Galois representations in arithmetic geometry

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When he formulated an analogue of the Riemann hypothesis for congruence zeta functions of varieties over finite fields, Weil predicted that a reasonable cohomology theory should lead us to a proof of the Weil conjecture. The dream was realized when Grothendieck defined etale cohomology. Since then, ℓ-adic etale cohomology has been a fundamental object in arithmetic geometry. It enables us to investigate the arithmetic of algebraic varieties and also to construct representations of the absolute Galois group of a field of arithmetic interest.

With etale cohomology, geometric problems can be studied using linear algebra as in the diagram

$$\begin{array}{ccc} \text{Algebraic Geometry} & & \text{Linear Algebra} \\ (\text{Varieties over a field } K) & \longrightarrow & \begin{pmatrix} \ell\text{-adic representations of the absolute} \\ \text{Galois group } G_K = \mathrm{Gal}(\bar K/K) \text{ of } K \end{pmatrix} \\ X & \mapsto & H^q(X_{\bar K}, \mathbb{Q}_\ell). \end{array}$$

For example, the number of rational points of a variety X over a finite field K is expressed as the alternating sum of traces:

$$\# X(K) = \sum_{q=0}^{2\dim X} \mathrm{Tr}(Fr_K : H^q(X_{\bar K}, \mathbb{Q}_\ell).$$

The relative version of the diagram

$$\begin{array}{ccc} (\text{Fiber spaces over } S) & \longrightarrow & (\ell\text{-adic sheaves on } S) \\ f : X \to S & \mapsto & R^q f_* \mathbb{Q}_\ell \end{array}$$

is important as well. The first diagram is in fact the special case $S = \mathrm{Spec}\, K$. A variety over a field K may be regarded as a fiber space over a $K(\pi, 1)$-space where $\pi = G_K$. Conversely, if K has a large Galois group G_K e.g. a number field, the right hand side (ℓ-adic representations of G_K) itself is of great interest. In this case, etale cohomology enables us to construct and study them in a geometric way.

Let X be a variety over a field K, ℓ be a prime number different from the characteristic of K and $\mathcal{F}$ be a smooth ℓ-adic sheaf on X. The fundamental object in this article is etale cohomology with compact support

$$H^q_c(X_{\bar K}, \mathcal{F})$$

[D1]. It is a $\mathbb{Q}_\ell$-vector space of finite dimension with a natural continuous action of the absolute Galois group $G_K = \mathrm{Gal}(\bar K/K)$ of K. It is 0 unless $0 \le q \le 2\dim X$. If K is a subfield of the complex number field $\mathbb{C}$, its underlying vector space is identified as follows. Let X^{an} be the complex manifold defined by X and let $\mathcal{F}_{\mathbb{Q}}$ be a linear system of $\mathbb{Q}$-vector spaces on X^{an} such that $\mathcal{F}|_{X^{\mathrm{an}}} = \mathcal{F}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then we may identify

$$H^q_c(X_{\bar K}, \mathcal{F}) = H^q_c(X^{\mathrm{an}}, \mathcal{F}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

The cohomology in the right hand side is the singular cohomology with the coefficient sheaf $\mathcal{F}_{\mathbb{Q}}$ on the topological space X^{an} with the complex topology.

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The contents of this survey article is summarized as follows. In the first section, we study Galois representations of dimension 1 defined as the determinant of etale cohomology. Here, theorems are stated for an arbitrary base field. In the second section, we study 2-dimensional representations where the base field is a number field. Contrary to the 1-dimensional case, a description of 2-dimensional representations depends on the base field. In this case, it is important to know the relations with the corresponding modular form. In section 3, we consider a generalization of a result in section 1 to higher dimension in view of characteristic classes of an orthogonal representation. The characteristic class of degree 2 studied there is closely related to the functional equation of L-function. In section 4, we consider the case where the base field is a local field. A remarkable feature there is the relation of the degeneration of a variety with the ramification of its cohomology. In the text, we omit proofs. We only mention here that all the results in the first three sections are proved by studying the ramification.

It is difficult to understand etale cohomology as a Galois representation just from its abstract definition. However, by studying its relation with many of mathematical objects such as differential forms, characteristic classes, Jacobi sums, modular forms, intersection theories etc., we understand it little by little. Among others, a deep relation with de Rham cohomology is impressive. Many of the results introduced in the text e.g. in 1a, 1c, 3c and 4b are on such a relation. They represent relations between the topological properties and the geometric properties of arithmetic varieties. When the base field is the complex number field $\mathbb{C}$, they are directly connected by the de Rham theorem, which is the foundation of Hodge theory. When the base field is a p-adic field, they are also directly connected by p-adic Hodge theory. However, if the characteristic of the base field p is positive, no direct relation among them is possible. In fact, one is a vector space over a field of characteristic 0 and the other is over a field of characteristic p. Nonetheless, they are miraculously deeply related to each other.

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1. Determinant. [S4], [S5]

In this section, K denotes an arbitrary field. For the results introduced in the following, the case where K is finite is essential. We keep the notation that X is a variety over a field K and $\mathcal{F}$ is a smooth ℓ-adic sheaf on X. We consider the character $\lambda_c(X,\mathcal{F})$ of G_K defined as the alternating product

$$\lambda_c(X,\mathcal{F}) : G_K \to \mathbb{Q}_\ell^\times, \qquad \sigma \mapsto \prod_{q=0}^{2\dim X} \det(\sigma : H_c^q(X_{\bar{K}},\mathcal{F}))^{(-1)^q}$$

of the determinant representations. This representation of G_K of dimension 1 is described as follows.

1a. Constant sheaves.

