}^{2d}(-1)^q\mathrm{Tr}(\Gamma^* : H^q_c(U,\mathbb{Q}_\ell))$.

Lemma 7 *p_2 is proper if and only if*

$$\tilde\Gamma \cap (D \times X) \subset \tilde\Gamma \cap (X \times D). \tag{4}$$

To have a nice formula, we need more assumption. Assume $D = D_1 \cup \cdots \cup D_m$ is a divisor with simple normal crossings and define

$p : (X \times X)' \to X \times X$: the blow-up at $D_1 \times D_1, \ldots , D_m \times D_m$

$\Delta'_X = X \to (X \times X)'$: the log diagonal.

Theorem 8 *Let $\tilde\Gamma'$ be the closure of Γ in $(X \times X)'$ and assume*

$$\tilde\Gamma' \cap (D \times X)' \subset \tilde\Gamma' \cap (X \times D)' \tag{5}$$

where $(D \times X)'$ and $(X \times D)'$ are the proper transforms of $D \times X$ and $X \times D$. Then, $p_2 : \Gamma \to U$ is proper and we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(U,\mathbb{Q}_\ell)) = \deg(\tilde\Gamma', \Delta'_X)_{(X\times X)'}.$$

The assumption is satisfied in our case: $\overline{W\times_U W} \cap (D\times Y)' = \overline{W\times_U W} \cap (Y \times D)'$.

Can not replace (5) $\tilde\Gamma' \cap D^{(1)\prime} \subset \tilde\Gamma' \cap D^{(2)\prime}$ by (4) $\tilde\Gamma \cap D^{(1)} \subset \tilde\Gamma \cap D^{(2)}$.

Example 1. $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $F : U \to U$ Frobenius. $\Gamma = \Gamma_F$

Then, $\mathrm{Tr}(F^* : H^*_c(U,\mathbb{Q}_\ell)) = p$ and $(\Gamma,\Delta)_{(X\times X)'} = p$. On the other hand, $\mathrm{Tr}(F_* : H^*_c(U,\mathbb{Q}_\ell)) = 1$ and $(\Gamma^t,\Delta)_{(X\times X)'} = p$.

Example 2. $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $\sigma : U \to U$ defined by $x \mapsto x + 1$. $\Gamma = \Gamma_\sigma$.

Then, $\mathrm{Tr}(\sigma^* : H^*_c(U,\mathbb{Q}_\ell)) = 1$ and $(\Gamma,\Delta)_{(X\times X)'} = 1$. There is an intersection at ∞. $t = \frac{1}{x} \mapsto \frac{1}{\frac{1}{t}+1} = \frac{t}{1+t}$. $\frac{s}{1+s} - t = t \cdot \frac{u-(1+ut)}{1+ut}$.

Proof of Theorem 8.

Classical case ($X = U$ is proper).

$$\begin{array}{ccccc}
\Gamma^* \in & \bigoplus \mathrm{End} H^q(X,\mathbb{Q}) & \xrightarrow{\simeq} & H^{2d}(X\times X,\mathbb{Q}_\ell(d)) & \ni [\Gamma] \\
 & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \Delta^*} & \\
\mathrm{Tr}\ \Gamma^* \in & \mathbb{Q}_\ell & \xleftarrow{\mathrm{Tr}} & H^{2d}(X,\mathbb{Q}_\ell(d)) & \ni \Delta^*[\Gamma]
\end{array}$$

The isomorphism in the upper line is given by the Poincaré duality and Künneth formula.

$\Delta^*[\Gamma] = [(\Delta, \Gamma)]$: compatibility of the cup-product with the intersection product.

deg = Tr.

Our case.

$$\begin{array}{ccccc} \Gamma^* \in & \bigoplus \mathrm{End} H^q_c(U, \mathbb{Q}) & \xrightarrow{\simeq} & H^{2d}_{!*}(U \times U, \mathbb{Q}_\ell(d)) & \ni [\Gamma] \\ & \Big\downarrow{\scriptstyle \mathrm{Tr}} & & \Big\downarrow{\scriptstyle \Delta^*} & \\ \mathrm{Tr}\ \Gamma^* \in & \mathbb{Q}_\ell & \xleftarrow{\mathrm{Tr}} & H^{2d}_c(U, \mathbb{Q}_\ell(d)) & \ni \Delta^*[\Gamma] \end{array}$$

where $H^{2d}_{!*}(U \times U, \mathbb{Q}_\ell(d)) = H^{2d}(X \times X, j_{1!}Rj_{2*}\mathbb{Q}_\ell(d))$, $j_2 : U \times U \to U \times X, j_1 : U \times X \to X \times X$.

Need to relate $\Delta^*[\Gamma]$ with $[(\Delta, \tilde{\Gamma})_{(X\times X)'}]$. We have a commutative diagram

$$\begin{array}{ccccc} [\Gamma] \in & H^{2d}_{!*}(U \times U, \mathbb{Q}_\ell(d)) & \longrightarrow & H^{2d}_{!\emptyset*}(U \times U, \mathbb{Q}_\ell(d)) & \ni [\tilde{\Gamma}] \\ & \Big\downarrow{\scriptstyle \Delta^*} & & \Big\downarrow{\scriptstyle \Delta^*} & \\ \Delta^*[\Gamma] \in & H^{2d}_c(U, \mathbb{Q}_\ell(d)) & \longrightarrow & H^{2d}(X, \mathbb{Q}_\ell(d)) & \ni \Delta^*[\tilde{\Gamma}] \end{array}$$

where $H^{2d}_{!\emptyset*}(U \times U, \mathbb{Q}_\ell(d)) = H^{2d}((X \times X)', j_{1!}Rj_{2*}\mathbb{Q}_\ell(d))$, $k_2 : (X \times X)' - (D \times X)' \cup (X \times D)') \to (X \times X)' - (D \times X)', k_1 : (X \times X)' - (D \times X)' \to (X \times X)'$. The assumption implies that $[\tilde{\Gamma}] \in H^{2d}_{!\emptyset*}(U \times U, \mathbb{Q}_\ell(d))$ is defined. The key point is that the upper horizontal map sends $[\Gamma] \to [\tilde{\Gamma}]$. This follows from the fact that the map is an isomorphism observed by Faltings and Pink. ■

5.

Conjecture 9 *Let A be a regular local ring with perfect residue field and G be a finite group of automorphisms of A. Assume that, for $\sigma \in G, \neq 1$, $A/(\sigma(a) - a : a \in A)$ is of finite length. Then the function $a_G : G \to \mathbb{Z}$ defined by*

$$a_G(\sigma) = \begin{cases} -\text{length } A/(\sigma(a) - a : a \in A) & \text{if } \sigma \neq 1 \\ -\sum_{\tau \in G, \neq 1} a_G(\tau) & \text{if } \sigma = 1. \end{cases}$$

is a character of G.

Corollary 10 of Theorem 5.2 ([KSS]) *Conjecture 9 is true if A is the local ring at a closed point of a smooth surface over a perfect field.*