Bad reduction of varieties over a local field ([33], [32], [19], [17], [15])

The stable reduction theorem for curves, proved by Deligne-Mumford, asserts that a proper smooth curve over a local field has semi-stable reduction if and only if the ℓ-adic monodromy action is unipotent. A new cohomological proof without using the Jacobian is given in [32]. An equivalent condition for the ℓ-adic monodromy action to be tame is also given. Another proof of the latter using log geometry is given in [15].

The trace of an automorphism on the space of vanishing cycles of a normal curve over a dvr at an isolated singular point in the closed fiber is computed in [33]. The method of proof using the semi-stable reduction theorem of curves is later used by Kato to prove a generalization of the dimension formula by Deligne-Laumon for the space of vanishing cycles. A certain inequality on the conductor of a curve over a dvr is proved in [19].

The weight spectral sequence defined by Rapoport-Zink is a basic tool in the study of the monodromy of ℓ-adic cohomology. A new construction using perverse sheaves is given in [17]. As an application, some results on the independence of ℓ of traces are deduced from the Lefschetz trace formula. A p-adic counterpart is obtained by Tsuji.

A purity result proved in [15] asserts that a family of curves on the interior of a log regular scheme has a log smooth extension if the monodromy is tame.

Conductor formula ([30], [29], [28], [14], [13], [2])

The Tate-Ogg formula for elliptic curve comparing the discriminant with the conductor is reproved and is generalized to higher genus by using the Mumford isomorphism in [30], as an application of the conductor formula of Bloch. For curves of genus 2, the Mumford isomorphism is compared with the discriminant in [28].

The conductor formula is originally formulated by Bloch using the localized Chern class. It is interpreted in terms of coherent sheaves in [29]. This approach was developed as K-theoretic intersection product and eventually lead to a proof of large cases of the conductor formula in [13]. Another key ingredient in the proof is the logarithmic method. Some inaccuracies in [29] are corrected also in [13]. Shortly before the proof, the formula modulo 2 was proved in [14].

The conductor formula is generalized to allow coefficient sheaves in [2], by developing the method inaugurated in [13] and defining the Swan class of an ℓ-adic sheaf measuring the wild ramification. It is generalized to an ℓ-adic Riemann-Roch formula and gives an arithmetic analogue of [11]. The integrality of the Swan class is also proved for arithmetic surfaces by the reduction to the rank one case.

Euler number and the characteristic class of an ℓ-adic sheaf on a variety of positive characteristic ([31], [27], [12], [11], [9], [5], [1], [−1])

A conjecture of Serre on the Artin character of an isolated fixed point of an algebraic surface is deduced in [31] from a trace formula proved by Shuji Saito. A proof of the conjecture for arithmetic surface was annonced by Kato at ICM 90 and a proof is given in [2].

A ramification theoretic formula for the Euler number of a rank one sheaf on an algebraic surface was first proved by Kato. It is generalized to higher dimension in [27] and to higher rank in [9].

The Euler number of an ℓ-adic sheaf is refined as the characteristic class. Using the new method of blow-up at the ramification locus in the diagonal, the characteristic class is computed for a rank 1 sheaf in [12]. The proof relies on computations on Witt vectors in [10].

Using ideas from log geometry, the Swan class of an ℓ-adic sheaf is defined as an invariant measuring the wild ramification in [11]. The Grothendieck-Ogg-Shafarevich formula for the Euler number of a sheaf on a curve is generalized to higher dimension. The integrality of the Swan class is also proved for surfaces by the reduction to rank one case. An arithmetic version is given in [2]

After establishing a relation of the graded piece of the ramification group to differential forms, the characteristic variety of an ℓ-adic sheaf of higher rank is defined in [9] using the relation, under an assumption. The Euler number is computed as the intersection product with the 0-section in the logarithmic cotangent bundle. This advances the project sketched in [10]. It is refined and generalized in [5].

A non-logarithmic version of the results in [9], [5] is given in [−1]. In particular, the characteristic cycle is defined as a cycle on the cotangent bundle and the Euler number is computed as the intersection number with the 0-section, under a certain assumption. The non-logarithmic version has an advantage to behave better with the method of cutting by curves. For a sheaf on a surface, the characteristic cycle is defined unconditionally in [1] using the result in [−1] and the Radon transform and formulas for the Euler number and the total number of vanishing cycles are proved.