First, we consider the case where the sheaf $\mathcal{F}$ is the constant sheaf $\mathbb{Q}_\ell$. The case where X is proper and smooth is essential and we assume this. Then, the character

$\lambda_c(X, \mathbb{Q}_\ell)$ is the product of the $-\frac{n\chi}{2}$-th power of the cyclotomic character with a character ε of order at most 2. Here n is the dimension of X and χ denotes the Euler number $\chi = \sum_{q=0}^{2n}(-1)^q \dim H^q(X_{\bar{K}}, \mathbb{Q}_\ell)$. This fact is an immediate consequence of Poincaré duality. Further, if the dimension n is odd, the character ε is trivial and there is nothing more to study. In this case, the Euler number χ is even.

In the following, we assume $n = \dim X$ is even. Our task is to determine the quadratic extension L (which may be trivial) of K corresponding to the character ε. We assume the characteristic of K is different from 2. The cup product on the de Rham cohomology $H^n_{dR}(X/K)$ of the middle degree defines a non-degenerate quadratic form. Hence its discriminant $d \in K^\times/K^{\times 2}$ is defined.

Theorem 1. [S5 Theorem 2] *Assume X is projective and smooth over K. Then the quadratic extension L defined by the determinant $\lambda(X, \mathbb{Q}_\ell)$ is $K(\sqrt{\pm d})$. The sign ± 1 is determined by the parity of the integer $\frac{n\chi}{2} + \sum_{q\leq n, \text{odd}} \dim H^q_{dR}(X/K)$.*

As an application of Theorem 1, we obtain Ogus's conjecture ([Ou] Conjecture 3.11) on the crystalline determinant under the assumption that $\mathbb{F}_{p^2}$ is not a subfield of K.

1b. Coefficient sheaves.

Next, we turn to the case where we have coefficient sheaves.

Theorem 2. [S4 Theorem 1] *Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on a smooth variety X over a field K. Let $\bar{X}$ be a smooth projective variety including X as the complement of a divisor D with normal crossings. Assume that the ramification of $\mathcal{F}$ along D is tame (see 2c) and that the sheaf $\mathcal{F}$ satisfies a certain finiteness assumption. If* $\dim X \geq 3$, *we assume that the characteristic of K is not 2. Then the determinant character $\lambda_c(X, \mathcal{F})$ of G_K is equal to the product*

$$\lambda_c(X, \mathbb{Q}_\ell)^{\text{rank}\, \mathcal{F}} \times \det \mathcal{F}(c_{\bar{X},D}) \times J_{D,\mathcal{F}}.$$

The second term $\det \mathcal{F}(c_{\bar{X},D})$ is the character of G_K corresponing to the pull-back to the base field K of the rank 1 sheaf $\det \mathcal{F}$ by the relative canonical 0-cycle $c_{\bar{X},D}$. The third term $J_{D,\mathcal{F}}$ is the representation of dimension 1 defined by the algebraic Hecke character of Jacobi sums determined by the ramification of $\mathcal{F}$ along D. The tameness assumption is satisfied if the characteristic of K is 0. The precise finiteness assumption on $\mathcal{F}$ is not recalled here. It is satisfied for example if $\mathcal{F}$ has a geometric origin. When $X = \bar{X}$ is projective and K is finite, Theorem 2 is proved by Shuji Saito [SS]. Anderson [An] and Loeser [Lo] have closely related studies on trigonometric sums. Theorem 2 is applied in the study of the relation between Galois module sturcture and the signs in functional equations for variety over finite field by Chinburg [Ch].

The second and the third terms in are defined as follows. Definition of the second term requires class field theory in higher dimension. The third term is the Jacobi sum grössencharactere arising from the ramification. The definition is also explained in [Te].

First we consider the unramified case where $X = \bar{X}$. Let $\pi_1(X)^{ab}$ be the abelianized fundamental group of X. For each closed point $i_x : x \to X$, we consider the composite map $G_K^{ab} \xrightarrow{\text{transfer}} G_{K(x)}^{ab} \xrightarrow{i_{x*}} \pi_1(X)^{ab}$. The residue field at x is denoted by $K(x)$. The maps induce a pairing $G_K^{ab} \times CH_0(X) \to \pi_1(X)^{ab}$. The Chow group $CH_0(X)$ is the quotient of the free abelian group generated by the closed points in X divided by the rational equivalence. When K is finite, the pairing is equivalent to the reciprocity map of unramified class field theory in higher dimension.

The locally free sheaf $\Omega^1_{X/K}$ of rank $n = \dim X$ of differential 1-forms has top Chern class $c_n(\Omega^1_{X/K}) \in CH_0(X)$. The pairing with the canonical class $c_X = (-1)^n c_n(\Omega^1_{X/K})$ defines a homomorphism $G_K^{ab} \to \pi_1(X)^{ab}$. By pulling-back the representation $\det \mathcal{F}$ of $\pi_1(X)^{ab}$ by this homomorphism, we obtain a character $\det \mathcal{F}(c_X)$ of G_K. It is the second term for proper X. The third term is trivial in this case.

We proceed to the general case. The relative Chow group $CH_0(\bar{X}, D)$ is defined as an extension of $CH_0(\bar{X})$. The top Chern class $c_n(\Omega^1_{\bar{X}}(\log D))$ of the sheaf of 1-forms with log poles along D has a canonical lifting to $CH_0(\bar{X}, D)$. To define the lifting, we use the residue map $\Omega^1_{\bar{X}}(\log D)|_{D_i}$ for irreducible components D_i of D. Let $\pi_1(X)^{ab,tame}$ be the abelianization of the tame quotient $\pi_1(X)^{tame}$ of the fundamental group $\pi_1(X)$ of X classifying the coverings at most tamely ramified along D. The pairing above is refined to a pairing $G_K^{ab} \times CH_0(\bar{X}, D) \to \pi_1(X)^{ab,tame}$. Taking the pull-back of $\det \mathcal{F}$ by the pairing $G_K^{ab} \to \pi_1(X)^{ab,tame}$ with the relative canonical 0-cycle $c_{X,D} = (-1)^n c_n(\Omega^1_{\bar{X}}(\log D)) \in CH_0(\bar{X}, D)$, we obtain a character $\det \mathcal{F}(c_{X,D})$ of G_K.

The definition of the third term $J_{D,\mathcal{F}}$ is more complicated and we shall recall it only briefly. For simplicity, we assume that K is finite and that the constant field of each irreducible component D_i of D is K. The ramification of $\mathcal{F}$ along D_i gives rise to a representation ρ_i of $\varprojlim_n \mu_n(\bar{K})$. By the finiteness assumption on $\mathcal{F}$, its semi-simplification is the direct sum $\rho_i^{ss} = \bigoplus_{j=1}^r \chi_{ij}$ ($r = \text{rank } \mathcal{F}$) of characters of finite order. Let q denote the order of K and assume for simplicity that the order of each χ_{ij} divides $q-1$. Namely, the characters χ_{ij} are characters of $\mu_{q-1} = K^\times$. Further, we assume that the Euler number $c_i = \chi_c(D_i^o) = \sum_r (-1)^r \dim H_c^r(D^o_{i,\bar{K}}, \mathbb{Q}_\ell)$ of $D_i^o = D_i - \bigcup_{k \neq i} D_k$ is non negative for simplicity. Then the product $\prod_{ij} \chi_{ij}^{c_i}$ is the trivial character. The Jacobi sum $J_{D,\mathcal{F}}$ is defined by

$$J_{D,\mathcal{F}} = (-1)^{N-1} \sum_{(a_{ijk})} \prod_{i,j,k} \chi_{ij}(a_{ijk}).$$

Here (a_{ijk}) runs through $(\prod_{ij} K^{\times c_i})/K^\times \subset \mathbb{P}^{N-1}(K)$ and $N = r \sum_i c_i$. We also let $J_{D,\mathcal{F}}$ denote the character of G_K whose value at the geometric Frobenius Fr_K is the Jacobi sum $J_{D,\mathcal{F}}$. For general K, we define the representation $J_{D,\mathcal{F}}$ of G_K by reducing to this case.

1c. Analogy with Hodge structures.

A counterpart of Theorem 2 for Hodge structures was studied jointly with Terasoma [ST]. A realization of a motive over a number field K consists of ℓ-adic representations of G_K and Hodge structures at infinite places. Hence it is natural to investigate the analogy between ℓ-adic representations and Hodge structures [D2]. In the Hodge analogue, the canonical class in Theorem 2 is explained by Riemann-Roch.

A motive of rank 1 over a number field is conjectured to be associated to an algebraic Hecke character, i.e. comes from CM abelian varieties [D7]. Theorem 2 together with its Hodge analogue supports this conjecture.

This analogy to Hodge theory is not only the theorems themselves, but the proofs are also completely parallel. In both cases, we apply product formulas taking a projection to $\mathbb{P}^1$. The product formulas themselves are consequences of Fourier transform in both cases [La]. We refer [Te] for more detail.

The Hodge version of Theorem 2 is a formula for the determinant of period matrix. A p-adic Hodge version should give a formula for the determinant of p-adic period matrix. Using p-adic Hodge theory, one can deduced it from Theorem 2. However, we expect to have a proof using Fourier transform for p-adic $\mathcal{D}$-modules.

For the de Rham analogue, S.Bloch and H.Esnault proved a determinant formula for integrable connnections [BE].

2. Galois representations associated to modular forms, Langlands correspondence.

In this section, K denotes a totally real number field. For Hilbert modular forms, a commutative diagram

$$\begin{array}{ccc}
\left(\begin{array}{c}\ell\text{-adic representations}\\ \text{of } G_K \text{ of dimension } 2\end{array}\right) & \xleftarrow{\left(\begin{array}{c}\text{Langlands}\\ \text{correspondence}\end{array}\right)} & \left(\begin{array}{c}\text{Automorphic representa-}\\ \text{tions of } GL_2(\mathbb{A}_K)\end{array}\right) \\
\downarrow & & \downarrow \\
\left(\begin{array}{c}\ell\text{-adic representations}\\ \text{of } G_{K_p} \text{ of dimension } 2\end{array}\right) & \xleftarrow[\left(\begin{array}{c}\text{local Langlands}\\ \text{correspondence}\end{array}\right)]{} & \left(\begin{array}{c}\text{Irreducible admissible}\\ \text{representations of } GL_2(K_p)\end{array}\right)
\end{array}$$

is known as the compatibility between the global and local Langlands correspondences for a place p of K not dividing ℓ. Such a compatibility is proved also in the case $p|\ell$ using p-adic Hodge theory. The compatibility is a 2-dimensional version of the compatibility

$$\begin{array}{ccc}
(\text{characters of } G_K) & \xleftarrow{\left(\begin{array}{c}\text{global class}\\ \text{field theory}\end{array}\right)} & (\text{characters of } \mathbb{A}_K^\times) \\
\downarrow & & \downarrow \\
(\text{characters of } G_{K_p}) & \xleftarrow[\left(\begin{array}{c}\text{local class}\\ \text{field theory}\end{array}\right)]{} & (\text{characters of } K_p^\times)
\end{array}$$

of global and local class field theory.

2a. ℓ-adic representations associated to modular forms. [D3]

It is conjectured that there exists a correspondence between ℓ-adic representations of the absolute Galois group G_K of a number field K and automorphic forms of an algebraic group G over K. It is called the Langlands correspondence. Class field theory gives a one-to-one correspondence between the representations of G_K of degree 1 and the automorphic forms of the multiplicative group $\mathbb{G}_{m,K}$. The Langlands correspondence is a non-abelian generalization of class field theory.

When $K = \mathbb{Q}$ and $G = GL_2$, Fontaine and Mazur formulated a precise conjecture.