Filtration by ramification groups ([18], [16], [9], [6], [−1])

The upper numbering filtration of the absolute Galois group of a local field with not necessarily perfect residue field is defined and studied in [18] and [16]. It is first defined in [18] using rigid geometry. It is also proved in [18] that the jumps are rational numbers. In [16], the graded pieces are proved to be abelian and are related to certain tangent spaces. The graded pieces are shown to be killed by p and the character groups are related to differential forms; in the equal characteristic case in [9] for the logarithmic version and in [−1] for the non-logarithmic version; in the mixed characteristic case in [6] for the logarithmic version. This description is used in [9] and [−1] to define the characteristic cycle of an ℓ-adic sheaf.

Epsilon factors, the determinant and the Stiefel-Whitney class of cohomology ([26], [25], [24], [23], [22], [7], [4])

The constant term of the functional equation of the L-function of an ℓ-adic sheaf on a variety over a finite field gives a refinement of the Euler number. In the case where the ramification is tame, a formula is proved in [26] by taking a Lefschetz pencil and applying Laumon's product formula.

The alternating sum of the inertia action on the cohomology of a variety over a local field with 'tame' reduction is computed also in [26]. The epsilon factor of the cohomology is computed using the fact that the local epsilon factor of a tamely ramified representation of a local field is determined by the restriction to the inertia subgroup.

If we consider the Euler number as an invariant of degree 0, the determinant will be an invariant of degree 1. For an ℓ-adic sheaf tamely ramified along the boundary, the determinant of cohomology is computed in [25]. The ramification along the boundary contributes as the Jacobi sum. Since the constant term of the functional equation studied in [26] is the determinant of the Frobenius, this is a generalization of the result there.

For the cohomology with constant coefficients, the determinant is compared with the discriminant of the de Rham cohomology also in [25]. If the variety is a hypersurface, the determinant is computed using the discriminant of the defining polynomial in [3]. If we regard the determinant of the cohomology of the constant sheaf as an invariant of degree 1 modulo 2, an invariant of degree 2 will be defined as the second Stiefel-Whitney class. A formula for the second Stiefel-Whitney class for a finite extension of a field was proved by Serre. Some computation of the second Stiefel-Whitney class for a finite extension of a local field is done in [24] and improved in [7]. The degree 2 version of the formula comparing ℓ-adic cohomology with de Rham cohomology is studied in [4].

Deligne proved that the local epsilon factor of an orthogonal representation is essentially the second Stiefel-Whitney class and deduced that the sign of the functional equation of an Artin orthogonal L-function is positive. Using the Fontaine-Lafaille theory for the places dividing p, it is generalized to an orthogonal motive of even weight in [23].

For Hodge structures, an analogue of Galois action is given by period integrals. By transporting the method in [26], an analogous formula for periods is proved in [22].

The local Fourier transform of an ℓ-adic representation of the absolute Galois group of a local field of characteristic p is computed explicitly in [7], under a certain assumption. This gives a new geometric and local proof of the formula of Laumon for the local epsilon factors. A computation of the epsilon factors module roots of unity of p-power orders is given in [−2].

Galois representation associated to modular forms ([21], [20], [8])

For an ℓ-adic Galois representation associated to an elliptic modular form or to a Hilbert modular form, the local-global compatibility was proved by Deligne-Langlands-Carayol at the places prime to ℓ, under a certain condition in the case where the degree of the totally real field is even. Using the p-adic Hodge theory, the compatibility at a place dividing ℓ is similarly formulated. This compatibility is proved for elliptic modular forms in [21] and some inaccuracies on F-isocrystals are corrected in [20]. It is proved for Hilbert modular forms in [8].

The method relies on the comparison of p and ℓ using the weight spectral sequences. It uses a purely geometric construction of the representation and on the proof of the weight-monodromy conjecture proved using the fact that the base modular variety is a curve. The condition in the case of even degree has been removed by Skinner and by Tong Liu by different methods.

References

[1] Characteristic cycle and the Euler number of a constructible sheaf on a surface, Kodaira Centennial issue of the Journal of Mathematical Sciences, the University of Tokyo, vol 22, 2015 pp. 387-442.

[2] Ramification theory for varieties over a local field, (with Kazuya Kato), Publications Mathematiques, IHES. 117, Issue 1 (2013), 1-178.