Conjecture. (Fontaine-Mazur) [FM] *The map*

$$\left\{ \begin{array}{c} \textit{normalized eigen new} \\ \textit{form of weight } k \geq 1 \end{array} \right\} \to \left\{ \begin{array}{c} \textit{isomorphism class of } \ell\textit{-adic} \\ \textit{representation of } G_{\mathbb{Q}} \textit{ of degree 2} \\ \textit{satisfying the following condition } * \end{array} \right\}$$

sending f to the ℓ-adic representation $V_{\ell,f}$ associated to f is a bijection.

* *$V_{\ell,f}$ is odd, unramified at almost finite places and, at the place ℓ, pst and of Hodge-Tate number $(0, k-1)$.*

An ℓ-adic representation of $G_{\mathbb{Q}}$ of degree 2 is called odd if the determinant of the complex conjugate is -1. The definition of being pst will be recalled later in 2b. The definition of the Hodge-Tate number $(0, k-1)$ is omitted.

An ℓ-adic representation $V = V_{\ell,f}$ of $G_{\mathbb{Q}}$ is said to be associated to a normalized eigen new form $f = \sum_{n=1} a_n q^n$ if its L-function $L(V, s)$ is equal to $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$. For an ℓ-adic representation V unramified at almost finite places, its L-function $L(V, s)$ is defined by

$$L(V, s) = \prod_{p\text{: prime}} \det(1 - Fr_p p^{-s} : V^{I_p})^{-1}.$$

In the right hand side, V^{I_p} denotes the inertia fixed part and Fr_p denotes the geometric Frobenius at a prime p. If $k \geq 2$, the ℓ-adic representation $V_{\ell,f}$ associated to a modular form f is constructed by using the etale cohomology of modular curves and Kuga-Sato varieties, which are the fiber products of the universal elliptic curves over them [D3]. The Ramanujan conjecture asserts that, for the modular form $\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_n \tau(n) q^n$, the inequality $\tau(p) \leq 2p^{\frac{11}{2}}$ is satisfied for each prime p. It is proved by reducing it to the Weil conjecture on congruence zeta functions using the construction of $V_{\ell,\Delta}$ as etale cohomology.

The theorem of Wiles [W] asserting that a (semi-stable) elliptic curve E over $\mathbb{Q}$ is modular may be considered as a part of the correspondence. In fact, etale cohomology $V = H^1(E_{\bar{Q}}, \mathbb{Q}_\ell)$ is an ℓ-adic representation of $G_{\mathbb{Q}}$ satisfying the condition $*$ in Conjecture above. The existence of f such that $V = V_{\ell,f}$ means that E is modular. The core in Wiles's proof is an essential surjectivity of the map in Conjecture above. The compatibility with the local Langlands correspondence plays an important role in the proof.

2b. Weil-Deligne group of local fields. [D5]

The local Langlands correspondence is described in terms of the Weil-Deligne group of a local field, which we recall briefly. If K is a finite field, an ℓ-adic representation of the absolute Galois group G_K is determined by the action of the geometric Frobenius Fr_K. Or equivalently, by the representation of the Weil group

$W_K = \langle Fr_K \rangle \subset G_K$. The role of the Weil group for a finite field is played by the Weil-Deligne group for a local field. The compatibility with the local Langlands correspondence mentioned above is described in terms of the representations of the Weil-Deligne group associated to ℓ-adic or p-adic representations of the Galois group. If $\ell \neq p$, ℓ-adic representations of the Galois group is directly related to representations of the Weil-Deligne group by the monodromy theorem of Grothendieck. In this case, these representations carry equivalent informations. If $\ell = p$, we need p-adic Hodge theory to link p-adic representations of the Galois group with representations of the Weil-Deligne group. In this case, p-adic representations of the Galois group carry more informations on the Hodge filtrations.

First, we briefly recall the structure of the absolute Galois group of a local field. It is described in the following picture

$$\begin{array}{ccccccc} G_K & \supset & I & \supset & P & \supset & 1 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ K & \subset & K^{ur} & \subset & K^{tr} & \subset & \bar{K}. \end{array}$$

Here $K^{ur} = K(\zeta_m, p \nmid m)$ denotes the maximum unramified extension of K and $K^{tr} = K^{ur}(\pi^{1/m}, p \nmid m)$ denotes the maximum tamely ramified extension of K in $\bar{K}$ where π is a uniformizer of K. Let F be the residue field of K. The kernel I of the canonical surjection $G_K \to G_F$ is called the inertia subgroup. A canonical surjection

$$\prod_\ell t_\ell : I \longrightarrow \varprojlim_{p\nmid n} \mu_n = \prod_{\ell \neq p} \mathbb{Z}_\ell(1); \sigma \mapsto (\sigma(\pi^{\frac{1}{n}})/\pi^{\frac{1}{n}})_n$$

is independent of the choice of a prime element π in K. The kernel P is a pro-p group and is called the ramification group. The condition in Theorem 2 that the ramification of $\mathcal{F}$ is tame means the following: For each irreducible component of $D = \bar{X} - X$, the restriction of the local monodromy to the ramification group is trivial.

The inverse image $W_K \subset G_K$ of the Weil group $W_F \subset G_F$ of the residue field F is called the Weil group of K. A representation of the Weil-Deligne group W'_K is a pair (ρ', N) satisfying a certain compatibility condition. Here $\rho' : W_K \to GL(V)$ is a representation of the Weil group W_K whose kernel in the inertial subroup I is open and $N \in \mathrm{End} V \otimes \mathbb{Q}_\ell(-1)$ is a nilpotent endomorphism. If $\rho : G_K \to GL(V)$ is an ℓ-adic representation of the absolute Galois group G_K of a local field K and $\ell \neq p$, there exists a unique representation (ρ', N) of the Weil-Deligne group satisfying

$$\rho(F^n\sigma) = \rho'(F^n\sigma)\exp(t_\ell(\sigma)N).$$

Here, $F \in W_K$ is an arbitrarily fixed lifting of the geometric Frobenius Fr_F, σ is an element in I and n is an integer. It is a consequence of the monodromy theorem of Grothendieck.

We consider the case $p = \ell$. Fontaine [Fo] defined a linear representation of the Weil-Deligne group on the $\hat{\mathbb{Q}}_p^{ur}$-vector space

$$D(V) = \bigcup_{J \subset I:\text{open}} (V \otimes B_{st})^J.$$

Here B_{st} denotes a huge p-adic ring defined by Fontaine [Bu]. We have an inequality

$$\dim_{\mathbb{Q}_p} V \geq \dim_{\hat{\mathbb{Q}}_p^{nr}} D(V)$$

in general. When we have an equality, we say V is potentially semistable (pst for short). The etale cohomology $V = H^q(X_{\bar{K}}, \mathbb{Q}_p)$ is a pst representation for a proper smooth variety over a local field K. It is a consequence of the so-called C_{st}-conjecture proved by Kato-Hyodo-Tsuji [Tj] and the alteration proved by de Jong [dJ]. If X has a semi-stable model X_{O_L} over the integer ring O_L of a finite extension L of K, we have a canonical isomorphism

$$D(V) = H^q_{\text{log cris}}(Y/W) \otimes_W \hat{\mathbb{Q}}_p^{ur}$$

where Y denotes the closed fiber of the semi-stable model X_{O_L}. We say X_{O_L} is a semi-stable model if it is a proper regular model and the closed fiber Y has normal crossings. The second condition is equivalent to that X_{O_L} is etale locally etale over Spec $O_L[T_1, \cdots, T_d]/(\prod_i T_i - \pi_L)$. The cohomology in the right hand side is log crystalline cohomology of Y with respect to the natural log structure on Y [K1].

2c. Compatibility with the local Langlands correpondence. [Ca], [S7]

We consider the ℓ-adic representation $V_{\ell,f}$ of degree 2 of $G_{\mathbb{Q}}$ associated to a modular form f. For each prime p, we obtain representations of the Weil-Deligne group $(\rho_{f,\ell}|'_{G_{\mathbb{Q}_p}}, N)$ if $\ell \neq p$ and $D_{pst}(\mathrm{Res}^{G_{\mathbb{Q}}}_{G_{\mathbb{Q}_p}} V_{f,p})$ if $\ell = p$. They are defined by applying the construction in the last subsection to the restriction $\rho_{f,\ell}|_{G_{\mathbb{Q}_p}}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$.

They are defined by using the Galois representations associated to a modular form f. On the other hand, there is another way to construct a representation of Weil-Deligne group using the automorphic representation associated to f. Let π_f be the automorphic representation of the adele group $GL_2(\mathbb{A}_{\mathbb{Q}})$ generated by f [D4]. It is factorized into tensor product $\pi_f = \bigotimes_{p\leq\infty} \pi_{f,p}$ of irreducible admissible representations $\pi_{f,p}$ of $GL_2(\mathbb{Q}_p)$. The local Langlands correspondence is a one-to-one correspondence

$$\begin{pmatrix} \text{irreducible admissible} \\ \text{representations of } GL_2(\mathbb{Q}_p) \end{pmatrix} \to \begin{pmatrix} \text{F-semi-simple representations of} \\ \text{Weil-Deligne group } W'_{\mathbb{Q}_p} \text{ of degree 2} \end{pmatrix}.$$

Applying it to the representation $\pi_{f,p}$ of $GL_2(\mathbb{Q}_p)$, we obtain a representation $\sigma(\pi_{f,p})$ of the Weil-Deligne group $W'_{\mathbb{Q}_p}$. A representation (ρ', N) of the Weil-Deligne group is called F-semi-simple if the representation ρ' of the Weil group is semi-simple.

It is known that the two construction give the same answer for $\ell \neq p$.

Theorem. (Deligne-Langlands-Carayol) [Ca] *Assume $p \neq \ell$. Then the F-semi-simplification of the representation $(\rho_{f,\ell}|'_{G_{\mathbb{Q}_p}}, N)$ is isomorphic to $\sigma(\pi_{f,p})$.*

This is the compatibility mentioned above. For $p = \ell$, we have the following result.

Theorem 3. [S7] *For a modular form f and a prime p, the F-semi-simplification of the representation $D_{pst}(Res^{G_{\mathbb{Q}}}_{G_{\mathbb{Q}_p}} V_{f,p})$ of the Weil-Deligne group is isomorphic to $\sigma(\pi_{f,p})$.*

This is proved by reducing it to Deligne-Langlands-Carayol Theorem above. Namely, we prove it by comparing $p \neq \ell$ and $p = \ell$. The main points are the following three.

(1) The p-adic representation $\mathrm{Res}^{G_{\mathbb{Q}}}_{G_{\mathbb{Q}_p}} V_{f,p}$ is pst.
(2) The traces of the action of $\sigma \in W_K$ are the same for p and ℓ.
(3) The ranks of the monodromy operator N are the same for p and ℓ.

The assertion (1) follows from the fact that the representation $V_{f,\ell}$ is constructed geometrically using etale cohomology and the theorem of Kato-Hyodo-Tsuji above. The assertion (2) is deduced from a geometric computation of the traces using the reduction mod p. The essential ingredient is that the Lefschetz trace formula takes the same form for $p \neq \ell$ and for $p = \ell$. We need a more detailed argument for (3) using that modular curves are of dimension 1 (see 4c).

An analogous statement as Theorem 3 for Hilbert modular forms in place of usual elliptic modular forms is proved under a certain condition. In this case, we consider a representation of the absolute Galois group G_F of a totally real number field F in place of a representation of $G_{\mathbb{Q}}$.

The p-adic representation $\mathrm{Res}^{G_{\mathbb{Q}}}_{G_{\mathbb{Q}_p}} V_{f,p}$ defines not only the representation of the Weil-Deligne group but also the Hodge filtration. Theorem 3 does not tell how one can determine the Hodge filtration.