[3] The determinant and the discriminant of a hypersurface of even dimension, Mathematical Research Letters. 19 (2012), no. 04, 855-871

[4] Stiefel-Whitney classes of ℓ-adic cohomology, Journal für die reine und angewandte Mathematik, Volume 2013, Issue 681, 101-147.

[5] Ramification and cleanliness (with Ahmed Abbes), Tohoku Mathematical Journal, Centennial Issue, 63 No. 4 (2011), 775-853.

[6] Ramification of local fields with imperfect residue fields III, Mathematische Annalen, 352, Issue 3 (2012), 567-580.

[7] Local Fourier transform and epsilon factors (with Ahmed Abbes) Compositio Mathematica, 146-6, (2010) 1507-1551.

[8] Hilbert modular forms and p-adic Hodge theory, Compositio Mathematica, 145-5, (2009) 1081-1113.

[9] Wild ramification and the characteristic cycle of an ℓ-adic sheaf, Journal de l'Institut de Mathematiques de Jussieu, (2009) 8(4), 769-829

[10] Analyse micro-locale ℓ-adique en caracteristique $p > 0$: Le cas d'un trait (with Ahmed Abbes) , Publication of the Research Institute for Mathematical Sciences 45-1 (2009) 25-74

[11] Ramification theory for varieties over a perfect field (with Kazuya Kato) , Annals of Mathematics, 168 (2008), 33-96.

[12] The characteristic class and ramification of an ℓ-adic etale sheaf (with Ahmed Abbes), Inventiones Mathematicae 168 No. 3 (2007) 567-612

[13] On the conductor formula of Bloch (with Kazuya KATO), Publications Mathematiques, IHES 100, (2004), 5-151.

[14] Parity in Bloch's conductor formula, Journal de Théorie des Nombres de Bordeaux, 16-2 (2004), 403-421.

[15] Log smooth extension of family of curves and semi-stable reduction, Journal of Algebraic Geometry, 13 (2004), 287-321

[16] Ramification of local fields with imperfect residue fields II (with Ahmed ABBES), Documenta Mathematica, Extra Volume Kato (2003), 5-72

[17] Weight spectral sequences and independence of ℓ, Journal de l'Institut de Mathematiques de Jussieu 2 (4), (2003), 583-634.

[18] Ramification of local fields with imperfect residue fields (with Ahmed ABBES), American Journal of Mathematics, 124.5 (2002), 879-920

[19] (with Qing Liu) Inequality for conductor and differentials of a curve over a local field, J. of Algebraic Geometry 9 (2000) 409-424

[20] Weight-monodromy conjecture for ℓ-adic representations associated to modular forms, A supplement to the paper [21],
in B.B.Gordon et al.(eds.), The arithmetic and geometry of algebraic cycles, 427-431.

[21] Modular forms and p-adic Hodge theory,
Inventiones Math. 129 (1997) 607-620

[22] (with T.Terasoma) Determinant of period integrals,
J. of American Math. Soc. 10 (1997) 865-937

[23] The sign of functional equation of the L-function of an orthogonal motive,
Inventiones Math. 120 (1995) 119-142

[24] Local constant of $\mathrm{Ind}_K^L 1$, Commentary Math. Helv. 70 (1995) 507-515

[25] Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies, J. of Algebraic Geometry 3 (1994) 411-434

[26] ϵ-factor of a tamely ramified sheaf on a variety, Inventiones Math. 113 (1993) 389-417

[27] The Euler numbers of ℓ-adic sheaves of rank 1 in positive characteristic,
ICM-90 Satellite conf. Proc. 1991 165-181

[28] The discriminants of curves of genus 2, Compositio Math. 69 (1989) 229-240

[29] Self-intersection 0-cycles and coherent sheaves on arithmetic schemes,
Duke Math. J. 57 (1988) 555-578

[30] Conductor, discriminant, and the Noether formula of arithmetic surfaces,
Duke Math. J. 57 (1988) 151-173

[31] (with K.Kato and S.Saito) Artin characters for algebraic surfaces,
American J. of Math. 110 (1988) 49-75

[32] Vanishing cycles and geometry of curves over a discrete valuation ring,
American J. of Math. 109 (1987) 1043-1085

[33] Trace formula for vanishing cycles of curves, Math. Ann. 276 (1987) 311-315

Preprints

[−1] Wild ramification and the cotangent bundle

[−2] Ramification groups and local constants