3. Sign in the functional equation of L-functions and Stiefel-Whitney classes.

The fields we consider are number fields in 3a and 3b and are arbitrary fields in 3c. For an orthogonal representation of the Galois group of a number field, it is shown that the sign in the functional equation of the L-function is positive. The proof uses the Stiefel-Whitney class. The Stiefel-Whitney class of a variety of even dimension is expected to be related to the Hasse-Witt class of the de Rham cohomology.

3a. Sign in functional equation of L-functions. [Se3]

Let us recall that the Hasse-Weil L-function $L(H^m(X), s)$ for a variety X over a number field K is defined as

$$L(H^m(X), s) = \prod_{p:\ \mathrm{prime}} \det(1 - Fr_p(Np)^{-s} : V^{I_p})^{-1}$$

in terms of the ℓ-adic representation $V = H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$ of G_K. Strictly speaking, the Euler factor for a prime p above ℓ is defined by the representation $D_{\mathrm{pst}}(V)$ of the Weil-Deligne group recalled in 2d. It is conjectured that the L-function is independent of the choice of the prime ℓ and has a meromorphic continuation to the whole s-plane. Further, if we define the completed L-function $\Lambda(H^m(X), s)$ by multiplying the Γ-factors, it is conjectured to satisfy a functional equation of the form

$$\Lambda(H^m(X), s) = \pm N^{\frac{m+1}{2} - s} \Lambda(H^m(X), m + 1 - s)$$

[Se3]. The Γ-factors are determined by the Hodge structures $H^m(X(\bar{K}_v), \mathbb{Q})$ for infinite places v of K. In the functional equation, N denotes a positive integer called the conductor and is determined by the ramification of $H^m(X)$ and of K over $\mathbb{Q}$. The sign ± 1 is called the sign in the functional equation and shall be denoted by w.

When $X = E$ is an elliptic curve and $m = 1$, Birch and Swinnerton-Dyer conjectured that $L(H^1(E), s)$ has a zero of order $r = \text{rank } E(K)$ at $s = 1$. According to this conjecture, we should have $w = (-1)^r$. In general, if m is odd, Bloch and Beilinson conjectured that $L(H^m(X), s)$ has a zero of order $r = \text{rank } CH^{\frac{m+1}{2}}(X)_{\text{hom}}$ at $s = \frac{m+1}{2}$, where $CH^i(X)_{\text{hom}}$ denotes the kernel of the cycle map $CH^i(X) \to H^{2i}(X_{\bar{K}}, \mathbb{Q}_\ell)$ from the Chow group of algebraic cycles of codimension i. The conjecture also implies that w should be determined by the parity of the rank r. When m is odd, the sign in the functional equation is conjectured to be determined by the parity of the rank of a group of arithmetic significance.

3b. Case of even degree.

On the other hand, when m is even, the sign should be always positive.

Theorem 4. [S6] *Assume ℓ is sufficiently large and that the constant in the functional equation is the product of the local ε-factors. Then the sign w in the functional equation is $+1$ if the degree m is even.*

We say ℓ is sufficiently large when ℓ is not 2, bigger than $2m$ and X has good reduction at the places above ℓ. The assumption that the constant is the product of the local ε-factors means that the sign w is defined by the formula

$$\pm N^{\frac{m+1}{2}} = \prod_{v:\text{ place of } K} \varepsilon_v(H^m(X_{\bar{K}}, \mathbb{Q}_\ell), \psi_v, dx_v).$$

This conjectural formula is called the product formula for the constant in the functional equation. In view of the analogy between number fields and function fields, the fact that the sign is easily determined in Theorem 4 for *even* degree corresponds to the fact that the determinant is easily determined in Theorem 1 for *odd* dimension.

The starting point of the proof of Theorem 4 is that the etale cohomology $H^m(X_{\bar{K}}, \mathbb{Q}_\ell)(\frac{m}{2})$ has the cup-product as a non-degenerate quadratic form as m is even and hence is an orthogonal representation of G_K. For an orthogonal representation V of G_K, its second Steifel-Whitney class $sw_2(V) \in H^2(K, \mathbb{Z}/2\mathbb{Z})$ is defined in the Galois cohomology. At places $p \nmid \ell$, it is known that the sign of the local ε-factor is determined by the second Steifel-Whitney $sw_2(\text{Res}^{G_{\mathbb{Q}}}_{G_{\mathbb{Q}_p}} V)$ [D6]. The main point in the proof is to show that it is so at places dividing ℓ using p-adic Hodge theory. Then since an element in $H^2(K, \mathbb{Z}/2\mathbb{Z}) = {}_2Br(K)$ is given by the Hilbert symbol, Theorem 4 follows from the quadratic reciprocity law $\prod_v (a, b)_v = 1$.

Theorem 4 is a statement for orthogonal representations. It is expected to have a generalization to certain similitude representations. It looks interesting to investigate the relation with the theory on Galois module structure and the sign in functional equations by Fröhlich, Taylor etc. [Fr], [Ta]. In its generalization to varieties over finite field by Chinburg [Ch], Theorem 2 is applied.

3c. Steifel-Whitney classes and de Rham cohomology.

The second Steifel-Whitney class of an orthogonal representation V plays an important role in the proof of Theorem 4. The first Steifel-Whitney class $sw_1(V)$ is nothing other than the character $G_K \to \{\pm 1\}$ defined by the determinant regarded as an element in $H^1(G_K, \mathbb{Z}/2)$. From this point of view, Theorem 1 is considered to give a relation between the first Steifel-Whitney class $sw_1(V)$ of an orthogonal representation $V = H^m(X_{\bar{K}}, \mathbb{Q}_\ell)(\frac{m}{2})$ of G_K and the discriminant of de Rham cohomology $H^m_{dR}(X/K)$. By Kummer theory, we identify $H^1(G_K, \mathbb{Z}/2) = K^\times/K^{\times 2}$.

If the discriminant is the first invariant of a quadratic form, the second invariant is the so-called Hasse-Witt invariant. Similarly as in Theorem 1, we expect that the second Steifel-Whitney class $sw_2(V) \in H^2(K, \mathbb{Z}/2)$ is related to the Hasse-Witt class $hw_2(D) \in H^2(K, \mathbb{Z}/2)$ of de Rham cohomology $D = H^m_{dR}(X/K)$ for a variety X over an arbitrary field K. An exact relation between $sw_2(V)$ and $hw_2(D)$ is conjectured and is proved for smooth hypersurfaces in [S8]. Contrary to $sw_1(V)$, the class $sw_2(V)$ does depend on ℓ. The relation is a generalization to higher dimension of a formula for a finite separable extension L of K [Se4]. Such a relation for a number field K will give another proof for Theorem 4. There is a related study on the relation between the Steifel-Whitney classes and and Hasse-Witt classes [EKV].

4. Ramification and degeneration.

Let K be a local field. Classical theory on ramification in finite extensions of local fields is found in the book [Se1]. We generalize it to varieties of higher dimension over a local field by regarding finite extensions as varieties of dimension 0. Generalization to finite covering schemes of higher dimensions is a similar problem.

4a. Conductor.

The basic invariants for ℓ-adic representations V of the absolute Galois group G_K of a local field K are as in the second line of the table below.

	F is algebraically closed	F is finite
V is a representation of G_F	dimension $\dim V$	determinant $\det(Fr_F : V)$
V is a representation of G_K	conductor $a(V), sw(V)$	ε-factor $\varepsilon(V), \varepsilon_0(V)$

The first line shows the corresponding invariants of ℓ-adic representations of the absolute Galois group G_F of the residue field. In the table, the invariants in the second column are ℓ-adic numbers and the invariants in the first column are non-negative integers. Those in the second column are often algebraic numbers in practice. Roughly speaking, their complex absolute value are powers of the order of F to half of the invariants in the first column. If K is the local field at a point of an algebraic curve over a field F, the invariants in the lower line appear as the local contribution to the global invariants in the upper line, in the formula of Grothendieck-Ogg-Shafarevich [G] and the product formula of Deligne-Laumon [D4], [La]. The Artin conductor $a(V)$ and the Swan conductor $sw(V)$ are related by $a(V) = \dim V - \dim V^I + sw(V)$. The Swan conductor $sw(V)$ is 0 if and only if the action of the ramification group P of V is trivial. The conductor in the functional equation of L-function in 3a is defined by the Artin conductor.

4b. Conductor formula of Bloch. [B], [KS]

A fundamental formula in ramification theory is the conductor-discriminant formula $a(V) = \mathrm{ord}_L D_{L/K} \cdot f_{L/K}$ for a finite extension L of a local field K. The left hand side is the Artin conductor of the representation $V = \mathbb{Q}_\ell^{\mathrm{Hom}_K(L,\bar{K})}$. In the right hand side, $D_{L/K}$ is the relative different and $f_{L/K}$ is the residual degree. The right hand side is equal to the valuation of the discriminant. Interpreting V as the etale cohomology $H^0(X_{\bar{K}}, \mathbb{Q}_\ell)$ for $X = \mathrm{Spec}\ L$, we consider a generalization to higher dimension.

Bloch formulated a conductor formula for a proper regular scheme X_O over the integer ring O of a local field K with smooth generic fiber X_K as follows. First using the idea of the localized Chern class of Fulton-MacPherson, he defines the self-intersection class

$$(\Delta, \Delta) = (-1)^n c_n(\Omega^1_{X_O/O_K}) \in CH_0(X_F)$$

as a 0-cycle class supported on the closed fiber X_F. Here n denotes the dimension of X_O. Let $SwH^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ denote the alternating sum of the Swan conductors and define the conductor of X by

$$Art(X) = \chi(X_{\bar{K}}) - \chi(X_{\bar{F}}) + SwH^*(X_{\bar{K}}, \mathbb{Q}_\ell).$$

Bloch conjectured the following conductor formula.

Conjecture. (Bloch) *The conductor $Art(X)$ of a regular proper scheme X over the integer ring is equal to the minus of degree of the canonical 0-cycle (Δ, Δ):*

$$Art(X) = -\deg(\Delta, \Delta).$$

The right hand side of the formula can be described in terms of de Rham cohomology. Hence it gives one more example of the relation between the etale cohomology and the de Rham cohomology. In the same paper [B], he proved it for curves.

Theorem. (Bloch) *Conjecture is true if* $\dim X_K = 1$.

Recently, the conjecture is proved under a mild assumption in a joint work with Kato.

Theorem 5. [KS] *Assume that the reduced closed fiber $X_{F,\mathrm{red}}$ is a divisor with normal crossings. Then Conjecture is true.*

The proof uses log geometry [K1], alteration [dJ] and a K-theoretic version of the localized intersection theory [Ab]. Theorem 5 implies that Conjecture is true if the reduced closed fiber admits an embedded resolution in the strong sense. Namely, if there is a sequence of blowing-ups X_i of X at non-singular closed subschemes of closed fibers such that the reduced closed fiber of X_n has normal crossings, then Conjecture for X is true. The tame case was proved earlier in [Ar], [CPT2]. An application of the conductor formula to the relation between the Galois module structure and the signs in the functional equations of L-functions is studied in [CPT2]. In [KS], the conductor formula is generalized to a formula involving algebraic correspondence. It was studied earlier in [Ab] for curves. A congruence modulo 2 was proved by a different method [S9].

Used the idea of Bloch on the conductor formula, Kato proved the Serre conjecture for the Artin character of a group action with isolated fixed point on regular local ring of dimension 2 [K2]. He also formulated and proved a Riemann-Roch formula for the Euler number of ℓ-adic sheaves of rank 1 with wild ramification on an algebraic surface [K2], [S3]. This is a generalization to higher dimension of the Grothendieck-Ogg-Shararevich formula.

4c. Conductor and discriminant.

The conductor formula of Bloch is a generalization to the higher dimension of the relation between the conductor and the different for finite extensions. For finite extensions, the different is related to the discriminant. The Tate-Ogg formula [Og] for elliptic curves is a generalization of the relation between the conductor and the discriminant. For a curve of genus $g \geq 1$, its discriminant is defined using the Mumford isomorphism. The Mumford isomorphism is a canonical isomorphism

$$(\det Rf_*\omega_{X/S})^{\otimes 13} \to \det Rf_*(\omega_{X/S}^{\otimes 2})$$

between the Hodge bundles defined for an arbitrary proper smooth family of curves $f : X \to S$. It is known to have a zero of order 1 at the boundary of the Deligne-Mumford compactification of the moduli space of curves.

Theorem 6. [S2] *Let X be a proper regular curve over the integer ring of a local field. Then the valuation of the discriminant is equal to the conductor of X.*

The Tate-Ogg formula is a special case of Theorem 6. For curves of genus 2, there is another definition of the discriminant using the fact that they are hypergeometric curves. Theorem 6 also implies a relation between this discriminant and the conductor [Li]. It gives the conductor which is of arithmetic nature a geometric meaning as the intersection number with the boundary of the moduli space of curves. The Noether formula for arithmetic surfaces with arbitrary bad reduction is also a consequence of it [Fa].

We do not know a definition of the discriminant for varieties of higher dimension, with which we can generalize Theorem 6.

4d. Stable reduction.

A fundamental theorem for curves over local fields is the stable reduction theorem. It asserts that a curve over a local field has a semi-stable model over a finite extension of the base field [DM]. It plays a crucial role in the proof of conductor formula for curves in [B]. Its generalization to higher dimension is an open problem except the case where the characteristic of the residue field F is 0 [KKMS].

Semi-stable reduction problem. *Let X_K be a proper smooth variety over a local field K. Is there a finite extension L of K, such that $X_L = X \otimes_K L$ has a semi-stable model?*

In many applications e.g. for the proof of the conductor formula in [KS], its weaker version, alteration [dJ], suffices. For example, by a similar argument as in (2) in the proof in Theorem 3, the following is proved.

Theorem. (Ochiai) [Oc] *For an arbitrary element σ in the Weil group, the alternating sum $\sum_q (-1)^q Tr\ (\sigma : H^q(X_{\bar{K}}, \mathbb{Q}_\ell))$ is independent of ℓ.*

However, the following conjecture remains open if the dimension is greater than 2, even assuming stable reduction.

Monodromy-weight conjecture. [D2] *Let X_K be a proper smooth variety over a local field K, q be the order of the residue field and F be a lifting to W_K of the geometric Frobenius. Then the complex absolute values of the eigenvalues of F acting on $Gr^i_M H^j(X_{\bar{K}}, \mathbb{Q}_\ell)$ are $q^{(i+j)/2}$.*

The monodromy filtration M on $V = H^q(X_{\bar{K}}, \mathbb{Q}_\ell)$ is the unique increasing filtration characterized by the following conditions: $NM^i \subset M^{i-2}$ and N^i induces an isomorphism $Gr^i_M \simeq Gr^{-i}_M$ for $i \geq 0$. Roughly speaking, the monodromy-weight conjecture implies that a representation of the Weil-Deligne group is determined by the underlying representation of the Weil group. The step (3) in the proof of Theorem 3 is proved by showing a special case of the monodromy-weight conjecture [S10].

A log smooth model is a model which is smooth in the sense of log structure [K1]. Log smoothness is a weaker condition than semi-stability. For example, a regular model is log smooth if the following condition is satisfied: The reduced closed fiber is a divisor with normal crossings and the multiplicities of the irreducible components are prime to the residual characteristic. In many aspect, ramification theory for log smooth models is as easy as in characteritic 0. For example, by a similar argument as in [KKMS], one can solve the semi-stable reduction problem for a log smooth model.

Theorem. (Yoshioka) [Y] *If X_K has a log smooth model, the semi-stable reduction problem for X_K has an affirmative answer.*

If X_K is a proper smooth variety over a local field K with a log smooth model, the alternating sum $\sum_q (-1)^q [H^q(X_{\bar{K}}, \mathbb{Q}_\ell)]$ has a simple description as an element in the Grothendieck group of the representations of the inertia group I. Let X_O be a regular proper model of X_K and assume that the reduced closed fiber is a divisor with normal crossings and that the multiplicities m_i of irreducible components D_i are prime to the residual characteristic. Then the alternating sum is $\sum_i c_i \mathrm{Ind}^I_{I_{m_i}} \mathbb{Q}_\ell$. Here $c_i = \chi_c(D^o_i)$ is the Euler number $D^o_i = D_i - \bigcup_{k \neq i} D_k$ appeared in the third term in the right hand side of Theorem 2. The ε-factor $\prod_q \varepsilon_{0,K}(H^q(X_{\bar{K}}, \mathbb{Q}_\ell), \psi, \mu)^{(-1)^q}$ may be computed easily from this. Using this computation, a relation between Galois module structure and the sign in functional equations is proved for arithmetic varieties in [CEPT], [CPT]. One can also compute the second Stiefel-Whitney class and compare it with the Hasse-Witt class by using the relation with ε-factor for log smooth scheme of relative even dimension (see 3d) [S8].

The stable reduction theorem for curves X_K of genus at least 2 is stated more precisely as follows: A curve X_K has a semi-stable model if and only if the action of the inertia group I on $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent. An analogue for log smooth models is the following

Theorem 7. [S1] *A proper smooth curve X_K over a local field K of genus at least 2 has a log smooth model if and only if the action of the ramification group P on $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ is trivial.*

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